On Godbersen's Conjecture

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Abstract

We provide a natural generalization of a geometric conjecture of Fáry and Rédei regarding the volume of the convex hull of $K \subset \mathbb{R}^n$, and its negative image $-K$. We show that it implies Godbersen's conjecture regarding the mixed volumes of the convex bodies K and $-K$. We then use the same type of reasoning to produce the currently best known upper bound for the mixed volumes $V(K[j], -K[n-j])$, which is not far from Godbersen's conjectured bound. To this end we prove a certain functional inequality generalizing Colesanti's difference function inequality.

1 Introduction and results

In this note we consider convex bodies $K \subset \mathbb{R}^n$, that is, compact convex sets with non-empty interior. The well known Rogers–Shephard bound for the volume of the so called "difference body", $K - K = \{x - y \,|\, x, y \in K\}$, states that

$$\mathrm{Vol}(K-K) \le \binom{2n}{n} \mathrm{Vol}(K). \tag{1}$$

This inequality was proved by Rogers and Shephard in [11], where it was also shown that equality is attained only for simplices. By a simplex we mean the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. Chakerian simplified their argument in [2], and in [12] they gave another variant of the proof, which we address in the appendix of this text.

A conjectured strengthening of the difference body inequality was suggested in 1938 by Godbersen [8] (and independently by Makai Jr. [9]).

Conjecture 1.1. *For any convex body $K \subset \mathbb{R}^n$ and any $1 \le j \le n-1$,*

$$V(K[j], -K[n-j]) \le \binom{n}{j} \mathrm{Vol}(K), \tag{2}$$

with equality attained only for simplices.

Here $V(K_1,\dots,K_n)$ denotes the mixed volume of the n convex bodies $K_1,\dots,K_n$, and $V(K[j],T[n-j])$ denotes the mixed volume of j copies of the convex body K and $n-j$ copies of the convex body T. We recall that for convex bodies $K_1,\dots,K_m\subset\mathbb{R}^n$, and non-negative real numbers $\lambda_1,\dots,\lambda_m$, a classical result of Minkowski states that the volume of $\sum\lambda_iK_i$ is a homogeneous polynomial of degree n in λ_i,

$$\mathrm{Vol}\left(\sum_{i=1}^m\lambda_iK_i\right)=\sum_{i_1,\dots,i_n=1}^m\lambda_{i_1}\cdots\lambda_{i_n}V(K_{i_1},\dots,K_{i_n}),\tag{3}$$

and the coefficient $V(K_{i_1},\dots,K_{i_n})$, which depends solely on $K_{i_1},\dots,K_{i_n}$, is called the mixed volume of $K_{i_1},\dots,K_{i_n}$. The mixed volume is a non-negative, translation invariant function, monotone with respect to set inclusion, invariant under permutations of its arguments, and positively homogeneous in each argument. Moreover, one has $V(K[n])=\mathrm{Vol}(K)$. For further information on mixed volumes and their properties, see Section §5.1 of [14].

The cases $j=1$ and $j=n-1$ of Conjecture 1.1 follow from the fact that $-K\subset nK$ for bodies with center of mass at the origin (see [1], page 57). The same argument gives the bound

$$V(K[j],-K[n-j])\le n^{\min\{j,n-j\}}\mathrm{Vol}(K),$$

for $0\le j\le n$. The only other cases for which Conjecture 1.1 is verified, are simplices (which are the equality case) and bodies of constant width, as shown in [8]. Godbersen's conjecture is indeed a strengthening of the difference body inequality (1) since, if Conjecture 1.1 holds true, one may write

$$\mathrm{Vol}(K-K)=\sum_{j=0}^n\binom{n}{j}V(K[j],-K[n-j])\le\sum_{j=0}^n\binom{n}{j}^2\mathrm{Vol}(K)=\binom{2n}{n}\mathrm{Vol}(K).$$

In 1950, Fáry and Rédei [6] conjectured that for all convex bodies K of fixed volume, one has

$$\min_{x\in K}\mathrm{Vol}((K-x)\vee(x-K))\le\min_{x\in S}\mathrm{Vol}((S-x)\vee(x-S)),\tag{4}$$

where S is a simplex with $\mathrm{Vol}(S)=\mathrm{Vol}(K)$. Here, $A\vee B$ denotes the convex hull of the set $A\cup B$, tough we shall sometimes use $\mathrm{conv}(A,B)$ to denote this set. Moreover, they showed that the right-hand side of (4) is precisely $\binom{n}{[n/2]}\mathrm{Vol}(S)$. We conjecture the following generalization of (4).

Conjecture 1.2. *For any convex body $K\subset\mathbb{R}^n$ and every $\lambda\in[0,1]$, there exists $x\in K$ such that*

$$\mathrm{Vol}((1-\lambda)(K-x)\vee\lambda(x-K))\le\mathrm{Vol}((1-\lambda)S\vee-\lambda S),\tag{5}$$

where S is a centered simplex, and $\mathrm{Vol}(S)=\mathrm{Vol}(K)$.

Note that the case $\lambda = 1/2$ implies the above mentioned conjecture by Fáry and Rédei (since for the simplex, the convex hull of minimal volume is attained when it is centered, see [4]). The numerical value of the right-hand side of (5) can be computed explicitly (see Section 2), so that Conjecture 1.2 would give a numerical upper bound for the quantity on the left-hand side of (5). Moreover, we remark that in dimension $n = 2$, Conjecture 1.2 holds true (see Section 4 for a discussion of the planar case).

Our first result in this paper states that Conjecture 1.2 implies Godbersen's conjecture.

Theorem 1.3. *Conjecture 1.2 implies Conjecture 1.1.*

Our second result is, to the best of our knowledge, the smallest upper bound for $V(K[j], -K[n-j])$ currently known for $2 < j < n-2$.

