On the extremality of Hofer's metric on the group of Hamiltonian diffeomorphisms

Yaron Ostrover and Roy Wagner

May 29, 2005

Abstract

Let M be a closed symplectic manifold, and let $\|\cdot\|$ be a norm on the space of all smooth functions on M, which are zero-mean normalized with respect to the canonical volume form. We show that if $\|\cdot\| \leq C\|\cdot\|_\infty$, and $\|\cdot\|$ is invariant under the action of Hamiltonian diffeomorphisms, then it is also invariant under all volume preserving diffeomorphisms. We also prove that if $\|\cdot\|$ is, additionally, not equivalent to $\|\cdot\|_\infty$, then the induced pseudo-distance function on the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms of M vanishes identically. These results provide partial answers to questions raised by Eliashberg and Polterovich in [4]. Both results rely on an extension of $\|\cdot\|$ to the space of essentially bounded measurable functions, which is invariant under all measure preserving bijections.

1 Introduction and Results

Let (M,ω) be a closed connected symplectic manifold of dimension $2n$. Denote by $\mathcal{A}$ the space of all smooth functions on M which are zero-mean normalized with respect to the canonical volume form ω^n. The main object of our study is the infinite-dimensional Lie group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms of M. We refer the readers to [6], [10] and [12] for symplectic preliminaries and further discussions on the group of Hamiltonian diffeomorphisms.

It is well known that the Lie algebra of $\mathrm{Ham}(M,\omega)$, that is the space of all Hamiltonian vector fields, can be identified with the space $\mathcal{A}$. Moreover, the adjoint action of $\mathrm{Ham}(M,\omega)$ on its Lie algebra $\mathcal{A}$ is the standard action of diffeomorphisms on functions. The choice of any norm $\|\cdot\|$ on $\mathcal{A}$ gives rise to a pseudo-distance function on $\mathrm{Ham}(M,\omega)$ in the following way: we define the length of a path $\alpha : [0,1] \to \mathrm{Ham}(M,\omega)$ as

$$\mathrm{length}\{\alpha\} = \int_0^1 \|\dot{\alpha}\| dt = \int_0^1 \|F_t\| dt,$$

where $F_t(x) = F(t,x)$ is the Hamiltonian function generating the path α. This is the usual definition of Finsler length. The distance between two Hamiltonian diffeomorphisms is given by

$$\rho(\psi,\varphi) = \inf \operatorname{length}\{\alpha\},$$

where the infimum is taken over all Hamiltonian paths α connecting ψ and φ. It is not hard to check that ρ is non-negative, symmetric and satisfies the triangle inequality. Moreover, a norm on $\mathcal{A}$ which is invariant under the adjoint action yields a bi-invariant pseudo-distance function, i.e. $\rho(\psi,\varphi) = \rho(\theta\,\psi,\theta\,\varphi) = \rho(\psi\,\theta,\varphi\,\theta)$ for every $\psi,\varphi,\theta \in \mathrm{Ham}(M,\omega)$. From now on we will deal only with such norms and we will refer to ρ as the pseudo-distance generated by the norm $\|\cdot\|$.

It is highly non-trivial to check whether such a distance function is non-degenerate, that is $\rho(\mathbb{1},\psi) > 0$ for $\psi \neq \mathbb{1}$. In fact, for compact symplectic manifolds, a bi-invariant pseudo-metric ρ on $\mathrm{Ham}(M,\omega)$ is either a genuine metric or identically zero. This is an immediate corollary of a well known theorem by Banyaga [1], which states that $\mathrm{Ham}(M,\omega)$ is a simple group, combined with the fact that the null-set

$$\mathrm{null}(\rho) = \{\psi \in \mathrm{Ham}(M,\omega) \mid \rho(\mathbb{1},\psi) = 0\}$$

is a normal subgroup of $\mathrm{Ham}(M,\omega)$.

A distinguished result by Hofer [5] states that the L_∞ norm $\|\cdot\|_\infty$ on $\mathcal{A}$ gives rise to a genuine distance function on $\mathrm{Ham}(M,\omega)$. This was discovered and proved by Hofer for the case of $\mathbb{R}^{2n}$, then generalized by Polterovich [13] to some larger class of symplectic manifolds, and finally proven in full generality by Lalonde and McDuff in [8]. The above mentioned distance function is known as Hofer's metric and has been intensively studied since its discovery (see e.g. [6], [10], [12]). We also refer the reader to Oh's paper [11] for another approach to the non-degeneracy of Hofer's metric, and to Chekanov's paper [3] for a proof that an analogue of Hofer's metric is (up to scaling) the only non-degenerate Hamiltonian-invariant Finsler metric (see [3] for the precise definition) on the space of Lagrangian submanifolds Hamiltonian isotopic to a given closed Lagrangian. In the opposite direction, Eliashberg and Polterovich showed in [4] that for $1 \leq p < \infty$, the pseudo-distances on $\mathrm{Ham}(M,\omega)$ which correspond to the L_p norms on $\mathcal{A}$ vanish identically. Thus, the following question arises from [4] and [12]:

Question: What are the invariant norms on $\mathcal{A}$, and which of them give rise to genuine bi-invariant metrics on $\mathrm{Ham}(M,\omega)$?

Our main contributions towards answering this question are

Theorem 1.1. *Let $\|\cdot\|$ be a $\mathrm{Ham}(M,\omega)$-invariant norm on $\mathcal{A}$ such that $\|\cdot\| \leq C\|\cdot\|_\infty$ for some constant C. Then $\|\cdot\|$ is invariant under all measure preserving diffeomorphisms of M.*

Theorem 1.2. *Let $\|\cdot\|$ be a $\mathrm{Ham}(M,\omega)$-invariant norm on $\mathcal{A}$ such that $\|\cdot\| \leq C\|\cdot\|_\infty$ for some constant C, but the two norms are not equivalent. Then the associated pseudo-distance function ρ on $\mathrm{Ham}(M,\omega)$ vanishes identically.*

Here, two norms are said to be *equivalent*, if each bounds the other up to a multiplicative constant.

The next result is a strengthened formulation of Theorem 1.1 and a key ingredient in the proof of Theorem 1.2. As the discussion below explains, it also bears on the question of classifying $\mathrm{Ham}(M,\omega)$-invariant norms.

Theorem 1.3. *Let $\|\cdot\|$ be a $Ham(M,\omega)$-invariant norm on $\mathcal{A}$ such that $\|\cdot\| \leq C\|\cdot\|_\infty$ for some constant C. Then $\|\cdot\|$ can be extended to a semi-norm $|||\cdot||| \leq C\|\cdot\|_\infty$ on $L_\infty(M)$, which is invariant under all measure preserving bijections on M.*

The formulation of the theorem states only what is necessary for the proofs of Theorems 1.1 and 1.2. In fact we know more about $|||\cdot|||$. First, $|||\cdot|||$ is a norm, rather than just a semi-norm (namely, $|||\cdot|||$ does not vanish on non-zero functions). Second, $|||\cdot|||$, when restricted to zero-mean functions, coincides with the completion of $||\cdot||$. Third, this completion can be viewed as a dense subspace of the space of zero-mean functions in $L_1(M)$, equipped with a norm invariant under measure preserving bijections. The argument for the first claim is briefly sketched in Remark 5.1, and that for the second and third claims is outlined in the final section. The final section also refers to literature concerning the classification of such norms, and indicates their possible pathologies.

