

When Symplectic Topology Meets Banach Space Geometry

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Abstract. In this paper we survey some recent works that take the first steps toward establishing bilateral connections between symplectic geometry and several other fields, namely, asymptotic geometric analysis, classical convex geometry, and the theory of normed spaces.

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1. Introduction

In the last three decades, symplectic topology has had an astonishing amount of fruitful interactions with other fields of mathematics, including complex and algebraic geometry, dynamical systems, Hamiltonian PDEs, transformation groups, and low-dimensional topology; as well as with physics, where, for example, symplectic topology plays a key role in the rigorous formulation of mirror symmetry.

In this survey paper, we present some recent works that take first steps toward establishing novel interrelations between symplectic geometry and several fields of mathematics, namely, asymptotic geometric analysis, classical convex geometry, and the theory of normed spaces. In the first part of this paper (Sections 2 and 3) we concentrate on the theory of symplectic measurements, which arose from the foundational work of Gromov [34] on pseudoholomorphic curves; followed by the seminal works of Ekeland and Hofer [24] and Hofer and Zehnder [42] on variational theory in Hamiltonian systems, and Viterbo on generating functions [89]. This theory – also known as the theory of “symplectic capacities” – lies nowadays at the core of symplectic geometry and topology.

In Section 2, we focus on an open symplectic isoperimetric-type conjecture proposed by Viterbo in [88]. It states that among all convex domains with a given volume in the classical phase space \mathbb{R}^{2n} , the Euclidean ball has the maximal “symplectic size” (see Section 2 below for the precise statement). In a collaboration with S. Artstein-Avidan and V. D. Milman [6], we were able to prove an asymptotic version of Viterbo’s conjecture, that is, we proved the conjecture up to a universal (dimension-independent) constant. This has been achieved by adapting techniques

from asymptotic geometric analysis and adjusting them to a symplectic context, while working exclusively in the linear symplectic category.

The fact that one can get within a constant factor to the full conjecture using only linear embeddings is somewhat surprising from the symplectic-geometric point of view, as in symplectic geometry one typically needs highly nonlinear tools to estimate capacities. However, this fits perfectly into the philosophy of asymptotic geometric analysis. Finding dimension independent estimates is a frequent goal in this field, where surprising phenomena such as concentration of measure (see e.g. [67]) imply the existence of order and structures in high dimensions, despite the huge complexity it involves. It would be interesting to explore whether similar phenomena also exist in the framework of symplectic geometry. A natural important source for the study of the asymptotic behavior (in the dimension) of symplectic invariants is the field of statistical mechanics, where one considers systems with a large number of particles, and the dimension of the phase space is twice the number of degrees of freedom. It seems that symplectic measurements were overlooked in this context so far.

In Section 3 we go in the opposite direction: we show how symplectic geometry could potentially be used to tackle a 70-years-old fascinating open question in convex geometry, known as the Mahler conjecture. Roughly speaking, Mahler's conjecture states that the minimum of the product of the volume of a centrally symmetric convex body and the volume of its polar body is attained (not uniquely) for the hypercube. In a collaboration with S. Artstein-Avidan and R. Karasev [8], we combined tools from symplectic geometry, classical convex analysis, and the theory of mathematical billiards, and established a close relation between Mahler's conjecture and the above mentioned symplectic isoperimetric conjecture by Viterbo. More precisely, we showed that Mahler's conjecture is equivalent to a special case of Viterbo's conjecture (see Section 3 for details).

In the second part of the paper (Section 4), we explain how methods from functional analysis can be used to address questions regarding the geometry of the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms associated with a symplectic manifold (M, ω) . One of the most striking facts regarding this group, discovered by Hofer in [40], is that it carries an intrinsic geometry given by a Finsler bi-invariant metric, nowadays known as Hofer's metric. This metric measures the time-averaged minimal oscillation of a Hamiltonian function that is needed to generate a Hamiltonian diffeomorphism starting from the identity. Hofer's metric has been intensively studied in the past twenty years, leading to many discoveries covering a wide range of subjects from Hamiltonian dynamics to symplectic topology (see e.g., [43, 59, 75] and the references therein). A long-standing question raised by Eliashberg and Polterovich in [26] is whether Hofer's metric is the only bi-invariant Finsler metric on the group $\text{Ham}(M, \omega)$. Together with L. Buhovsky [17], and based on previous results by Ostrover and Wagner [72], we used methods from functional analysis and the theory of normed function spaces to affirmatively answer this question. We proved that any non-degenerate bi-invariant Finsler metric on $\text{Ham}(M, \omega)$, which is generated by a norm that is continuous in the C^∞ -topology, gives rise to the same topology on $\text{Ham}(M, \omega)$ as the one induced by Hofer's metric.