Theorem 1.4. *Let $K \subset \mathbb{R}^n$ be a convex body, and $1 \le j \le n-1$. Then*

$$V(K[j], -K[n-j]) \le \frac{n^n}{j^j (n-j)^{n-j}} \mathrm{Vol}(K) \simeq \binom{n}{j} \sqrt{2\pi \frac{j(n-j)}{n}} \mathrm{Vol}(K).$$

The proof requires some preparation. Rogers and Shephard showed in [12] that if $0 \in K$, then

$$\mathrm{Vol}(K \vee -K) \le 2^n \mathrm{Vol}(K), \tag{6}$$

and that the bound is attained only when K is a simplex with 0 as a vertex. Another proof for this bound was given by Colesanti in [3] (see Section 3 below). Colesanti's proof is based on a functional analogue of the difference body, which he calls the "difference function". Using monotonicity of mixed volumes, (6) implies that for every $1 \le j \le n-1$,

$$V(K[j], -K[n-j]) \le \mathrm{Vol}(K \vee -K) \le 2^n \mathrm{Vol}(K).$$

Our next result generalizes inequality (6).

Theorem 1.5. *For any convex body $K \subset \mathbb{R}^n$ containing the origin and every $\lambda \in [0, 1]$*

$$\mathrm{Vol}((1-\lambda)K \vee -\lambda K) \le \mathrm{Vol}(K). \tag{7}$$

Theorem 1.4 is an immediate corollary of Theorem 1.5.

Proof of Theorem 1.4: Let $1 \le j \le n-1$, and set $\lambda = (n-j)/n$. Assume without loss of generality that $0 \in K$. Since $(1-\lambda)K$ and $-\lambda K$ are contained in

$(1-\lambda)K \vee -\lambda K$, the monotonicity and homogeneity properties of the mixed volume imply:

$$\begin{aligned} V(K[j], -K[n-j]) &= \frac{1}{(1-\lambda)^j \lambda^{n-j}} V((1-\lambda)K[j], -\lambda K[n-j]) \\ &\le \frac{1}{(1-\lambda)^j \lambda^{n-j}} \mathrm{Vol}((1-\lambda)K \vee -\lambda K) \le \frac{1}{(1-\lambda)^j \lambda^{n-j}} \mathrm{Vol}(K), \end{aligned}$$

where the last inequality follows from Theorem 1.5. Plugging in our choice of λ yields precisely the desired bound of the theorem. □

Theorem 1.5 follows as a special case (where $K = (1-\lambda)K'$, $L = \lambda K'$, and $\theta = \lambda$) from the following theorem, which is a variation of a result by Rogers and Shephard [12].

Theorem 1.6. *Let $K, L \subseteq \mathbb{R}^n$ be two convex bodies such that $0 \in K \cap L$. For every $\theta \in [0,1]$,*

$$\mathrm{Vol}(L \vee -K)\, \mathrm{Vol}(\theta K \cap (1-\theta)L) \le \mathrm{Vol}(K)\mathrm{Vol}(L).$$

We will prove in the appendix that equality in Theorem 1.6 holds if and only if K and L are simplices with a common vertex at the origin and such that $(1-\theta)L = \theta K$. Likewise, in Theorem 1.5, equality holds if and only if K is a simplex with a vertex at the origin.

We offer two different proofs of Theorem 1.6. The first follows from a functional inequality which we turn now to describe. A second proof, which closely follows Rogers and Shephard's original argument from [12] and is more geometric in nature, is presented in the appendix of this paper for completeness. We start with the notion of a "λ-difference function".

Definition 1.7. *Let $\lambda \in (0,1)$, and $f, g\colon \mathbb{R}^n \to \mathbb{R}^+$. The λ-difference function $\Delta_\lambda^{f,g} : \mathbb{R}^n \to \mathbb{R}^+$ associated with f and g is defined by*

$$\Delta_\lambda^{f,g}(z) = \sup_{(1-\lambda)x + \lambda y = z} f^{1-\lambda}\left(\tfrac{x}{1-\lambda}\right) g^{\lambda}\left(\tfrac{-y}{\lambda}\right).$$

Theorem 1.8. *Let $f, g : \mathbb{R}^n \to \mathbb{R}^+$ be log-concave functions, and $\lambda \in (0,1)$. Then*

$$\int_{\mathbb{R}^n} \Delta_\lambda^{f,g} \int_{\mathbb{R}^n} f^\lambda g^{1-\lambda} \le \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g. \tag{8}$$

The proof of Theorem 1.8 is similar to Colesanti's proof of Theorem 1.1 in [3]. An immediate corollary of Theorem 1.8 is the following result. Let K° stand for the polar body of K (see definition below).

Theorem 1.9. *Let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin. Then*

$$\mathrm{Vol}(K \vee -L)\,\mathrm{Vol}((K^\circ + L^\circ)^\circ) \le \mathrm{Vol}(K)\,\mathrm{Vol}(L).$$

Note that Theorem 1.9 it strictly stronger than Theorem 1.6. Indeed, one can readily check that for every $\theta \in [0,1]$, one has $\theta K \cap (1-\theta)L \subseteq (K^\circ + L^\circ)^\circ$.

Notations: A convex body with center of mass at the origin is said to be *centered*. Given a convex body $K \subseteq \mathbb{R}^n$, we denote by $h_K\colon \mathbb{R}^n \to \mathbb{R}$ its *support function*, that is, the 1-homogeneous convex function given by $h_K(u) = \sup\{\langle x, u\rangle \mid x \in K\}$. The *polar body* of K is defined as $K^\circ = \{y \in \mathbb{R}^n \mid \langle x, y\rangle \le 1, \forall x \in K\}$, and is also a convex body. The *convex indicator function* $\mathbb{1}_K^\infty$ *of* K is defined to be zero for $x \in K$ and $+\infty$ otherwise. The *Legendre transform* of $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is given by $\mathcal{L}\varphi(x) = \sup_{y\in\mathbb{R}^n}(\langle x, y\rangle - \varphi(x))$. Finally, the *inf convolution* of two functions $f, g\colon \mathbb{R}^n \to \mathbb{R}$ is defined by $(f\square g)(z) = \inf_{x+y=z}\{f(x) + g(y)\}$.

Organization of the paper: In Section 2 we prove Theorem 1.3. In Section 3 we prove Theorem 1.8 and its consequence, Theorem 1.9. In Section 4 we discuss the planar case of Conjecture 1.2. Finally, in the appendix, we give for completeness another proof of Theorem 1.6, which is more geometric in nature, and follows Rogers and Shephard's arguments from [12].