Structure of the paper:

The next section contains a fairly detailed outline of the proofs of our main theorems, stressing the main ingredients involved. The following two sections present complete proofs of Theorem 1.3 and Theorem 1.2 respectively. Section 5 contains proofs of some lemmas. The last section contains a sketchy treatment of some additional properties of the norm $|||\cdot|||$, together with some references concerning the classification of such norms.

2 Outline of the Proofs

As explained in the introduction, the degeneracy of the pseudo-distance function ρ (Theorem 1.2) is proved in [4] for L_p norms, $1 \leq p < \infty$. The only property of L_p actually used in that proof is, roughly speaking, that uniformly bounded functions with small support have small norm. More precisely, in Section 4 we reproduce an argument from [4] to show that the proof of Theorem 1.2 can be reduced to the following

Claim 2.1. *If $\sup\{\|F_n\|_\infty\} < \infty$ and $\mathrm{Vol}\big(\mathrm{Support}(F_n)\big) \to 0$, then $\|F_n\| \to 0$.*

Therefore, our main task is to prove this property for any norm which satisfies the requirements of Theorem 1.2. As will be explained below, Theorem 1.3 allows us to carry out the proof of this claim in a more amenable setting.

A natural approach to Claim 2.1 would be to consider characteristic functions with small-measure support first, then make the standard move to step functions, and conclude with any smooth bounded function with small-measure support. The obvious obstacle is that characteristic functions are not smooth, and are therefore outside our space. Here one may choose to approximate them by smooth functions and work from there. We chose, however, to extend our setting so as to include genuine characteristic functions. This is where Theorem 1.3 comes in. We will interrupt the discussion on the proof of Claim 2.1 to discuss the proof of Theorem 1.3.

Recall that our aim in Theorem 1.3 is to extend the given norm $\|\cdot\|$ to $L_\infty(M)$. For this purpose, we first extend our norm to all smooth functions, with average not necessarily zero (since this adds just one dimension to our original space of functions, any two extensions are equivalent). Next, we take advantage of the fact that $C^\infty(M)$ is dense in $L_\infty(M)$ with respect to the topology of convergence in measure. We define

$$|||F||| = \inf\{\liminf_{n\to\infty} ||F_n||\},$$

where the infimum is taken over all sequences $\{F_n\}$ of uniformly bounded smooth functions which converge in measure to F.

Such constructions occur occasionally in functional analysis, for instance in the extension of the Riemann integral from continuous to semi-continuous functions (using pointwise convergence from above/below), and in the extension of operator norms from finite-rank operators on a Banach space to approximable operators (using uniform convergence on compacta). However, we are not aware of any similar construction which relies on convergence in measure.

We study $|||\cdot|||$ in Section 3. First we confirm that $|||\cdot|||$ is a semi-norm on $L_\infty(M)$ which is dominated from above by $\|\cdot\|_\infty$. We then go on to prove the non-trivial properties of $|||\cdot|||$: it coincides with $\|\cdot\|$ on smooth functions, and is invariant under measure preserving bijections. Formally:

Claim 2.2. *For every $F \in \mathcal{A}$ we have $\|F\| = |||F|||$.*

Claim 2.3. *For every $F \in L_\infty(M)$ and every measure preserving bijection φ on M we have*

$$|||F \circ \varphi||| = |||F|||$$

In order to prove this second property, recall that our original norm $\|\cdot\|$ is already invariant under Hamiltonian diffeomorphisms. To extend the invariance we invoke Katok's "Basic Lemma" from [7], which allows to approximate in measure any measure preserving bijection by a Hamiltonian diffeomorphism. More precisely, fix an arbitrary Riemannian metric d on M. We claim

Lemma 2.4. *For every measure preserving bijection φ of M (not necessarily continuous) and every $\varepsilon > 0$, there exists a Hamiltonian diffeomorphism g on M which ε-approximates φ in measure, namely*

$$\mathrm{Vol}\big(\{x \in M\,;\, d(\varphi(x), g(x)) > \varepsilon\}\big) < \varepsilon$$

This result is of course independent of the specific Riemannian structure chosen. We postpone the proof of the lemma to the last section of this paper. The proof of Claim 2.3 follows easily from Lemma 2.4 and the definition of $|||\cdot|||$. Claim 2.2 and Claim 2.3 conclude the proof of Theorem 1.3.

With a measure-preserving-bijection-invariant extension of $||\cdot||$ at our disposal, let's return to the proof of Claim 2.1. Note that Claim 2.2 implies that it is sufficient to prove Claim 2.1 for the norm $|||\cdot|||$. The rest of this section is devoted to this issue.

Our argument depends on the fact, inspired by an argument from [14], that an operator, which performs piecewise averaging on functions, is bounded. More precisely, relying on the fact that $|||\cdot|||$ is invariant under measure preserving bijections, we prove that

Lemma 2.5 (Piecewise-Averaging property). *For every continuous F and every measurable partition $\{S_i\}$ of M, we have*

$$|||\sum_i \langle F\rangle_{S_i}\mathbb{1}_{S_i}||| \leq |||F|||,$$

where $\langle F\rangle_{S_i} = \frac{1}{\mathrm{Vol}(S_i)}\int_{S_i} F\omega^n$ denotes the average of F over S_i.

The proof of the lemma is postponed to the last section. Let us now explain how this property serves to prove Claim 2.1. Fix $\varepsilon > 0$. The hypothesis of Theorem 1.2 provides us with smooth functions F such that $\|F\|_\infty = 1$ while $\|F\| = |||F||| \leq \varepsilon$. Partition M into A and $A^c = M \setminus A$, where A is a small enough neighborhood of the maximum of F, such that $||\mathbb{1}_A - \langle F\rangle_A\mathbb{1}_A + \langle F\rangle_{A^c}\mathbb{1}_{A^c}||_\infty < \varepsilon$. Next, it follows from Lemma 2.5, the fact that $|||\cdot|||$ is dominated from above by $\|\cdot\|_\infty$, and the triangle inequality that

$$|||\mathbb{1}_A||| \leq ||\mathbb{1}_A - \langle F\rangle_A\mathbb{1}_A + \langle F\rangle_{A^c}\mathbb{1}_{A^c}||_\infty + |||\langle F\rangle_A\mathbb{1}_A + \langle F\rangle_{A^c}\mathbb{1}_{A^c}||| \leq \varepsilon + |||F||| \leq 2\varepsilon$$

Since $|||\cdot|||$ is invariant under measure preserving bijections, this applies to every set B with the same measure as A. Thus, we establish Claim 2.1 for sequences of characteristic functions on sets with measure tending to zero. It is now a simple approximation argument, which establishes Claim 2.1 as stated for smooth functions. The details are given in Section 4.