As mentioned before, the outlined interdisciplinary connections described above are just the first few steps in what seems to be a promising new direction. We hope that further exploration of these connections will strengthen the dialogue between these fields and symplectic geometry, and expand the range of methodologies alongside research questions that can be tackled through these means.

We end this paper with several open questions and speculations regarding some of the mentioned topics (see Section 5).

2. A Symplectic Isoperimetric Inequality

A classical result in symplectic geometry (Darboux's theorem) states that symplectic manifolds - in a sharp contrast to Riemannian manifolds - have no local invariants (except, of course, the dimension). The first examples of global symplectic invariants were introduced by Gromov in his seminal paper [34], where he developed and used pseudoholomorphic curve techniques to prove a striking symplectic rigidity result. Nowadays known as Gromov's "non-squeezing theorem", this result states that one cannot map a ball inside a thinner cylinder by a symplectic embedding. This theorem paved the way to the introduction of global symplectic invariants, called symplectic capacities which, roughly speaking, measure the symplectic size of a set.

We will focus here on the case of the classical phase space $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ equipped with the standard symplectic structure $\omega = dq \wedge dp$. We denote by $B^{2n}(r)$ the Euclidean ball of radius r , and by $Z^{2n}(r)$ the cylinder $B^2(r) \times \mathbb{C}^{n-1}$. Gromov's non-squeezing theorem asserts that if $r < 1$ there is no symplectomorphism ψ of \mathbb{R}^{2n} such that $\psi(B^{2n}(1)) \subset Z^{2n}(r)$. The following definition, which crystallizes the notion of "symplectic size", was given by Ekeland and Hofer in their influential paper [24].

Definition: A symplectic capacity on $(\mathbb{R}^{2n}, \omega)$ associates to each subset $U \subset \mathbb{R}^{2n}$ a number $c(U) \in [0, \infty]$ such that the following three properties hold:

- (P1) $c(U) \leq c(V)$ for $U \subseteq V$ (monotonicity);
- (P2) $c(\psi(U)) = |\alpha| c(U)$ for $\psi \in \text{Diff}(\mathbb{R}^{2n})$ such that $\psi^*\omega = \alpha\omega$ (conformality);
- (P3) $c(B^{2n}(r)) = c(Z^{2n}(r)) = \pi r^2$ (nontriviality and normalization).

Note that (P3) disqualifies any volume-related invariant, while (P1) and (P2) imply that for $U, V \subset \mathbb{R}^{2n}$, a necessary condition for the existence of a symplectomorphism ψ with $\psi(U) = V$, is $c(U) = c(V)$ for any symplectic capacity c .

It is a priori unclear that symplectic capacities exist. The above mentioned non-squeezing result naturally leads to the definition of two symplectic capacities: the Gromov radius, defined by $\underline{c}(U) = \sup\{\pi r^2 \mid B^{2n}(r) \xrightarrow{s} U\}$; and the cylindrical capacity, defined by $\bar{c}(U) = \inf\{\pi r^2 \mid U \xrightarrow{s} Z^{2n}(r)\}$, where \xrightarrow{s} stands for symplectic embedding. It is easy to verify that these two capacities are the smallest and largest possible symplectic capacities, respectively. Moreover, it is also known that the existence of a single capacity readily implies Gromov's non-squeezing theorem,

as well as the Eliashberg-Gromov C^0 -rigidity theorem, which states that for any closed symplectic manifold (M, ω) , the symplectomorphism group $\text{Symp}(M, \omega)$ is C^0 -closed in the group of all diffeomorphisms of M (see e.g., Chapter 2 of [43]).

Shortly after Gromov's work, other symplectic capacities were constructed, such as the Hofer-Zehnder [43] and the Ekeland-Hofer [24] capacities, the displacement energy [40], the Floer-Hofer capacity [27, 28], spectral capacities [29, 70, 89], and, more recently, Hutchings's embedded contact homology capacities [44]. Nowadays, symplectic capacities are among the most fundamental objects in symplectic geometry, and are the subject of intensive research efforts (see e.g., [45, 47, 52, 55–57, 60, 63, 82], and [20] for a recent detailed survey and more references). However, in spite of the rapidly accumulating knowledge regarding symplectic capacities, they are notoriously difficult to compute, and there are no general methods even to effectively estimate them.