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2 Proof of Theorem 1.3

In this section we prove Theorem 1.3. We start with the following lemma regarding the volume of the convex hull of homothetic copies of a simplex S and of $-S$, which is the expression appearing in Conjecture 1.2.

Lemma 2.1. *Let $S \subset \mathbb{R}^n$ be a centered simplex. For $\lambda \in (0,1)$, let $k \in \mathbb{N}$ such that $(n+1)(1-\lambda) - 1 \le k \le (n+1)(1-\lambda)$. Then*

$$\frac{\mathrm{Vol}((1-\lambda)S \vee (-\lambda S))}{\mathrm{Vol}(S)} = \binom{n}{k}(1-\lambda)^k\lambda^{n-k}. \tag{9}$$

Remark 2.2. Typically, the above inequalities determine $k = k(\lambda)$ uniquely. When they do not (i.e. there are two such k's), the corresponding expressions on the right hand side of (9) coincide.

Proof of Lemma 2.1. Note first that by symmetry, it is enough to assume that $\lambda \leq \frac{1}{2}$. Moreover, since S is centered, for $\lambda \leq \frac{1}{n+1}$ one has $-\lambda S \subseteq (1-\lambda)S$ (see [1], page 57), and hence (9) holds trivially. Thus, we can further assume that $\frac{1}{n+1} \leq \lambda$. Next, since

$$\mathrm{Vol}((1-\lambda)S \vee (-\lambda S)) = (1-\lambda)^n \mathrm{Vol}(S \vee (-\tfrac{\lambda}{1-\lambda}S)), \tag{10}$$

it is enough to compute the quantity $\frac{\mathrm{Vol}(S\vee(-tS))}{\mathrm{Vol}(S)}$ for $t = \frac{\lambda}{1-\lambda}$. In these notations, our assumptions become $t \in [\frac{1}{n}, 1]$ and $\frac{n+1}{1+t} - 1 \leq k \leq \frac{n+1}{1+t}$. Moreover, for simplicity, we may take S to be the convex hull of $\{e_j\}_{j=1}^{n+1}$ (vectors of the standard basis of $\mathbb{R}^{n+1}$) embedded in $\mathbb{R}^{n+1}$, which has $a = (\frac{1}{n+1}, \ldots, \frac{1}{n+1})$ as its center of mass. We then wish to compute the n-dimensional volume of $K_t := S \vee S_t$, with S_t being the convex hull of the vectors $v_j = (1+t)a - te_j$, where $j = 1, \ldots, n+1$.

First, we study the facets of K_t. A facet of K_t is the convex hull of (at least n) vertices of S and S_t. Since for every $1 \leq j \leq n+1$ the line between e_j and v_j intersects the interior of K_t, two such vertices never participate in the same facet, and hence every facet consists of either n or $n+1$ vertices. We will consider convex hulls of only n vertices, and see in the proof that typically, K_t has no facets with $n+1$ vertices.

Consider $F_k = \mathrm{conv}\{e_1, \ldots, e_k, v_{k+1}, \ldots, v_n\}$, for some $k \in \{1, \ldots n\}$. Note that F_k lies in the intersection of the two affine hyperplanes $a + a^{\perp}$ and $e_1 + (u_k)^{\perp}$, where

$$u_k = \Big(\underbrace{t, \ldots, t}_{k \text{ times}}, \underbrace{-1, \ldots, -1}_{n-k \text{ times}}, n - (1+t)k\Big)$$

is the normal to F_k, i.e. $\langle u_k, \nu \rangle \equiv t$ for ν which is a vertex of F_k. Clearly, F_k is a facet of K_t if and only if $K_t \setminus F_k \subseteq \{x \,|\, \langle u_k, x\rangle < t\}$, and it suffices to check this condition only for vertices of K_t. A direct computation shows that this holds if and only if $\frac{n+1}{1+t} - 1 < k < \frac{n+1}{1+t}$ which, assuming $\frac{n+1}{1+t} \notin \mathbb{N}$, has a (unique) solution in k. Thus, the $(n+1)\binom{n}{k}$ ways to choose k vertices of S and $n-k$ vertices of S_t correspond to all the facets of K_t, in the case $\frac{n+1}{1+t} \notin \mathbb{N}$ (we shall address the other case at the end of the proof).

The contribution of $F_k \vee \{a\}$ to the ratio $\frac{\mathrm{Vol}(K_t)}{\mathrm{Vol}(S)}$ can be easily computed by noticing that, by invariance under translation, $\mathrm{Vol}_n(F_k \vee \{a\}) = \mathrm{Vol}_n(V)$, where

$$V = \mathrm{conv}\{0, e_1 - a, \ldots, e_k - a, -t(e_{k+1} - a), \ldots, -t(e_n - a)\}.$$

Moreover, since

$$\begin{aligned} \frac{\mathrm{Vol}_n(V)|a|}{n+1} &= \mathrm{Vol}_{n+1}(\mathrm{conv}\{0, e_1 - a, \ldots, e_k - a, -t(e_{k+1} - a), \ldots, -t(e_n - a), a\}) \\ &= t^{n-k}\mathrm{Vol}_{n+1}(\mathrm{conv}\{0, e_1 - a, \ldots, e_k - a, e_{k+1} - a, \ldots, e_n - a, a\}), \end{aligned}$$

and $\frac{1}{n+1}\mathrm{Vol}_n(S)|a| = \mathrm{Vol}_{n+1}(\mathrm{conv}\{0, e_1, \ldots, e_{n+1}\})$, one has

$$\frac{\mathrm{Vol}_n(F_k \vee \{a\})}{\mathrm{Vol}_n S} = t^{n-k}\det(e_1 - a, \ldots, e_n - a, a) = \frac{t^{n-k}}{n+1}.$$

Finally, summing over all of the facets of K_t, we get

$$\frac{\mathrm{Vol}(K_t)}{\mathrm{Vol}(S)} = (n+1)\binom{n}{k} t^{n-k} \frac{1}{n+1} = \binom{n}{k} t^{n-k},$$

which by the definition of t and (10), completes the proof for $\frac{n+1}{1+t} \notin \mathbb{N}$. The other case, where $t = \frac{n+1}{m} - 1$ for an integer $m \in \{\lceil \frac{n+1}{2} \rceil, \ldots, n\}$, follows by continuity of volume. □