3 Proof of Theorem 1.3

In this section we construct the semi-norm $|||\cdot|||$, and prove its properties as stated in Theorem 1.3. The first step towards the construction of $|||\cdot|||$ is an extension of the given norm to $C^\infty(M)$. Let C be a constant such that $\|\cdot\| \leq C\|\cdot\|_\infty$. Endow the space $C^\infty(M)$ of all smooth function on M with the norm $\|\cdot\|'$ defined by

$$\|F\|' = \inf\big\{\|F_1\| + C\|F_2\|_\infty \ ; \ F = F_1 + F_2, \quad F_1 \in \mathcal{A}, \quad F_2 \in C^\infty(M)\big\}.$$

The above definition is just the analytic presentation of the norm corresponding to the convex hull of the unit ball of $(\mathcal{A}, \|\cdot\|)$ with the unit ball of $(C^\infty(M), \|\cdot\|_\infty)$, the latter homothetically shrunk so as to fit inside the former when restricted to $\mathcal{A}$. The homogeneity of the new norm is clear. To see that the new norm satisfies the triangle inequality, let $F = F_1 + F_2$ and $G = G_1 + G_2$ such that $\|F_1\| + C\|F_2\|_\infty \leq \|F\|' + \varepsilon$, and $\|G_1\| + C\|G_2\|_\infty \leq \|G\|' + \varepsilon$. Then

$$\begin{aligned}\|F+G\|' &\leq \|F_1+G_1\| + C\|F_2+G_2\|_\infty \\ &\leq (\|F_1\| + C\|F_2\|_\infty) + (\|G_1\| + C\|G_2\|_\infty) \\ &\leq \|F\|' + \|G\|' + 2\varepsilon.\end{aligned}$$

The new norm is obviously $\mathrm{Ham}(M,\omega)$-invariant. To see that $\|F\|' \leq C\|F\|_\infty$, just substitute $F_1 = 0$ and $F_2 = F$ in the definition. To see that $\|\cdot\|' = \|\cdot\|$ on $\mathcal{A}$, let $F = F_1 + F_2$ where $F_1 \in \mathcal{A}$ and $F_2 \in C^\infty(M)$. Choosing $F_1 = F$ and $F_2 = 0$ proves that $\|F\|' \leq \|F\|$. For the opposite direction note that since $F, F_1 \in \mathcal{A}$, and since $F_2 = F - F_1$, the function F_2 must also be in $\mathcal{A}$. Therefore

$$\|F\|' = \inf_{F=F_1+F_2} \left\{ \|F_1\| + C\|F_2\|_\infty \right\} \geq \inf_{F=F_1+F_2} \left\{ \|F_1\| + \|F_2\| \right\} \geq \inf_{F=F_1+F_2} \|F_1 + F_2\| = \|F\|$$

Now we are ready to extend our norm to the entire $L_\infty(M)$. Using the same convex-hull trick won't do (it would fail invariance under measure preserving bijections). Instead, we take advantage of the classical fact that any measurable function can be approximated in measure arbitrarily well by smooth functions (see e.g. [15]). We define a new functional by taking the least $\|\cdot\|'$ norm among all such approximations. Formally, we endow the space $L_\infty(M)$ with

$$|||F||| = \inf \left\{ \liminf_{n\to\infty} \|F_n\|' \right\},$$

where the infimum is taken over all sequences of uniformly bounded smooth functions $\{F_n\}$ which converge in measure to F.

It is clear that the new functional is homogeneous. To see that it obeys the triangle inequality, take $\{F_n\}$ and $\{G_n\}$ which satisfy $\liminf \|F_n\|' \leq |||F||| + \varepsilon$ and $\liminf \|G_n\|' \leq |||G||| + \varepsilon$. Then

$$|||F+G||| \leq \liminf \|F_n + G_n\|' \leq \liminf(\|F_n\|' + \|G_n\|') \leq |||F||| + |||G||| + 2\varepsilon.$$

To see that the new functional is still bounded by $C\|\cdot\|_\infty$, note that any essentially bounded function F can be approximated in measure by smooth F_n's with at most the same essential supremum. Indeed, take any approximation in measure F_n of F, and replace it with $\mathrm{sign}(F_n) \cdot (f_n \circ |F_n|)$, where f_n is a good enough smooth approximation from below of the function $f(s) : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(s) = \min\{s, \|F\|_\infty\}$. Taking such F_n's we get

$$|||F||| \leq \liminf \|F_n\|' \leq C \liminf \|F_n\|_\infty \leq C\|F\|_\infty.$$

In order to complete the proof of Theorem 1.3 we need the following two claims.

Claim 3.1. *For every $F \in \mathcal{A}$ we have $\|F\| = \|F\|' = |||F|||$*

Claim 3.2. *For every $F \in L_\infty(M)$ and every measure preserving bijection φ on M we have*

$$|||F \circ \varphi||| = |||F|||$$

In order to prove the first claim a certain technical lemma is needed. To state the lemma, fix from now on an arbitrary Riemannian structure on M, and denote by d the corresponding distance function. Our results are, of course, independent of the specific Riemannian structure chosen.

Lemma 3.3 (Covering Evenly by Many Packings). *For every $\delta > 0$ and $\varepsilon > 0$ there exists a covering of M by connected open subsets $\{U_i^j\}$, where $j = 1, \dots, J$ and $i = 1, \dots, L_j$, such that*

(i) for every fixed j, each pair of sets $\{U_i^j\}$ have a positive distance from each other.

(ii) the diameter of U_i^j with respect to d is at most δ for all i and j.

(iii) for every $x \in M$, the number of j's for which $x \notin \cup_i U_i^j$ is at most εJ.

The proof of the lemma is postponed to the last section of this paper.