In [88], Viterbo investigated the relation between the symplectic way of measuring the size of sets using symplectic capacities, and the classical approach using volume. Among many other inspiring results, in that work he conjectured that in the class of convex bodies in \mathbb{R}^{2n} with fixed volume, the Euclidean ball B^{2n} maximizes any given symplectic capacity. More precisely,

Conjecture 2.1 (Viterbo's volume-capacity inequality conjecture). *For any convex body K in \mathbb{R}^{2n} and any symplectic capacity c ,*

$$\frac{c(K)}{c(B)} \leq \left(\frac{\text{Vol}(K)}{\text{Vol}(B)} \right)^{1/n}, \quad \text{where } B = B^{2n}(1).$$

Here and henceforth a convex body of \mathbb{R}^{2n} is a compact convex set with non-empty interior. The isoperimetric inequality above was proved in [88] up to a constant that depends linearly on the dimension using the classical John ellipsoid theorem. In a joint work with S. Artstein-Avidan and V. D. Milman (see [6]), we made further progress towards the proof of the conjecture. By customizing methods and techniques from asymptotic geometric analysis and adjusting them to the symplectic context, we were able to prove Viterbo's conjecture up to a universal (i.e., dimension-independent) constant. More precisely, we proved that

Theorem 2.2. *There is a universal constant A such that for any convex domain K in \mathbb{R}^{2n} , and any symplectic capacity c , one has*

$$\frac{c(K)}{c(B)} \leq A \left(\frac{\text{Vol}(K)}{\text{Vol}(B)} \right)^{1/n}, \quad \text{where } B = B^{2n}(1).$$

We emphasize that in the proof of Theorem 2.2 we work exclusively in the category of linear symplectic geometry. It turns out that even in this limited category of linear symplectic transformations, there are tools which are powerful enough to obtain a dimension-independent estimate as above. While this fits with the philosophy of asymptotic geometric analysis, it is less expected from a symplectic geometry point of view, where one expects that highly nonlinear methods, such

as folding and wrapping techniques (see e.g., the book [82]), would be required to effectively estimate symplectic capacities.

The proof of Theorem 2.2 above is based on two ingredients. The first is the following simple geometric observation (see Lemma 3.3 in [6], cf. [1]).

Lemma 2.3. *If a convex body $K \subset \mathbb{C}^n$ satisfies $K = iK$, then $\bar{c}(K) \leq \frac{4}{\pi} c(K)$.*

Sketch of Proof. Let rB^{2n} be the largest multiple of the unit ball contained in K , and let $x \in \partial K \cap rS^{2n-1}$ be a contact point between the boundary of K and the boundary of rB^{2n} . It follows from the convexity assumption that the body K lies between the hyperplanes $x + x^\perp$ and $-x + x^\perp$. Moreover, since $K = iK$, it lies also between $-ix + ix^\perp$ and $ix + ix^\perp$. Thus, the projection of K onto the plane spanned by x and ix is contained in a square of edge length $2r$. This square can be turned into a disc with area $4r^2$, after applying a non-linear symplectomorphism which is essentially two-dimensional. Therefore, K is contained in a symplectic image of the cylinder $Z^{2n}(\sqrt{4/\pi}r)$, and the lemma follows. \square

Since by monotonicity, Conjecture 2.1 trivially holds for the Gromov radius c , it follows from Lemma 2.3 that

Corollary 2.4. *Theorem 2.2 holds for convex bodies $K \subset \mathbb{C}^n$ such that $K = iK$.*

The second ingredient in the proof is a profound result in asymptotic geometric analysis discovered by V.D. Milman in the mid 1980's called the "reverse Brunn-Minkowski inequality" (see [65, 66]). Recall that the classical Brunn-Minkowski inequality states that if A and B are non-empty Borel subsets of \mathbb{R}^n , then

$$\text{Vol}(A + B)^{1/n} \geq \text{Vol}(A)^{1/n} + \text{Vol}(B)^{1/n},$$

where $A + B = \{x + y \mid x \in A, y \in B\}$ is the Minkowski sum. Although at first glance it seems that one cannot expect any inequality in the reverse direction (consider, e.g., two very long and thin ellipsoids pointing in orthogonal directions in \mathbb{R}^2), it turns out that for convex bodies, if one allows for an extra choice of "position", i.e., a volume-preserving linear image of the bodies, then one can reverse the Brunn-Minkowski inequality up to a universal constant factor.