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Let $K \subset \mathbb{R}^n$ be a convex body, and assume that Conjecture 1.2 holds. We wish to show that Conjecture 1.1 holds as well. Note that our assumption is that for every $\lambda \in [0,1]$ there exists $x \in K$ such that

$$\frac{\mathrm{Vol}((1-\lambda)(K-x) \vee \lambda(x-K))}{\mathrm{Vol}(K)} \le \frac{\mathrm{Vol}((1-\lambda)S \vee -\lambda S)}{\mathrm{Vol}(S)},$$

where S is a centered simplex. Using the same argument as in the proof of Theorem 1.4, the monotonicity, homogeneity in each argument, and translation invariance of the mixed volume yield, for every $\lambda \in (0,1)$:

$$\frac{V(K[j], -K[n-j])}{\mathrm{Vol}(K)} \le \frac{\mathrm{Vol}((1-\lambda)(K-x) \vee \lambda(x-K))}{(1-\lambda)^j \lambda^{n-j} \mathrm{Vol}(K)}.$$

We may then use Lemma 2.1 with $k = j$ and $\lambda = \frac{n+1-j}{n+1} \in (0,1)$, together with the last two inequalities, to get

$$\frac{V(K[j], -K[n-j])}{\mathrm{Vol}(K)} \le \frac{\mathrm{Vol}((1-\lambda)S \vee -\lambda S)}{(1-\lambda)^j \lambda^{n-j} \mathrm{Vol}(S)} = \binom{n}{j}.$$

This completes the proof of Theorem 1.3. □

Remark 2.3. Note that in the above proof we apply Conjecture 1.2 only for finitely many values of the parameter λ.

3 The λ-difference function

In [3], Colesanti introduced the so called "difference function", which is a functional analogue of the difference body notion. He then proved a functional version of the difference body inequality (1) and used it to provide yet another proof of (6). In this section, we generalize Colesanti's definition and, using similar methods, extend the main result of [3].

First, we recall our notion of a λ-difference function (Definition 1.7). For two functions $f, g : \mathbb{R}^n \to \mathbb{R}_+$, and $\lambda \in (0,1)$, the λ-difference function $\Delta_\lambda^{f,g}$ of f and g is

$$\Delta_\lambda^{f,g}(z) = \sup_{(1-\lambda)x+\lambda y=z} f^{1-\lambda}\left(\tfrac{x}{1-\lambda}\right) g^\lambda\left(\tfrac{-y}{\lambda}\right).$$

Alternatively if $f = e^{-\varphi}$, $g = e^{-\psi}$ we may write

$$\Delta_\lambda^{f,g} = e^{-\delta_\lambda^{\varphi,\psi}}, \text{ where } \delta_\lambda^{\varphi,\psi}(z) = \inf_{(1-\lambda)x+\lambda y=z} \left\{(1-\lambda)\varphi(\tfrac{x}{1-\lambda}) + \lambda\psi(\tfrac{-y}{\lambda})\right\}.$$

Remark 3.1.

1) The λ-difference function is compatible with translations, and multiplications by positive constants. More precisely, denoting $f_a(x) := f(x+a)$, one has $\Delta_\lambda^{f_a,g_b} = (\Delta_\lambda^{f,g})_{(1-\lambda)a+\lambda b}$ and $\Delta_\lambda^{af,bg} = a^{1-\lambda}b^\lambda\Delta_\lambda^{f,g}$.

2) Letting $\Phi(x) = \varphi(x/(1-\lambda))$, $\Psi(x) = \psi(-x/\lambda)$, we may apply the Prékopa-Leindler inequality for the functions $e^{-\Phi}, e^{-\Psi}$, and $e^{-\delta_\lambda^{\varphi,\psi}}$, and obtain the following estimate:

$$\int_{\mathbb{R}^n} \Delta_\lambda^{f,g} \geq \left((1-\lambda)^{1-\lambda}\lambda^\lambda\right)^n \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^\lambda.$$

Theorem 1.8, which we now turn to prove, gives a complementary upper bound.

Proof of Theorem 1.8: Assume that $f = e^{-\varphi}$, $g = e^{-\psi}$, for some convex functions φ, ψ. Let $z \in \mathbb{R}^n$. First, assume that there exist $x^*, y^* \in \mathbb{R}^n$ such that

$$(1-\lambda)x^* + \lambda y^* = z, \text{ and } \delta_\lambda^{\varphi,\psi}(z) = (1-\lambda)\varphi(x^*/(1-\lambda)) + \lambda\psi(-y^*/\lambda).$$

Using the convexity of φ and ψ, for every $y \in \mathbb{R}^n$ one has

$$\psi\left((1-\lambda)y - z/\lambda\right) \leq (1-\lambda)\psi(y - x^*/\lambda)) + \lambda\psi(-y^*/\lambda),$$

and

$$\varphi(\lambda y) \leq (1-\lambda)\varphi(x^*/(1-\lambda)) + \lambda\varphi(y - x^*/\lambda).$$

Summing these two inequalities, we obtain

$$\psi\left((1-\lambda)y - z/\lambda\right) + \varphi(\lambda y) \leq \delta_\lambda^{\varphi,\psi}(z) + \lambda\varphi(y - x^*/\lambda) + (1-\lambda)\psi(y - x^*/\lambda). \quad (11)$$

As this holds for every $y \in \mathbb{R}^n$, we exponentiate and integrate (11) over $y \in \mathbb{R}^n$ to obtain

$$\Delta_\lambda^{f,g}(z) \int_{\mathbb{R}^n} f^\lambda g^{1-\lambda} \leq \int_{\mathbb{R}^n} g\left((1-\lambda)y - z/\lambda\right) f(\lambda y)\, dy. \quad (12)$$

If the case where the infimum in the definition of $\delta_\lambda^{\varphi,\psi}(z)$ is not attained, there are sequences $(x_j)_{j=1}^\infty, (y_j)_{j=1}^\infty \subset \mathbb{R}^n$ such that for every j, $(1-\lambda)x_j + \lambda y_j = z$, and