Proof of Claim 3.1: The restricted equality $\|\cdot\| = \|\cdot\|'$ has been proved along with the definition of $\|\cdot\|'$ above. Let's prove the restricted equality $|||\cdot||| = \|\cdot\|'$. By choosing $F_n = F$ for all n in the definition of $|||\cdot|||$, we get $|||\cdot||| \leq \|\cdot\|'$. In order to show that $\|\cdot\|' \leq |||\cdot|||$, let $F \in \mathcal{A}$ and let $\{F_n\}$ be a sequence of uniformly bounded smooth functions, which converges in measure to F. We need to show that

$$\liminf_{n\to\infty} \|F_n\|' \geq \|F\|'.$$

For this purpose we will construct a sequence $\{\widetilde{F}_n\}$ which converges uniformly to F, such that $\|F_n\|' \geq \|\widetilde{F}_n\|'$. Since $\|\cdot\|' \leq C\|\cdot\|_\infty$, uniform convergence implies convergence in $\|\cdot\|'$, and we can conclude

$$\liminf_{n\to\infty} \|F_n\|' \geq \liminf_{n\to\infty} \|\widetilde{F}_n\|' = \|F\|'.$$

Let us construct the sequence $\{\widetilde{F}_n\}$. Fix $\varepsilon > 0$, and let $\delta > 0$ such that every open neighborhood of diameter 2δ in M can be viewed as a neighborhood in $\mathbb{R}^{2n}$ such that the original d and the Euclidian distance are equivalent up to a factor 2. Take a covering $\{U_i^j\}$ of M as in Lemma 3.3 with the given ε and δ. Take $\eta < \delta/6$ such that the 3η-extensions of any two sets U_i^j with the same j still have a positive distance between them, and such that

$$d(x,y) \leq 2\eta \;\Rightarrow\; |F(x) - F(y)| \leq \varepsilon. \tag{1}$$

Set V_i^j to be the 3η-extension of U_i^j with respect to the distance d on M. The radius η was chosen such that V_i^j has diameter at most 2δ, and can therefore be viewed as a neighborhood in $\mathbb{R}^{2n}$ where d and the Euclidean distance are equivalent up to a factor 2. In particular, any closed Euclidean ball of radius η centered inside U_i^j is contained in V_i^j. Denote by $B_\eta(x)$ the Euclidean ball of radius η around x. Requirement (1) guarantees that

$$\left|\langle F\rangle_{B_\eta(x)} - F(x)\right| \leq \varepsilon. \tag{2}$$

Next fix n such that

$$\mathrm{Vol}(\{x : |F_n(x) - F(x)| > \varepsilon\}) \;<\; \frac{\varepsilon \cdot |B_\eta|}{\max\{||F_n||_\infty, ||F||_\infty\}},$$

where $|B_\eta|$ is the measure of a Euclidean ball of radius η. This is possible since $\{F_n\}$ converges to F in measure, and since the F_n's are uniformly bounded. This choice of n implies that

$$\left|\langle F_n\rangle_{B_\eta(x)} - \langle F\rangle_{B_\eta(x)}\right| \leq 3\varepsilon. \tag{3}$$

By the definition of the integral, and the uniform continuity of F_n, there exist points $\{x^k\}_{k=1}^K \subseteq B_\eta(0)$, where K may be depend on n, such that for every $x \in U_i^j$

$$\left|\frac{1}{K}\sum_{k=1}^{K} F_n\left(x + x^k\right) - \langle F_n\rangle_{B_\eta(x)}\right| \leq \varepsilon.$$

Note that we have arranged that V_i^j contains the closure of the η-extension of U_i^j. Thus, using a standard cut-off argument, we consider Hamiltonian diffeomorphisms $g_{i,j}^1, \ldots, g_{i,j}^K$, all supported inside V_i^j, defined by $g_{i,j}^k(x) = x + x^k$ inside U_i^j and $g_{i,j}^k(x) = x$ outside a small neighborhood of U_i^j. We therefore get for all $x \in U_i^j$

$$\left|\frac{1}{K}\sum_{k=1}^{K} F_n\left(g_{i,j}^k(x)\right) - \langle F_n\rangle_{B_\eta(x)}\right| \leq \varepsilon. \tag{4}$$

Note that for fixed j and k, the Hamiltonian diffeomorphisms $\{g_{i,j}^k\}$ have disjoint supports, and can therefore be bundled together to form a single diffeomorphism. We set

$$\widetilde{F_n}(x) = \frac{1}{J}\sum_{j=1}^{J}\Big(\frac{1}{K}\sum_{k=1}^{K} F_n\big(\prod_i g_{i,j}^k(x)\big)\Big).$$

From the triangle inequality and the fact that the norm $\|\cdot\|'$ is invariant under Hamiltonian diffeomorphisms we conclude that $\|\widetilde{F_n}\|' \leq \|F_n\|'$. Hence, we need only to show that $\|\widetilde{F_n} - F\|_\infty \to 0$ as $\varepsilon \to 0$. Indeed,

$$\widetilde{F_n}(x) = \frac{1}{J}\Big(\sum_{j\in\mathcal{J}(x)}\Big(\frac{1}{K}\sum_k F_n\big(\prod_i g_{i,j}^k(x)\big)\Big) + \sum_{j\in\mathcal{J}^c(x)}\Big(\frac{1}{K}\sum_k F_n\big(\prod_i g_{i,j}^k(x)\big)\Big)\Big),$$

where $\mathcal{J}(x) = \{j \mid x \in \cup_i U_i^j\}$, $\mathcal{J}^c(x) = \{j \mid x \notin \cup_i U_i^j\}$. Recall that the third item of Lemma 3.3 limited the cardinality of $\mathcal{J}^c(x)$ to at most εJ for all x. Together with (4) this implies that

$$\left|\widetilde{F_n}(x) - \frac{1}{J}\sum_{j=1}^{J}\big(\langle F_n\rangle_{B_\eta(x)}\big)\right| \leq \varepsilon\frac{|\mathcal{J}(x)|}{J} + \frac{|\mathcal{J}^c(x)|}{J}\cdot 2\max\|F_n\|_\infty \leq \varepsilon + 2\varepsilon\cdot\max\|F_n\|_\infty$$

Together with (2) and (3) we conclude that

$$\begin{aligned}
\left|\widetilde{F_n}(x) - F(x)\right| &\leq \left|\widetilde{F_n}(x) - \frac{1}{J}\sum_{j=1}^{J}\big(\langle F_n\rangle_{B_\eta(x)}\big)\right| + \left|\frac{1}{J}\sum_{j=1}^{J}\big(\langle F_n\rangle_{B_\eta(x)}\big) - \frac{1}{J}\sum_{j=1}^{J}\big(\langle F\rangle_{B_\eta(x)}\big)\right| \\
&+ \left|\frac{1}{J}\sum_{j=1}^{J}\big(\langle F\rangle_{B_\eta(x)}\big) - \frac{1}{J}\sum_{j=1}^{J}\big(F(x)\big)\right| + \left|\frac{1}{J}\sum_{j=1}^{J}\big(F(x)\big) - F(x)\right| \\
&\leq \varepsilon + 2\varepsilon\cdot\max\|F_n\|_\infty + 3\varepsilon + \varepsilon \leq 5\varepsilon + 2\varepsilon\cdot\max\|F_n\|_\infty
\end{aligned}$$

Since the F_n's are uniformly bounded, $\widetilde{F_n}$ indeed converges uniformly to F as ε goes to zero. $\square$

As explained in Section 2, the proof of Claim 3.2 is based on a powerful result by Katok [7] which is used for the proof of Lemma 2.4.