Theorem 2.5 (Milman's reverse Brunn-Minkowski inequality). *For any two convex bodies K_1, K_2 in \mathbb{R}^n , there exist linear volume preserving transformations T_{K_i} ($i = 1, 2$), such that for $\tilde{K}_i = T_{K_i}(K_i)$ one has*

$$\text{Vol}(\tilde{K}_1 + \tilde{K}_2)^{1/n} \leq C \left(\text{Vol}(\tilde{K}_1)^{1/n} + \text{Vol}(\tilde{K}_2)^{1/n} \right),$$

for some absolute constant C .

We emphasize that the transformation T_{K_i} ($i = 1, 2$) in Theorem 2.5 depends solely on the body K_i , and not on the joint configuration of the bodies K_1 and K_2 . For more details on the reverse Brunn-Minkowski inequality see [66, 75].

We can now sketch the proof of Theorem 2.2 (for more details see [6]). Since every symplectic capacity is bounded above by the cylindrical capacity \bar{c} , it is enough to prove the theorem for \bar{c} . For the sake of simplicity, we assume in what follows that K is centrally symmetric, i.e., $K = -K$. This assumption is not too restrictive, since by a classical result of Rogers and Shephard [79] one has that $\text{Vol}(K + (-K)) \leq 4^n \text{Vol}(K)$. After adjusting Theorem 2.5 to the symplectic context, one has that for any convex body $K \subset \mathbb{R}^{2n}$, there exists a linear symplectomorphism $S \in \text{Sp}(2n)$ such that SK and iSK satisfy the reverse Brunn-Minkowski inequality, that is, the volume $\text{Vol}(SK + iSK)$ is less than some constant times $\text{Vol}(K)$. Combining this with the properties of symplectic capacities and Corollary 2.4, we conclude that

$$\frac{\bar{c}(K)}{\bar{c}(B)} \leq \frac{\bar{c}(SK + iSK)}{\bar{c}(B)} \leq A \left(\frac{\text{Vol}(SK + iSK)}{\text{Vol}(B)} \right)^{\frac{1}{n}} \leq A' \left(\frac{\text{Vol}(K)}{\text{Vol}(B)} \right)^{\frac{1}{n}},$$

for some universal constant A' , and thus Theorem 2.2 follows.

In the next section we will show a surprising connection between Viterbo's volume-capacity conjecture and a seemingly remote open conjecture from the field of convex geometric analysis: the Mahler conjecture on the volume product of centrally symmetric convex bodies.

3. A Symplectic View on Mahler's Conjecture

Let $(X, \|\cdot\|)$ be an n -dimensional normed space and let $(X^*, \|\cdot\|_*)$ be its dual space. Note that the product space $X \times X^*$ carries a canonical symplectic structure, given by the skew-symmetric bilinear form $\omega((x, \xi), (x', \xi')) = \xi(x') - \xi'(x)$, and a canonical volume form, the *Liouville* volume, given by $\omega^n/n!$. A fundamental question in the field of convex geometry, raised by Mahler in [58], is to find upper and lower bounds for the Liouville volume of $B \times B^\circ \subset X \times X^*$, where B and B° are the unit balls of X and X^* , respectively. In what follows we shall denote this volume by $\nu(X)$. The quantity $\nu(X)$ is an affine invariant of X , i.e. it is invariant under invertible linear transformations. We remark that in the context of convex geometry $\nu(X)$ is also known as the *Mahler volume* or the *volume product* of X .