$\delta_\lambda^{\varphi,\psi}(z) = \lim_{j\to\infty}((1-\lambda)\varphi(x_j/(1-\lambda)) + \lambda\varphi(-y_j/\lambda))$. Using the same argument as before, one has,

$$f^{1-\lambda}(x_j/(1-\lambda))g^\lambda(-y_j/\lambda)\int_{\mathbb{R}^n} f^\lambda g^{1-\lambda} \le \int_{\mathbb{R}^n} g\left((1-\lambda)y - z/\lambda\right) f\left(\lambda y\right) dy,$$

and by taking the limit we get that (12) holds in this case as well. We may therefore integrate this inequality with respect to $z \in \mathbb{R}^n$, and get

$$\begin{aligned}
\int_{\mathbb{R}^n} f^\lambda g^{1-\lambda}\int_{\mathbb{R}^n}\Delta_\lambda^{f,g} &\le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} g\left((1-\lambda)y - z/\lambda\right) f\left(\lambda y\right) dy\right) dz \\
&= \int_{\mathbb{R}^n} f\left(\lambda y\right)\left(\int_{\mathbb{R}^n} g\left((1-\lambda)y - z/\lambda\right) dz\right) dy \\
&= \int_{\mathbb{R}^n} f\left(\lambda y\right) dy \int_{\mathbb{R}^n} g\left(-z/\lambda\right) dz \\
&= \left(\int_{\mathbb{R}^n} f\right)\left(\int_{\mathbb{R}^n} g\right).
\end{aligned}$$

This completes the proof of Theorem 1.8. $\square$

Remark 3.2. For $\lambda = 1/2$ and $f = g$, Colesanti showed in [3] that the bound (8) is sharp and attained for the function

$$g(x) = \begin{cases} e^{-(x_1+\cdots+x_n)}, & \text{if } x_j \ge 0, \quad \forall j = 1,\dots,n \\ 0, & \text{otherwise.} \end{cases}$$

In fact, this function shows that the bound is sharp for every $\lambda \in (0,1)$. Indeed, one can easily verify that $\Delta_\lambda^{g,g}(z) = e^{-\sum_{i=1}^n |z_j|_\lambda}$, where

$$|a|_\lambda = \begin{cases} \frac{|a|}{1-\lambda}, & \text{if } a \ge 0 \\ \frac{|a|}{\lambda}, & \text{otherwise.} \end{cases}$$

Thus, a direct computation gives

$$\int_{\mathbb{R}^n}\Delta_\lambda^{g,g}(z)dz = \int_{\mathbb{R}^n} e^{-\sum |z_j|_\lambda} dz = \sum_{j=0}^n \binom{n}{j}\lambda^j(1-\lambda)^{n-j}\left(\int_0^\infty e^{-x}dx\right)^n = 1 = \int_{\mathbb{R}^n} g.$$

Theorem 1.8 provides, as an immediate corollary, a proof of Theorem 1.9.

Proof of Theorem 1.9. Let $K, L \subseteq \mathbb{R}^n$ be two convex bodies containing the origin. Let $\lambda \in (0,1)$. Using the homogeneity of the support function $h_{K^\circ}(x)$ (both in K and in x), one has

$$\delta_\lambda^{h_{K^\circ},h_{L^\circ}}(z) = \inf_{(1-\lambda)x+\lambda y = z}((1-\lambda)h_{K^\circ}(x/(1-\lambda)) + \lambda h_{L^\circ}(-y/\lambda)) = h_{\frac{1}{1-\lambda}K^\circ}\Box h_{-\frac{1}{\lambda}L^\circ}(z),$$

where $f\Box g$ is the infimal convolution of f and g. It is well known (see e.g., [14], Section 1.7), that for lower semi continuous convex functions f, g we have $f\Box g = \mathcal{L}(\mathcal{L}f + \mathcal{L}g)$ and that $\mathcal{L}h_K = \mathbf{1}_K^\infty$. Thus,

$$\begin{aligned}\delta_\lambda^{h_{K^\circ}, h_{L^\circ}} &= \mathcal{L}(\mathbf{1}^\infty_{\frac{1}{1-\lambda}K^\circ} + \mathbf{1}^\infty_{-\frac{1}{\lambda}L^\circ}) = \mathcal{L}(\mathbf{1}^\infty_{\frac{1}{1-\lambda}K^\circ \cap -\frac{1}{\lambda}L^\circ}) \\ &= h_{\frac{1}{1-\lambda}K^\circ \cap -\frac{1}{\lambda}L^\circ} = h_{((1-\lambda)K \vee -\lambda L)^\circ}.\end{aligned}$$

Moreover, note that $(e^{-h_{K^\circ}})^\lambda (e^{-h_{L^\circ}})^{1-\lambda} = e^{-h_{\lambda K^\circ + (1-\lambda)L^\circ}}$. Thus, combining the fact that $\int_{\mathbb{R}^n} e^{-h_{K^\circ}} = n!\mathrm{Vol}(K)$ with Theorem 1.8 for $f = e^{-h_{K^\circ}}$ and $g = e^{-h_{L^\circ}}$, yields

$$\mathrm{Vol}((1-\lambda)K \vee -\lambda L)\,\mathrm{Vol}((\lambda K^\circ + (1-\lambda)L^\circ)^\circ) \le \mathrm{Vol}(K)\,\mathrm{Vol}(L),$$

or, equivalently,

$$\mathrm{Vol}(K \vee -L)\,\mathrm{Vol}((K^\circ + L^\circ)^\circ) \le \mathrm{Vol}(K)\,\mathrm{Vol}(L).$$

The proof of Theorem 1.9 is thus complete. $\square$

4 The planar case

While Godbersen's conjecture is clearly true for $n = 2$ by the inclusion $-K \subset 2K$ for centered convex regions in the plane, the validity of the other two conjectures presented in Section 1 is not as self-evident. In this section we assert the validity of Conjecture 1.2 when $n = 2$. We note that in the plane, the case $\lambda = \frac{1}{2}$ (along with a characterization of the equality case) was first established by Estermann [5], and later by Levi [10], Fáry [7] and Yaglom and Boltyanskiĭ [16]. We refer the reader to [4] for a detailed survey, which contains not only the history of these problems and several proofs, but also similar related problems dealing with intersections of the convex bodies K and $-tK$.