Proof of Claim 3.2: Take $F \in L_\infty(M)$ and φ a measure-preserving bijection on M. Consider a sequence $\{F_n\}$ of uniformly bounded smooth functions which converges in measure to F. Let ε_n such that F_n is an ε_n-approximation in measure of F. Choose positive numbers δ_n so that $d(x,y) \leq \delta_n \Rightarrow |F_n(x) - F_n(y)| \leq \varepsilon_n$. By repeatedly using Lemma 2.4 we get a family of Hamiltonian diffeomorphisms $\{g_n\}$ such that

$$\mathrm{Vol}\big(\{x \mid d\left(g_n(x), \varphi(x)\right) > \delta_n\}\big) \leq \varepsilon_n.$$

Obviously

$$\left|F_n\big(g_n(x)\big) - F\big(\varphi(x)\big)\right| \leq \left|F_n\big(g_n(x)\big) - F_n\big(\varphi(x)\big)\right| + \left|F_n\big(\varphi(x)\big) - F\big(\varphi(x)\big)\right|.$$

Our choice of ε_n, δ_n and g_n guarantees that the above sum is smaller than $2\varepsilon_n$ outside a $2\varepsilon_n$-measure exceptional set, and therefore that $\{F_n \circ g_n\}$ converges in measure to $F \circ \varphi$. This and the invariance of $\|\cdot\|'$ imply that

$$|||F\circ\varphi||| \leq \liminf_n \|F_n \circ g_n\|' = \liminf_n \|F_n\|'.$$

Since this is true for any sequence $\{F_n\}$ of uniformly bounded smooth functions which converges in measure to F, we conclude that $|||F\circ\varphi||| \leq |||F|||$. Moreover, by applying the same argument to $F\circ\varphi$ and φ^{-1} we obtain that $|||F||| \leq |||F\circ\varphi|||$, and the proof is complete. $\square$

4 Proof of Theorem 1.2

Let ρ be an intrinsic bi-invariant pseudo-distance function on $\mathrm{Ham}(M,\omega)$ induced by some invariant norm on $\mathcal{A}$. In order to determine whether ρ is degenerate or not we will use a criterion by Eliashberg and Polterovich [4]. This criterion is based on the following notion of "displacement energy" introduced by Hofer [5].

Definition 4.1. *For every open subset $A \subset M$ define its displacement energy with respect to the pseudo-distance ρ as*

$$e(A) = \inf \{\rho(\mathbb{1}, \psi) \mid \psi \in \mathrm{Ham}(M,\omega),\ \psi(A) \cap A = \emptyset\},$$

and set $e(A) = \infty$ if the above set is empty.

Theorem 4.2 (Eliashberg-Polterovich). *If ρ is a genuine metric on Ham(M,ω) then the displacement energy of every non-empty open set is strictly positive.*

This theorem allows to reduce the proof of Theorem 1.2 to showing that the displacement energy of some small ball vanishes. An argument borrowed from [4], to be presented immediately below, further reduces the problem to

Claim 4.3. *For every $C' > 0$ and every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, C')$ such that for every function F with $\|F\|_\infty \le C'$ and $\mathrm{Vol}\big(\mathrm{Support}(F)\big) < \delta$ we have $\|F\| < \varepsilon$.*

Indeed, choose an embedded open ball $B \subset M$ such that its boundary ∂B is an embedded sphere, and such that there exists some Hamiltonian isotopy $\{g_t\}$, $t \in [0,1]$, generated by a Hamiltonian function $G(t,x)$ with $g_1(B) \cap B = \emptyset$. Denote by Σ_t the sphere $g_t(\partial B)$. Consider the function $K(t,x)$ obtained from G by smoothly cutting-off outside a neighborhood U_t of Σ_t. Note that the time-one-map of $K(t,x)$ also displaces B, i.e. $k_1(B) \cap B = \emptyset$. This is true since for every $t \in [0,1]$ we have $k_t(\partial B) = g_t(\partial B)$. Using Claim 4.3, we note that by decreasing the sizes of the neighborhoods U_t we can make $\|K_t\|$ arbitrarily small. Hence the displacement energy of the ball B vanishes.

We are thus left with proving Claim 4.3. As explained in Section 2, instead of proving it for $\|\cdot\|$, we shall prove it for the extension $|||\cdot|||$ announced in Theorem 1.3.

Proof of Claim 4.3. Let $\mathbb{1}_V$ stand for the characteristic function of the set V. We first prove that

$$|||\mathbb{1}_V||| \to 0 \text{ as } \mathrm{Vol}(V) \to 0. \tag{5}$$

Since $\|\cdot\|$ is not equivalent to $\|\cdot\|_\infty$, and since $|||\cdot|||$ is an extension of $\|\cdot\|$, for every $\varepsilon > 0$ there exists some function $F \in \mathcal{A}$ with $\|F\| = |||F||| \le \varepsilon$, while $\|F\|_\infty = 1$. Assume that the maximum of F is obtained at some point $x_0 \in M$ and set U to be a small-radius open set around x_0. Continuity allows us to choose U in such a way that $|F(x)| > 1 - \varepsilon$ for every $x \in U$. Using the triangle inequality we obtain:

$$|||\langle F\rangle_U \cdot \mathbb{1}_U||| \le |||\langle F\rangle_U \cdot \mathbb{1}_U + \langle F\rangle_{U^c} \cdot \mathbb{1}_{U^c}||| + |||\langle F\rangle_{U^c} \cdot \mathbb{1}_{U^c}|||,$$

where $U^c = M \setminus U$. The left summand is estimated via Lemma 2.5. To estimate the right summand, recall that $\mathrm{Vol}(U)\langle F\rangle_U + \mathrm{Vol}(U^c)\langle F\rangle_{U^c} = \langle F\rangle_M = 0$. Together with Lemma 2.5 we therefore get:

$$|||\langle F\rangle_U \cdot \mathbb{1}_U||| \le |||F||| + |||\frac{\langle F\rangle_U \cdot \mathrm{Vol}(U)}{\mathrm{Vol}(U^c)} \cdot \mathbb{1}_{U^c}|||.$$

Now, since $|||\cdot||| \le C\|\cdot\|_\infty$, and since $\|\frac{\langle F\rangle_U \cdot \mathrm{Vol}(U)}{\mathrm{Vol}(U^c)} \cdot \mathbb{1}_{U^c}\|_\infty$ goes to zero with $\mathrm{Vol}(U)$, for U with small enough measure we get

$$|||\langle F\rangle_U \cdot \mathbb{1}_U||| \le |||F||| + \varepsilon \le 2\varepsilon.$$

Due to the fact that $|\langle F\rangle_U| > 1-\varepsilon$, taking $\varepsilon < 1/2$ we get $|||\mathbb{1}_U||| < 4\varepsilon$. Since $|||\cdot|||$ is invariant under measure preserving bijections, this applies to every set V with the same measure as U .