The Blaschke-Santaló inequality asserts that the maximum of $\nu(X)$ is attained if and only if X is a Euclidean space. This was proved by Blaschke [14] for dimensions two and three, and generalized by Santaló [81] to higher dimensions. The following sharp lower bound for $\nu(X)$ was conjectured by Mahler [58] in 1939:

Conjecture 3.1 (Mahler's volume product conjecture). *For any n -dimensional normed space X one has $\nu(X) \geq 4^n/n!$.*

The conjecture has been verified by Mahler [58] in the two-dimensional case. In higher dimensions it is proved only in a few special cases (see e.g., [33, 49, 64, 69, 76–78, 80, 86]). A major breakthrough towards answering Mahler's conjecture is a result due to Bourgain and Milman [16], who used sophisticated tools from

functional analysis to show that the conjecture holds asymptotically, i.e., up to a factor γ^n , where γ is a universal constant. This result has been re-proved later on, with entirely different methods, by Kuperberg [51], using differential geometry, and independently by Nazarov [68], using the theory of functions of several complex variables. A new proof using simpler asymptotic geometric analysis tools has been recently discovered by Giannopoulos, Paouris, and Vritsiou [32]. The best known constant today, $\gamma = \pi/4$, is due to Kuperberg [51].

Despite great efforts to deal with the general case, a proof of Mahler's conjecture has been insistently elusive so far, and is currently the subject of intensive research. A possible reason for this, as pointed out for example by Tao in [87], is that, in contrast with the above mentioned Blaschke-Santaló inequality, the equality case in Mahler's conjecture, which is obtained for example for the space l_∞^n of bounded sequences with the standard maximum norm, is not unique, and there are in fact many distinct extremizers for the (conjecturally) lower bound of $\nu(X)$ (see, e.g., the discussion in [87]). This practically renders impossible any proof based on currently known optimisation techniques, and a radically different approach seems to be needed.

We refer the reader to Section 5 below for further discussion on the characterization of the equality case of Mahler's conjecture, and its possible connection with symplectic geometry.

In a recent work with S. Artstein-Avidan and R. Karasev [8], we combined tools from symplectic geometry, convex analysis, and the theory of mathematical billiards, and established a close relationship between Mahler's conjecture and Viterbo's volume-capacity conjecture. More precisely, we proved in [8] that

Theorem 3.2. *Viterbo's volume-capacity conjecture implies Mahler's conjecture.*

In fact, it follows from our proof that Mahler's conjecture is equivalent to a special case of Viterbo's conjecture, where the latter is restricted to the Ekeland-Hofer-Zehnder symplectic capacity, and to domains in the classical phase space of the form $\Sigma \times \Sigma^\circ \subset \mathbb{R}^{2n} = \mathbb{R}_q^n \times \mathbb{R}_p^n$ (for more details see [8], and in particular Remark 1.9 *ibid.*). Here, $\Sigma \subset \mathbb{R}_q^n$ is a centrally symmetric convex body, the space \mathbb{R}_p^n is identified with the dual space $(\mathbb{R}_q^n)^*$, and

$$\Sigma^\circ = \{p \in \mathbb{R}_p^n \mid p(q) \leq 1 \text{ for every } q \in \Sigma\}$$

Theorem 3.2 is a direct consequence of the following result proven in [8].

Theorem 3.3. *There exists a symplectic capacity c such that $c(\Sigma \times \Sigma^\circ) = 4$ for every centrally symmetric convex body $\Sigma \subset \mathbb{R}_q^n$.*

With Theorem 3.3 at our disposal, it is not difficult to derive Theorem 3.2.

Proof of Theorem 3.2. Assume that Viterbo's volume-capacity conjecture holds. From Theorem 3.3 it follows that there exists a symplectic capacity c such that for every centrally symmetric convex body $\Sigma \subset \mathbb{R}_q^n$ one has

$$\frac{4^n}{\pi^n} = \frac{c^n(\Sigma \times \Sigma^\circ)}{\pi^n} \leq \frac{\text{Vol}(\Sigma \times \Sigma^\circ)}{\text{Vol}(B^{2n})} = \frac{n! \text{Vol}(\Sigma \times \Sigma^\circ)}{\pi^n},$$

which is exactly the bound for $\text{Vol}(\Sigma \times \Sigma^\circ)$ required by Mahler's conjecture. \square

In the rest of this section we sketch the proof of Theorem 3.3 (see [8] for a detailed exposition). We remark that an alternative proof, based on an approach to billiard dynamics developed in [11], was recently given in [3]. We start with recalling the definition of the Ekeland-Hofer-Zehnder capacity, which is the symplectic capacity that appears in Theorem 3.3.