We show that Conjecture 1.2 holds in the plane (for every $\lambda \in [0, 1]$):

Theorem 4.1. *Let $K \subseteq \mathbb{R}^2$ be a convex body, and let $\lambda \in [0, 1]$. If x is the center of mass of K, then*

$$\mathrm{Area}((1-\lambda)(K - x) \vee \lambda(x - K)) \le \mathrm{Area}((1-\lambda)S \vee -\lambda S),$$

where S is a centered triangle such that $\mathrm{Area}(K) = \mathrm{Area}(S)$.

Proof of Theorem 4.1. Without loss of generality, we may assume that K is centered and has unit area, and show that

$$\mathrm{Area}((1-\lambda)K \vee -\lambda K) \le \mathrm{Area}((1-\lambda)S \vee -\lambda S).$$

We may further assume that K is a polygon, and by a standard continuity argument, the result will hold for a general convex body.

We will describe an inductive process, in which we remove the vertices of K one by one until reaching a triangle. In each step, we replace the body K with a body K' which has one vertex less, without changing the area or the center of mass, and without decreasing the area of $(1-\lambda)K \vee -\lambda K$. Let $K = \mathrm{conv}\{x_1, \dots, x_N\}$, where $N \geq 4$. For $t \in \mathbb{R}$, consider the body

$$\widetilde{K}_t = \mathrm{conv}\{x_1, x_2 + tu, x_3, \dots, x_N\}, \text{ with } u = x_3 - x_1.$$

Denote by l_1 and l_3 the lines containing the segments $[x_N, x_1]$ and $[x_3, x_4]$ respectively. Let $\alpha < 0 < \beta$ be such that $x_2 + \alpha u \in l_1$, and $x_2 + \beta u \in l_3$. Note that for $t \in [\alpha, \beta]$ one has $\widetilde{K}_t = \mathrm{conv}\{x_1, x_3, \dots, x_N\} \cup \mathrm{conv}\{x_1, x_2 + tu, x_3\}$, and thus $\mathrm{Area}(\widetilde{K}_t) = \mathrm{Area}(K)$. Moreover, the center of mass of $\widetilde{K}_t$ is $\theta t u$, where $\theta = \frac{1}{3}\mathrm{Area}(\mathrm{conv}\{x_1, x_2, x_3\})$. Indeed, the center of mass of a union of two planar sets is the average of the centers of mass, weighted by their respective areas. Denote $K_t = \widetilde{K}_t - \theta t u$. Note that for $t \in [\alpha, \beta]$, K_t has its center of mass at the origin, and $\mathrm{Area}(K_t) = \mathrm{Area}(K) = 1$.

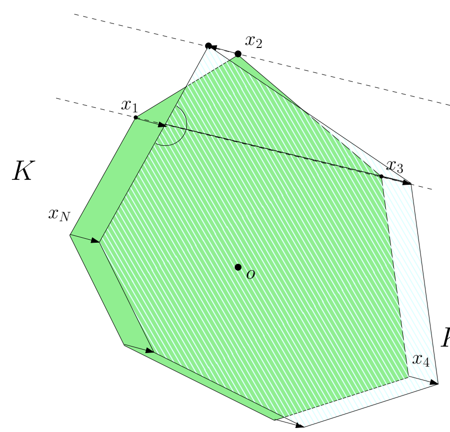

Figure 1: The area and center of mass are preserved, and K' has one vertex less than K.

The family of bodies $\{(1-\lambda)K_t \vee -\lambda K_t\}_{t \in [\alpha, \beta]}$ is a linear parameter system (see [13] for the definition), since all vertices are moving parallel to u. Therefore the area of $(1-\lambda)K_t \vee -\lambda K_t$ is a convex function of t, and attains its maximum in one of the edges of the interval $[\alpha, \beta]$. Assume the maximum is achieved at α, and set $K' = K_\alpha$. Then, in addition to having unit area and center of mass at the origin, K' satisfies

$$\mathrm{Area}((1-\lambda)K \vee -\lambda K) \leq \mathrm{Area}((1-\lambda)K' \vee -\lambda K'),$$

as $K = K_0$. Since $x_2 + \alpha u \in l_1$, K' has one less vertex than K, and thus the proof is complete. $\square$

5 Appendix

We give here another proof of Theorem 1.6. The proof is attained by essentially repeating the arguments from [12], but instead of considering K and $-K$, we consider two general convex bodies K and L. Moreover, we will be able to characterize the equality case in Theorem 1.5.

5.1 The Rogers–Shephard body

We consider the following $(2n+1)$-dimensional body (see Figure 2), a special case of which, where $K = L$, plays a central role in [12].

$$G(K,L) := \{(x,y,\theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \,|\, \theta \in [0,1]\,,\, x \in \theta K,\, x+y \in (1-\theta)L\}\,.$$

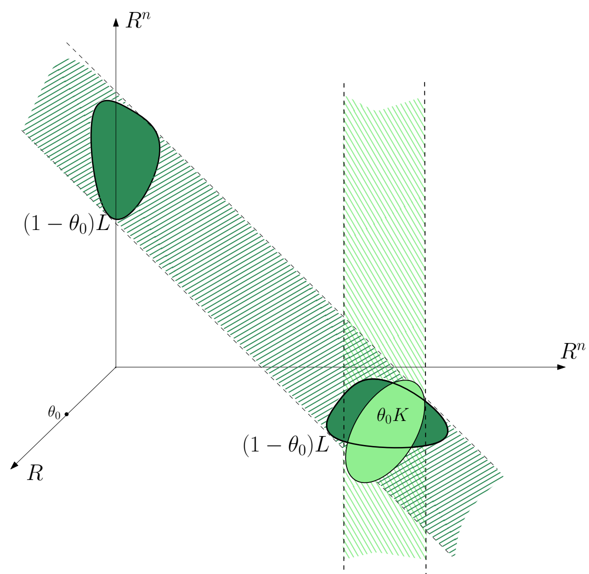

Figure 2: The section of $G(K,L)$ where $\theta = \theta_0$, $y = 0$, is an intersection of two cylinders.