Now we can complete the proof of Claim 4.3. Let $F \in C^\infty(M)$ be supported in some compact set $U \subset M$ with measure ε. Consider a finite partition of U into measurable sets $\{S_i\}_{i=1}^N$ with radius so small that uniform continuity affirms $\max(F|_{S_i}) - \min(F|_{S_i}) \le \varepsilon$ for every $1 \le i \le N$. We have

$$|||F||| = |||\sum_{i=1}^N F \cdot \mathbb{1}_{S_i}||| \le |||\sum_{i=1}^N \big(F - F(\eta_i)\big) \cdot \mathbb{1}_{S_i}||| + |||\sum_{i=1}^N F(\eta_i) \cdot \mathbb{1}_{S_i}|||,$$

where η_i is an arbitrary point in S_i. Without loss of generality we assume that $F(\eta_1) \le F(\eta_2) \le \ldots \le F(\eta_N)$. Using the fact that $|||\cdot||| \le C\|\cdot\|_\infty$ and the choice of the S_i's we get

$$|||F||| \le C\|\sum_{i=1}^N \big(F - F(\eta_i)\big) \cdot \mathbb{1}_{S_i}\|_\infty + |||\sum_{i=1}^N F(\eta_i) \cdot \mathbb{1}_{S_i}||| \le C\varepsilon + |||\sum_{i=1}^N F(\eta_i) \cdot \mathbb{1}_{S_i}|||$$

Next, in order to bound the last term on the right, we use Abel's summation trick

$$|||\sum_{i=1}^N F(\eta_i) \cdot \mathbb{1}_{S_i}||| = |||\sum_{i=1}^N \big(F(\eta_i) - F(\eta_{i-1})\big) \cdot \mathbb{1}_{\bigcup_{k=i}^N S_k}|||,$$

where $F(\eta_0)$ is defined to be zero. Substituting this in the above inequality we conclude

$$|||F||| \le C\varepsilon + \Big(\sum_{i=1}^N F(\eta_i) - F(\eta_{i-1})\Big) \cdot \max_i |||\mathbb{1}_{\bigcup_{k=i}^N S_k}||| \le C\varepsilon + 2\|F\|_\infty \cdot \max_i |||\mathbb{1}_{\bigcup_{k=i}^N S_k}|||.$$

Applying this estimate to a sequence of functions as in the statement of the claim, recalling that $\varepsilon = \mathrm{Vol}(\bigcup_{k=1}^N S_k)$ is the volume of the support, and relying on (5), the proof of the claim is complete.

$\square$

5 Lemmas

Here we prove Lemma 2.4, Lemma 2.5 and Lemma 3.3. Recall that M is a closed connected symplectic manifold and d is some Riemannian metric on M.

Proof of Lemma 2.4: Fix $\varepsilon > 0$. Let $\{A_i\}_{i=1}^N$ be a family of compact measurable disjoint subsets of M such that the following two conditions hold:

1. The diameter of each set A_i is at most ε,
2. $\mathrm{Vol}(\cup_{i=1}^N A_i) \geq \mathrm{Vol}(M) - \varepsilon$.

Next, let $\widetilde{B}_i = \varphi^{-1}(A_i)$, and let B_i be compact subsets of $\widetilde{B}_i$, such that $\sum_{i=1}^N \mathrm{Vol}(\widetilde{B}_i \setminus B_i) < \varepsilon$. From Katok's Basic Lemma [7] we get a Hamiltonian diffeomorphism g satisfying

$$\sum_{i=1}^N \mathrm{Vol}(g(B_i) \setminus A_i) \leq \varepsilon.$$

We claim that g is a good approximation in measure of φ. To see this, let $C_i = \{x \in B_i \,|\, g(x) \in A_i\}$ and denote by $C = \cup_{i=1}^N C_i$. Note that

$$\mathrm{Vol}(C) = \sum_{i=1}^N \mathrm{Vol}(C_i) \geq \sum_{i=1}^N \mathrm{Vol}(B_i) - \varepsilon \geq \sum_{i=1}^N \mathrm{Vol}(A_i) - 2\varepsilon \geq \mathrm{vol}(M) - 3\varepsilon.$$

Moreover, for every $x \in C$ the points $g(x)$ and $\varphi(x)$ belong to the same A_i. Since the diameter of the sets A_i is at most ε, we conclude that g is a 3ε-approximation in measure of φ. $\square$

Proof of Lemma 2.5: In order to keep notation simple, let's assume we have only two parts, S_1 and S_2. Fix $\varepsilon > 0$. Partition S_1 and S_2 into disjoint measurable sets $\{U_j^1\}_{j=1}^{J_1}$ and $\{U_j^2\}_{j=1}^{J_2}$, respectively, such that the following two conditions hold:

1. All U_j^1's have the same measure, and all U_j^2's have the same measure,
2. Inside all U_j^i's the function F does not oscillate by more than ε.

Choose $\varphi_{j,k}^i$ to be measure preserving bijections, not necessarily continuous, which map U_j^i onto U_k^i. For every permutation π of the set $\{1, \dots, J_i\}$, define $\varphi_\pi^i(x) = \varphi_{j,\pi(j)}^i(x)$ if $x \in U_j^i$, and $\varphi_\pi^i(x) = x$ if $x \notin S_i$. Finally, define the measure preserving bijections $\varphi_{\pi,\sigma} = \varphi_\pi^1 \circ \varphi_\sigma^2$. Since $|||\cdot|||$ is invariant under measure preserving bijections, the triangle inequality yields

$$|||\frac{1}{J_1!J_2!} \sum_{\pi,\sigma} F \circ \varphi_{\pi,\sigma}||| \leq |||F|||.$$

Now, our choice of U_j^i is such that for every $x \in U_j^i$ we have $|\langle F\rangle_{U_j^i} - F(x)| \leq \varepsilon$. Together with the equal measures of the U_j^i's, this means that for $x \in U_k^i$ we get

$$|\frac{1}{J_i}\sum_{j=1}^{J_i} F \circ \varphi_{k,j}^i(x) - \langle F\rangle_{S_i}| \leq \varepsilon.$$

We thus infer the inequality

$$\|\frac{1}{J_1!J_2!}\sum_{\pi,\sigma}(F \circ \varphi_{\pi,\sigma}) - (\langle F\rangle_{S_1}\mathbb{1}_{S_1} + \langle F\rangle_{S_2}\mathbb{1}_{S_2})\|_\infty \leq \varepsilon.$$

Since $|||\cdot||| \leq C\|\cdot\|_\infty$, taking ε to zero concludes the proof. □

Remark 5.1. If F is not continuous, then the choice of equal measure U_j^i's where F has small oscillations may not be possible. The argument, however, can be easily adapted to include the non-continuous case as well. Lemma 2.5 is also the key behind the proof that $|||\cdot|||$ is indeed a norm, namely that it vanishes only on the zero function. Indeed, if $|||F|||$ were zero for a non-zero F, a piecewise averaging of F would generate a non-zero step function with vanishing norm. Then, further piecewise averagings may be used to produce a sequence of zero norm step functions, which converge uniformly to a non-zero smooth function. This would mean that the original norm, which coincides with $|||\cdot|||$ on smooth functions, was already only a seminorm. We omit the details, because our main results still hold even if $|||\cdot|||$ were only a seminorm.