The restriction of the standard symplectic form $\omega = dq \wedge dp$ to a smooth closed connected hypersurface $\mathcal{S} \subset \mathbb{R}^{2n}$ defines a 1-dimensional subbundle $\ker(\omega|_{\mathcal{S}})$, whose integral curves comprise the characteristic foliation of \mathcal{S} . In other words, a *closed characteristic* of \mathcal{S} is an embedded circle in \mathcal{S} tangent to the canonical line bundle

$$\mathfrak{S}_{\mathcal{S}} = \{(x, \xi) \in T\mathcal{S} \mid \omega(\xi, \eta) = 0 \text{ for all } \eta \in T_x\mathcal{S}\}.$$

Recall that the symplectic action of a closed curve γ is defined by $A(\gamma) = \int_{\gamma} \lambda$, where $\lambda = pdq$ is the Liouville 1-form. The action spectrum of \mathcal{S} is

$$\mathcal{L}(\mathcal{S}) = \{|A(\gamma)|; \gamma \text{ closed characteristic on } \mathcal{S}\}.$$

The following theorem, which is a combination of results from [24] and [43], states that on the class of convex domains in \mathbb{R}^{2n} , the Ekeland-Hofer capacity c_{EH} and Hofer-Zehnder capacity c_{HZ} coincide, and are given by the minimal action over all closed characteristics on the boundary of the corresponding convex body.

Theorem 3.4. *Let $K \subseteq \mathbb{R}^{2n}$ be a convex bounded domain with smooth boundary. Then there exists at least one closed characteristic $\tilde{\gamma} \subset \partial K$ satisfying*

$$c_{\text{EH}}(K) = c_{\text{HZ}}(K) = A(\tilde{\gamma}) = \min \mathcal{L}(\partial K).$$

We remark that although the above definition of closed characteristics, as well as Theorem 3.4, were given only for the class of convex bodies with smooth boundary, they can naturally be generalized to the class of convex sets in \mathbb{R}^{2n} with non-empty interior (see [7]). In what follows, we refer to the coinciding Ekeland-Hofer and Hofer-Zehnder capacities on this class as the Ekeland-Hofer-Zehnder capacity.

We turn now to show that for every centrally symmetric convex body $\Sigma \subset \mathbb{R}_q^n$, the Ekeland-Hofer-Zehnder capacity satisfies $c_{\text{EHZ}}(\Sigma \times \Sigma^\circ) = 4$. For this purpose, we now switch gears and turn to mathematical billiards in Minkowski geometry.

It is folklore to people in the field that billiard flow can be treated, roughly speaking, as the limiting case of geodesic flow on a boundaryless manifold. Indeed, let Ω be a smooth plane billiard table, and consider its “thickening”, i.e. an infinitely thin three dimensional body whose boundary Γ is obtained by pasting two copies of Ω along their boundaries and smoothing the edge. Thus, a billiard trajectory in Ω can be viewed as a geodesic line on the boundary of Γ , that goes from one copy of Ω to another each time the billiard ball bounces off the boundary. The main technical difficulties with this strategy is the rigorous treatment of the limiting process, and the analysis involved with the dynamics near the boundary. One approach to billiard dynamics and the existence question of periodic trajectories is

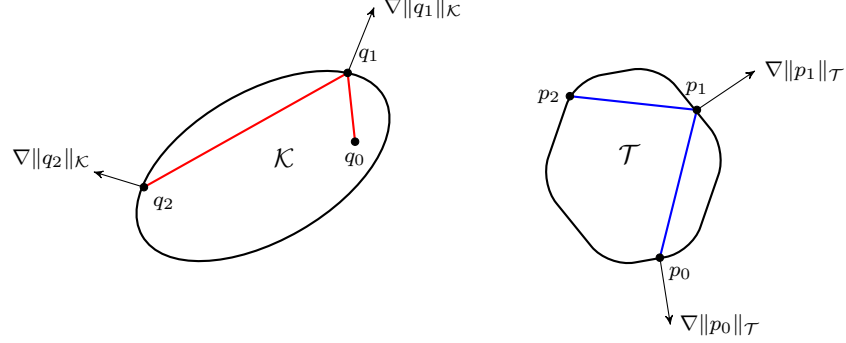
an approximation scheme which uses a certain “penalization method” developed by Benci and Giannoni in [10] (cf. [5, 48]). In what follows we present an alternative approach, and use characteristic foliation on singular convex hypersurfaces in \mathbb{R}^{2n} (see e.g., [21, 23, 50]) to describe Finsler type billiards for convex domains in the configuration space \mathbb{R}_q^n . The main advantage of this approach is that it allows one to use the natural one-to-one correspondence between the geodesic flow on a manifold and the characteristic foliation on its unit cotangent bundle, and thus provides a natural “symplectic setup” in which one can use tools such as Theorem 3.4 above in the context of billiard dynamics. In particular, we show that the Ekeland-Hofer-Zehnder capacity of certain Lagrangian product configurations $\mathcal{K} \times \mathcal{T}$ in the classical phase space \mathbb{R}^{2n} is the length of the shortest periodic \mathcal{T} -billiard trajectory in \mathcal{K} (see e.g., [7, 88]), which we turn now to describe.