The projection of $G(K,L)$ onto the $(n+1)$-dimensional subspace of points of the form $(0,y,\theta)$ is denoted by $C(K,L) = \{(0,y,\theta) \mid \theta \in [0,1],\ y \in (1-\theta)L - \theta K\}$. Equivalently,

$$C(K,L) = \{0\} \times \operatorname{conv}\left(L \times \{0\}\,,\, -K \times \{1\}\right) \subseteq \{0\} \times \mathbb{R}^{n+1}. \tag{13}$$

When $K = L$, this is exactly the body $C(K)$ used in [12].

The main tool used by Rogers and Shephard in [12] for finding an upper bound for the volume of the difference body is the following theorem which we provide along with its proof, for the convenience of the reader.

Theorem 5.1 (**Rogers and Shephard).** *Let $K \subset \mathbb{R}^n$ be a convex body, and let $H = K \cap E$ be a j-dimensional section of K, and L the orthogonal projection of K onto $E^{\perp}$. Then*

$$\frac{j!\,(n-j)!}{n!} \mathrm{Vol}_j\,(H)\,\mathrm{Vol}_{n-j}\,(L) \leq \mathrm{Vol}_n\,(K)\,. \tag{14}$$

Moreover, equality holds if and only if for every direction $v \in E^{\perp}$, the intersection of K with $E + \mathbb{R}^+ v$ is obtained by taking the convex hull of H and one more point.

Proof of Theorem 5.1. The proof consists of two arguments. The first one is that all of the quantities in (14) are invariant under a Schwarz symmetrization, so we may assume that all intersections of K parallel to H are (centered) dilates of a ball. That is, we may consider the body:

$$K^* = \left\{ (l, y) \in \mathbb{R}^{n-j} \times \mathbb{R}^j \quad | \quad l \in L,\ |y| \leq \left(\frac{\mathrm{Vol}_j(K_l)}{\mathrm{Vol}_j(B_2^j)} \right)^{1/j} \right\},$$

where $K_l = K \cap (E + l)$, for every $l \in L$. The second argument is that $H^* \vee L \subseteq K^*$, where H^* denotes the section $K^* \cap E$. Then, a simple computation shows that the left-hand side of (14) equals $\mathrm{Vol}_n(H^* \vee L)$, and thus inequality (14) holds.

Assume now that equality holds in (14). First note that $H^* \vee L \subseteq K^*$, thus equality in volumes implies $K^* = H^* \vee L = H^* \vee \partial L$. Moreover, for every direction $v \in E^{\perp}$, let $[0, l] = L \cap (\mathbb{R}^+ v)$, and for every $t \in [0, 1]$, let $f_v(t) = \mathrm{Vol}_j(K_{tl})^{1/j}$. Note that f_v is invariant under Schwartz symmetrization, hence for every $v \in E^{\perp}$, f_v is linear, and $f_v(1) = 0$. Since K_{tl} contains $tK_l + (1-t)H$, by the Brunn–Minkowski inequality one has

$$f_v(t) = \mathrm{Vol}_j(K_{tl})^{1/j} \geq t\mathrm{Vol}_j(K_l)^{1/j} + (1-t)\mathrm{Vol}_j(H)^{1/j} = (1-t)f_v(0).$$

Thus, from the equality case in the Brunn–Minkowski inequality, K_l is a homothety of H of zero volume, i.e. it is a point. Moreover, for every $t \in [0, 1]$, one has $K_{tl} = tK_l + (1-t)H$, and thus $K \cap (E + \mathbb{R}^+ v) = H \vee K_l$, and the proof is complete. □

5.2 An upper bound for the volume of $C(K, L)$

Theorem 5.2. *For convex bodies $K, L \subseteq \mathbb{R}^n$, let $C(K, L)$ be as defined in* (13). *For every $\theta \in [0, 1]$,*

$$\mathrm{Vol}_{n+1}\,(C\,(K, L)) \leq \frac{1}{n+1} \left(\frac{\mathrm{Vol}_n\,(K)\,\mathrm{Vol}_n\,(L)}{\mathrm{Vol}_n(\theta K \cap (1-\theta)L)} \right).$$

In [12] this bound was obtained for the specific case $K = L$ and $\theta = \frac{1}{2}$.

Proof of Theorem 5.2. In order to estimate $\mathrm{Vol}_{n+1}(C(K,L))$, we apply Theorem 5.1 for the body $G(K,L)$, and the n-dimensional affine subspace $E=\{\theta=\theta_0, y=0\}$. First, by Fubini's Theorem,

$$\begin{aligned}\mathrm{Vol}_{2n+1}(G(K,L)) &= \int_0^1 d\theta \int_{\theta K} \mathrm{Vol}_n((1-\theta)L)dx \\ &= \mathrm{Vol}_n(K)\,\mathrm{Vol}_n(L)\int_0^1 \theta^n(1-\theta)^n\,d\theta \\ &= \mathrm{Vol}_n(K)\,\mathrm{Vol}_n(L)\,\frac{n!n!}{(2n+1)!}.\end{aligned}$$

As in Theorem 5.1, set $H=G(K,L)\cap E$. Note that $\mathrm{Vol}_n(H)=\mathrm{Vol}_n(\theta_0 K\cap(1-\theta_0)L)$. As mentioned before, the projection of $G(K,L)$ on $E^\perp$ is exactly $C(K,L)$, and using Theorem 5.1 we get

$$\frac{n!\,(n+1)!}{(2n+1)!}\mathrm{Vol}_{n+1}(C(K,L))\,\mathrm{Vol}_n(\theta_0 K\cap(1-\theta_0)L)\le \mathrm{Vol}_{2n+1}(G(K,L)). \tag{15}$$

Plugging in the volume of $G(K,L)$ completes the proof of the theorem. □

5.3 A second proof of Theorem 1.6

Using Theorem 5.1, this time for the body $C(K,L)$, and the volume bound from Theorem 5.2, we can give yet another proof of Theorem 1.6.