Proof of Lemma 3.3: According to Whitney's embedding theorem there exists a smooth embedding $\Psi : M \to \mathbb{R}^N$ for some (large enough) N. Next, for $\alpha, \beta \in \mathbb{R}$ set $\alpha\mathbb{K}+\beta = \{x \in \mathbb{R}^N \mid \exists\, i \text{ such that } x_i - \beta \in \alpha\mathbb{Z}\}$. Roughly speaking, $\alpha\mathbb{K}+\beta$ stands for the homothetic image of the "standard" grid in $\mathbb{R}^N$ translated in the direction of the vector $(1,\ldots,1)$. Fix $J \in \mathbb{N}$. For every $1 \leq j \leq J$, let G_j be the $\frac{\alpha}{4J}$-extension of the grid $\alpha\mathbb{K} + \frac{\alpha j}{J}$. Set $\{V_i^j\}_{i=1}^{L_j}$ to be the connected components of $\Psi(M) \cap (G_j)^c$. Note that a single "cell" of $(G_j)^c$ may be split into several connected components when intersected with $\Psi(M)$. However, by choosing the embedding coordinate-functions Ψ_i to be Morse functions, we can guarantee that the number of connected components is indeed finite. It may well be the case that for a given j some V_i^j's are zero-distance apart, but since our coordinates are Morse functions, arbitrarily small translations of G_j suffice to guarantee positive distance separation between all V_i^j's.

Now set $U_i^j = \Psi^{-1}(V_i^j)$. The first property in the statement follows from the positive distance between the V_i^j's. Compactness guarantees that a small enough α implies the second property. The last property follows from the fact that, regardless of J, the intersection of any $N+1$ different extended grids G_j is empty. Taking J such that $\frac{N+1}{J} < \varepsilon$ we are done. □

6 Further Information Concerning our Norms

Let $\|\cdot\|$ be a $\mathrm{Ham}(M,\omega)$-invariant norm on $\mathcal{A}$ such that $\|\cdot\| \le C\|\cdot\|_\infty$ for some constant C. Let $|||\cdot|||$ be the extension of $\|\cdot\|$ to $L_\infty(M)$ constructed in the proof of Theorem 1.3. The main objective of this section is to place the normed spaces $(\mathcal{A},\|\cdot\|)$ and $(L_\infty(M),|||\cdot|||)$ in the context of Banach (i.e. topologically complete) spaces of functions, so that existing knowledge from this field be made applicable to our context. For this purpose, we need to be able to view our spaces as subspaces of Banach spaces of functions. It can be easily seen that if the original norm $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$, so is $|||\cdot|||$, and we are in a Banach space setting. In the non-equivalent case we claim the following. Here $\|\cdot\|'$ is the extensions of $\|\cdot\|$ to $C^\infty(M)$ constructed in the proof of Theorem 1.3.

Proposition 6.1. *Let $\|\cdot\|$ be a $\mathrm{Ham}(M,\omega)$-invariant norm on $\mathcal{A}$ which is dominated from above by $\|\cdot\|_\infty$, but not equivalent to it. The space $(L_\infty(M),|||\cdot|||)$ then coincides with a dense subspace of the completion of $(C^\infty(M),\|\cdot\|')$. Moreover, this completion can be viewed as a dense subspace of the space $L_1(M)$ of integrable measurable functions on M, equipped with a norm which is invariant under measure preserving bijections.*

Sketch of the proof: To establish the relation between $|||\cdot|||$ and the completion of $||\cdot||'$, we need to show that if $\{F_n\}$ is a sequence of uniformly bounded smooth functions tending in measure to F, then $\{F_n\}$ is a Cauchy sequence in $\|\cdot\|'$ (which is equivalent to showing that it is a Cauchy sequence in $|||\cdot|||$). Indeed, let F_n and F_m both ε-approximate F in measure for some arbitrary small ε. We can then write $F_n - F_m = G_{n,m} + H_{n,m}$, where $G_{n,m}$ and $H_{n,m}$ are smooth and uniformly bounded, $\|G_{n,m}\|_\infty \le 2\varepsilon$, and the measure of the support of $H_{n,m}$ is at most 2ε. Claim 4.3 now proves that $|||F_n - F_m||| \to 0$ as $n,m\to\infty$.

We turn now to the second part of the proposition. First we claim that there exists some constant C such that $|||F||| \ge C\|F\|_{L_1}$ for any essentially bounded measurable function F. Indeed, set M_F to be the median of F, namely the unique number for which both $\mathrm{Vol}(\{x\in M \,|\, F(x)\ge M_F\})$ and $\mathrm{Vol}(\{x\in M\,|\,F(x)\le M_F\})$ are at least half. Without loss of generality we may assume that $M_F\ge 0$. Let $\{x\in M\,|\,F(x)>M_F\}\subseteq A\subseteq\{x\in M\,|\,F(x)\ge M_F\}$, such that $\mathrm{Vol}(A)=\frac{1}{2}$. Finally, let $B=\{x\in M\,|\,F(x)\ge 0\}$. We will argue under the assumption that F is zero-mean; the extension to general F involves adding just one dimension to our space of functions, and therefore follows immediately. By Lemma 2.5 we obtain that

$$|||F||| \ge |||\langle F\rangle_A 1\!\!1_A + \langle F\rangle_{A^c} 1\!\!1_{A^c}||| = \langle F\rangle_A |||1\!\!1_A - 1\!\!1_{A^c}|||.$$

We clearly have

$$\langle F\rangle_A = 2\int_A F(x)\,\omega^n \ge 2\int_{B-A} F(x)\,\omega^n,$$

and therefore

$$\langle F\rangle_A \ge \int_B F(x)\,\omega^n = \frac{1}{2}||F||_{L_1}.$$

Together with the above estimate for $|||F|||$, this yields $|||F||| \geq \frac{1}{2}||F||_{L_1} \cdot |||\mathbb{1}_A - \mathbb{1}_{A^c}|||$. The invariance property of the norm $|||\cdot|||$ implies that the value of $|||\mathbb{1}_A - \mathbb{1}_{A^c}|||$ depends only on the fixed $\mathrm{Vol}(A) = \frac{1}{2}$. Thus, the inequality is proved.