The general study of billiard dynamics in Finsler and Minkowski geometries was initiated by Gutkin and Tabachnikov in [36]. From the point of view of geometric optics, Minkowski billiard trajectories describe the propagation of light in a homogeneous anisotropic medium that contains perfectly reflecting mirrors. Below, we focus on the special case of Minkowski billiards in a smooth convex body $\mathcal{K} \subset \mathbb{R}_q^n$. We equip \mathcal{K} with a metric given by a certain norm $\|\cdot\|$, and consider billiards in \mathcal{K} with respect to the geometry induced by $\|\cdot\|$. More precisely, let $\mathcal{K} \subset \mathbb{R}_q^n$, and $\mathcal{T} \subset \mathbb{R}_p^n$ be two convex bodies with smooth boundary, and consider the unit cotangent bundle

$$U_{\mathcal{T}}^* \mathcal{K} := \mathcal{K} \times \mathcal{T} = \{(q, p) \mid q \in \mathcal{K}, \text{ and } g_{\mathcal{T}}(p) \leq 1\} \subset T^* \mathbb{R}_q^n = \mathbb{R}_q^n \times \mathbb{R}_p^n.$$

Here $g_{\mathcal{T}}$ is the gauge function $g_{\mathcal{T}}(x) = \inf\{r \mid x \in r\mathcal{T}\}$. When $\mathcal{T} = -\mathcal{T}$ is centrally symmetric one has $g_{\mathcal{T}}(x) = \|x\|_{\mathcal{T}}$. For $p \in \partial\mathcal{T}$, the gradient vector $\nabla g_{\mathcal{T}}(p)$ is the outer normal to $\partial\mathcal{T}$ at the point p , and is naturally considered to be in $\mathbb{R}_q^n = (\mathbb{R}_p^n)^*$.

Motivated by the classical correspondence between geodesics in a Riemannian manifold and characteristics of its unit cotangent bundle, we define $(\mathcal{K}, \mathcal{T})$ -billiard trajectories to be characteristics in $U_{\mathcal{T}}^* \mathcal{K}$ such that their projections to \mathbb{R}_q^n are closed billiard trajectories in \mathcal{K} with a bouncing rule that is determined by the geometry induced from the body \mathcal{T} ; and vice versa, the projections to \mathbb{R}_p^n are closed billiard trajectories in \mathcal{T} with a bouncing rule that is determined by \mathcal{K} . More precisely, when we follow the vector fields of the dynamics, we move in $\mathcal{K} \times \partial\mathcal{T}$ from (q_0, p_0) to $(q_1, p_0) \in \partial\mathcal{K} \times \partial\mathcal{T}$ following the inner normal to $\partial\mathcal{T}$ at p_0 . When we hit the boundary $\partial\mathcal{K}$ at the point q_1 , the vector field changes, and we start to move in $\partial\mathcal{K} \times \mathcal{T}$ from (q_1, p_0) to $(q_1, p_1) \in \partial\mathcal{K} \times \partial\mathcal{T}$ following the outer normal to $\partial\mathcal{K}$ at the point q_1 . Next, we move from (q_1, p_1) to (q_2, p_1) following the opposite of the normal to $\partial\mathcal{T}$ at p_1 , and so on and so forth (see Figure 1). It is not hard to check that when one of the bodies, say \mathcal{T} , is a Euclidean ball, then when considering the projection to \mathbb{R}_q^n , the bouncing rule described above is the classical one (i.e., equal impact and reflection angles). Hence, the above reflection law is a natural variation of the classical one when the Euclidean structure on \mathbb{R}_q^n is replaced by the metric induced by the norm $\|\cdot\|_{\mathcal{T}}$. We continue with a more precise definition.

Figure 1. A proper $(\mathcal{K}, \mathcal{T})$ -Billiard trajectory.