Proof of Theorem 1.6. Let $K,L\subseteq\mathbb{R}^n$ be two convex bodies such that $0\in K\cap L$, and set $\theta\in[0,1]$. We need to show that

$$\mathrm{Vol}(L\vee -K)\,\mathrm{Vol}(\theta K\cap(1-\theta)L)\le \mathrm{Vol}(K)\mathrm{Vol}(L).$$

Let E be the 1-dimensional subspace of $\mathbb{R}^{n+1}$ given by $E=\{x=0\}$. The body $L\vee -K$ is the n-dimensional projection of $C(K,L)$ onto the subspace $E^\perp=\{(x,0)\,|\,x\in\mathbb{R}^n\}$. Since $0\in K\cap L$, the section $H=E\cap C(K,L)$ is a unit segment. By Theorem 5.1,

$$\frac{1}{n+1}\mathrm{Vol}_n(-K\vee L)\le \mathrm{Vol}_{n+1}(C(K,L)). \tag{16}$$

Combining this with the volume bound for $C(K,L)$ established in Theorem 5.2 above, we get the desired inequality. □

5.4 The equality case in Theorem 1.5 and in Theorem 1.6

Here we characterize the equality cases in Theorems 1.5 and 1.6. We start with the former, and show that equality holds in (7) if and only if K is a simplex with a vertex at the origin. Indeed, it is not hard to check that for the standard simplex S one has

$$\mathrm{Vol}((1-\lambda)S \vee -\lambda S) = \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} \mathrm{Vol}(S) = \mathrm{Vol}(S).$$

As for the other direction, assume that $\mathrm{Vol}((1-\lambda)K \vee -\lambda K) = \mathrm{Vol}(K)$. We wish to show that K is a simplex with 0 as one of its vertices. Recall from the introduction that Theorem 1.6, for $(1-\lambda)K, \lambda K$, and $\theta_0 = \lambda$, immediately yields inequality (7). Combining (15) and (16) in this case yields

$$\mathrm{Vol}_n(-(1-\lambda)K \vee \lambda K) \le (n+1)\mathrm{Vol}_{n+1}(C((1-\lambda)K, \lambda K)) \le \mathrm{Vol}_n(K). \tag{17}$$

From the assumption $\mathrm{Vol}((1-\lambda)K \vee -\lambda K) = \mathrm{Vol}(K)$ it follows that both inequalities in (17) are in fact equalities. In particular, equality holds in (15) for the bodies $(1-\lambda)K, \lambda K$, and $\theta_0 = \lambda$. By the equality condition in Theorem 5.1, this implies in particular that sections of the body $G((1-\lambda)K, \lambda K)$ by affine subspaces of the form $\{(x, y, \theta) | x \in \mathbb{R}^n, y = y_0, \theta = \lambda\}$, for any $y_0 \in \lambda(1-\lambda)K - \lambda(1-\lambda)K$ (which are given by $\lambda(1-\lambda)K \cap \lambda(1-\lambda)K - y_0$), are homothetic. Thus, K must be a simplex by the following lemma due to Rogers and Shephard.

Lemma 4 from [11]. *Let K be a convex body in $\mathbb{R}^n$. If the intersections $K \cap (K + x)$ are homothetic for all $x \in K - K$, then K is a simplex.*

Finally, in order to show that 0 is a vertex of K, note that equality holds also in (16), for the bodies $(1-\lambda)K, \lambda K$. Hence, among all sections of $C((1-\lambda)K, \lambda K)$ parallel to $H = E \cap C((1-\lambda)K, \lambda K)$, H is the only one with unit length. Since we assumed that K contains the origin, one has $(1-\lambda)K \cap -\lambda K = \{0\}$. This, together with the fact that K is a simplex, means that 0 must be one of the vertices of K.

We turn now to show that equality in Theorem 1.6 holds if and only if K and L are simplices with a common vertex at the origin, and such that $(1-\theta)L = \theta K$. To this end, we shall make use of Theorem 1.9. Assume that for some given $\theta \in (0, 1)$,

$$\mathrm{Vol}(L \vee -K)\,\mathrm{Vol}(\theta K \cap (1-\theta)L) = \mathrm{Vol}(K)\mathrm{Vol}(L). \tag{18}$$

Since the inclusion $\theta K \cap (1-\theta)L \subseteq (K^\circ + L^\circ)^\circ$ always holds, by Theorem 1.9 we then must have

$$\theta K \cap (1-\theta)L = (K^\circ + L^\circ)^\circ$$

(as both are compact convex sets, inclusion together with equality of volumes implies equality of sets). We claim that this equality implies that K and L are homothetic. Indeed, we may rewrite the above equality as

$$\theta^{-1}K^\circ \vee (1-\theta)^{-1}L^\circ = K^\circ + L^\circ,$$

so that in particular

$$\theta^{-1}K^\circ \subset K^\circ + L^\circ, \qquad (1-\theta)^{-1}L^\circ \subset K^\circ + L^\circ.$$

Thus, $\theta^{-1}h_{K^\circ} \le h_{K^\circ} + h_{L^\circ}$, and $(1-\theta)^{-1}h_{L^\circ} \le h_{K^\circ} + h_{L^\circ}$. Putting the two together one has

$$h_{L^\circ} = \frac{1-\theta}{\theta} h_{K^\circ},$$

or equivalently, $(1-\theta)L = \theta K$. This, together with (18) implies that equality in (7) holds for K. Thus, from the characterization of the equality case in Theorem 1.5, the body K must be a simplex with a vertex at the origin. This completes the proof for the equality case in Theorem 1.6.

Remark 5.3. It is worthwhile to notice that in Theorem 1.9 there are more equality cases than in Theorem 1.5. Indeed, one may readily check that for positive $\lambda_1, \dots, \lambda_n$, and the bodies $K = \text{conv}\{0, e_1, \dots, e_n\}$, $L = \text{conv}\{0, \lambda_1 e_1, \dots, \lambda_n\}$ we have that $\text{Vol}(K)\text{Vol}(L) = \frac{1}{n!^2}\prod_{i=1}^n \lambda_i$, that

$$\text{Vol}((K^\circ + L^\circ)^\circ) = \frac{1}{n!}\prod_{i=1}^n \frac{\lambda_i}{1+\lambda_i}$$

and that

$$\text{Vol}(K \vee -L) = \frac{1}{n!}\sum_{A\subset\{1,\dots n\}} \prod_{i\in A} \lambda_i = \frac{1}{n!}\prod_{i=1}^n (1+\lambda_i).$$

The characterization of the equality case in Theorem 1.9 is thus left open.

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