Now, every Cauchy sequence of smooth functions in $\|\cdot\|'$ (not necessarily uniformly bounded) is also a Cauchy sequence in L_1, and is therefore convergent in measure. In order to regard this limit in measure as an element of the completion of $\|\cdot\|'$, we need to show that if two Cauchy sequences in $\|\cdot\|'$, $\{F_n\}$ and $\{G_n\}$, converge in measure to the same F, then both sequences have the same limit in the completion, namely $\|F_n - G_n\|' \to 0$. For this purpose set $H_n = F_n - G_n$. By definition, H_n is a Cauchy sequence in $\|\cdot\|'$ converging in L_1 and in measure to the zero function. We need to prove that $\|H_n\|' \to 0$. Again, we will carry out the proof in $|||\cdot|||$. By taking small uniform perturbations we may also assume that $\mathrm{Vol}(A_n) \to 0$, where $A_n = \mathrm{Support}(H_n)$. Next, by applying a slight variant of Lemma 2.5 we get

$$|||H_n - H_m||| \geq |||(H_n - H_m)\mathbb{1}_{A_m} + \langle H_n - H_m \rangle_{A_m^c} \mathbb{1}_{A_m^c}|||.$$

Since $\langle H_n - H_m \rangle_{A_m^c} \leq \|H_n - H_m\|_{L_1} \cdot \mathrm{Vol}(A_m^c)$, this term goes to zero. We therefore conclude that $|||(H_n - H_m)\mathbb{1}_{A_m}||| = |||H_m - H_n \cdot \mathbb{1}_{A_m}|||$ converges to zero as n, m increase. Due to Claim 4.3, for every fixed n the term $|||H_n \cdot \mathbb{1}_{A_m}|||$ tends to zero with m. We therefore conclude, as announced, that $|||H_m||| \to 0$.

Since we already know that $|||\cdot|||$ is invariant under measure preserving bijections, the proof that the completion is also invariant is straightforward. Note that since we assume $\|\cdot\|$ is dominated by $\|\cdot\|_\infty$, but not equivalent to it, Banach's Open Map Theorem implies that the completion of $\|\cdot\|$ must in fact exceed the space of essentially bounded measurable functions. The proof is now complete. □

The literature contains much information concerning a special subclass of the class of Banach norms on spaces of functions, which are invariant under measure preserving bijections. This is the subclass of the so called *Rearrangement Invariant Function Spaces*. The main (but not only!) requirement is that the norm be monotone with respect to the natural partial order on non-negative functions. Since an explicit formulation will drag us into a long list of definitions which are not relevant for this paper, we will make do here with a reference. The book [2] introduces Rearrangement Invariant Function Spaces in Chapter 2, Definition 1.4 (which relies on Definitions 1.1 and 1.3 from Chapter 1). The main classification results are announced in Chapter 2, Theorem 5.15 and in Chapter 3, Theorem 2.12. Another thorough analysis from a somewhat different point of view is available in [9] (Definitions 1.b.17 and 2.a.1, and the results of the second section).

We cannot rule out the possibility that all normed spaces $(\mathcal{A}, \|\cdot\|)$, which are invariant under Hamiltonian diffeomorphisms, can be viewed as subspaces of Rearrangement Invariant Function Spaces. The following example, while not relating directly to the issue under discussion, serves to indicate the kind of pathologies one might expect from norms outside this class. Take the space $\mathcal{A} \bigoplus \mathcal{B}$, where $\mathcal{B}$ is the space of functions on M which attain only finitely many values. It is straightforward to see that the sum is indeed an algebraically direct sum. For an element $a+b \in \mathcal{A} \bigoplus \mathcal{B}$ consider the functional $\|a+b\| = \|a\|_1 + \|b\|_\infty$. It is easy to check that $\|\cdot\|$ is a norm invariant under measure preserving diffeomorphisms, but not under measure preserving bijections (we restrict our attention, of course, to measure preserving bijections which keep the 'rearranged' function inside our space). It is also not hard to see that this norm is not bounded by $\|\cdot\|_\infty$.

Acknowledgment: This work is a part of the first author's Ph.D. thesis, being carried out under the supervision of Professor Leonid Polterovich at Tel-Aviv University. The first author would like to thank Professor Polterovich for his guidance, encouragement and continual support. Both authors wish to thank Professor Leonid Polterovich for a thorough review of the paper and many valuable suggestions, and Professor Vitali Milman for useful comments.

References

[1] Banyaga, A. *Sur la structure du groupe des difféomorphisms qui préservent une forme symplectique.* Comment. Math. Helv. **53** (1978), no.2, 174-227.

[2] Bennett, C. and Sharpley, R. **Interpolation of Operators**, Academic Press (1988).

[3] Chekanov, Yu. V. *Invariant Finsler metrics on the space of Lagrangian embeddings,* Math. Z. **234** (2000), 605-619.

[4] Eliashberg, Y. and Polterovich, L. *Bi-invariant metrics on the group of Hamiltonian diffeomorphisms,* Internat. J. Math. **4** (1993), 727-738.

[5] Hofer, H. *On the topological properties of symplectic maps.* Proceedings of the Royal Society of Edinburgh, **115** (1990), 25-38.

[6] Hofer, H. and Zehnder, E. **Symplectic invariants and Hamiltoniam dynamics**, Birkhäuser Advanced Texts, Birkhäuser Verlag, 1994.

[7] Katok, A. *Ergodic perturbations of degenerate integrable Hamiltonian systems*, Math, USSR Izvestija **7** (1973), 535-571.

[8] Lalonde, F. and McDuff, D. *The geometry of symplectic energy*, Ann. of Math **141** (1995), 349-371.

[9] Lindenstrauss, J. and Tzafriri, L. **Classical Banach Spaces, Vol. 2**, Springer Verlag (1977).

[10] McDuff, D. and Salamon, D. **Introduction to Symplectic Topology**, 2nd edition, Oxford University Press (1998).

[11] Oh, Yong-Geun. *Lectures on the Floer theory and spectral invariants of Hamiltonian flows.* Preprint.

[12] Polterovich, L. **The Geometry of the group of Symplectic Diffeomorphisms**, Lectures in Math, ETH, Birkhäuser (2001).

[13] Polterovich, L. *Symplectic displacement energy for Lagrangian submanifolds.* Ergodic Theory and Dynamical Systems, **13** (1993), 357-67.

[14] Polterovich, L. *Hamiltonian loops from the ergodic point of view.* J. Eur. Math. Soc. **1** (1999), 87-107.

[15] Rudin, W. **Real and Complex analysis**, Third edition. McGraw-Hill Book Co., New York, 1987

Yaron Ostrover
School of Mathematical Sciences
Tel Aviv University
Tel Aviv 69978, Israel
yaronost@post.tau.ac.il

Roy Wagner
Computer Science Department
Academic College of Tel Aviv – Yaffo
4 Antokolsky St., Tel Aviv 64044, Israel
rwagner@mta.ac.il