Definition 3.5. Given two smooth convex bodies $\mathcal{K} \subset \mathbb{R}_q^n$ and $\mathcal{T} \subset \mathbb{R}_p^n$. A closed $(\mathcal{K}, \mathcal{T})$ -billiard trajectory is the image of a piecewise smooth map $\gamma: S^1 \rightarrow \partial(\mathcal{K} \times \mathcal{T})$ such that for every $t \notin \mathcal{B}_\gamma := \{t \in S^1 \mid \gamma(t) \in \partial\mathcal{K} \times \partial\mathcal{T}\}$ one has

$$\dot{\gamma}(t) = d\mathfrak{X}(\gamma(t)),$$

for some positive constant d and the vector field \mathfrak{X} given by

$$\mathfrak{X}(q, p) = \begin{cases} (-\nabla g_{\mathcal{T}}(p), 0), & (q, p) \in \text{int}(\mathcal{K}) \times \partial\mathcal{T}, \\ (0, \nabla g_{\mathcal{K}}(q)), & (q, p) \in \partial\mathcal{K} \times \text{int}(\mathcal{T}). \end{cases}$$

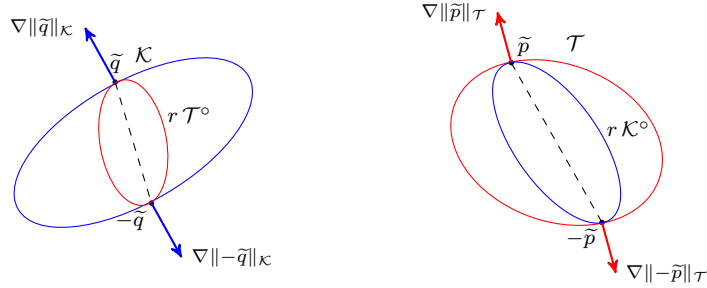
Moreover, for any $t \in \mathcal{B}_\gamma$, the left and right derivatives of $\gamma(t)$ exist, and

$$\dot{\gamma}^\pm(t) \in \{\alpha(-\nabla g_{\mathcal{T}}(p), 0) + \beta(0, \nabla g_{\mathcal{K}}(q)) \mid \alpha, \beta \geq 0, (\alpha, \beta) \neq (0, 0)\}.$$

Although in Definition 3.5 there is a natural symmetry between the bodies \mathcal{K} and \mathcal{T} , in what follows we shall assume that \mathcal{K} plays the role of the billiard table, while \mathcal{T} induces the geometry that governs the billiard dynamics in \mathcal{K} . We will use the following terminology: for a $(\mathcal{K}, \mathcal{T})$ -billiard trajectory γ , the curve $\pi_q(\gamma)$, where $\pi_q: \mathbb{R}^{2n} \rightarrow \mathbb{R}_q^n$ is the projection of γ to the configuration space, shall be called a \mathcal{T} -billiard trajectory in \mathcal{K} . Moreover, similarly to the Euclidean case, one can check that \mathcal{T} -billiard trajectories in \mathcal{K} correspond to critical points of a length functional defined on the j -fold cross product of the boundary $\partial\mathcal{K}$, where the distances between two consecutive points are measured with respect to the support function $h_{\mathcal{T}}$, where $h_{\mathcal{T}}(u) = \sup\{\langle x, u \rangle; x \in \mathcal{T}\}$.

Definition 3.6. A closed $(\mathcal{K}, \mathcal{T})$ -billiard trajectory γ is said to be *proper* if the set \mathcal{B}_γ is finite, i.e., γ is a broken bicharacteristic that enters and instantly exits the boundary $\partial\mathcal{K} \times \partial\mathcal{T}$ at the reflection points. In the case where $\mathcal{B}_\gamma = S^1$, i.e., γ is travelling solely along the boundary $\partial\mathcal{K} \times \partial\mathcal{T}$, we say that γ is a *gliding trajectory*.

The following theorem was proved in [7].

Figure 2. \mathcal{T} -billiard trajectory in \mathcal{K} of length $4 \operatorname{inrad}_{\mathcal{T}^\circ}(\mathcal{K})$.

Theorem 3.7. *Let $\mathcal{K} \subset \mathbb{R}_q^n$, $\mathcal{T} \subset \mathbb{R}_p^n$ be two smooth convex bodies. Then, every $(\mathcal{K}, \mathcal{T})$ -billiard trajectory is either a proper trajectory, or a gliding one. Moreover, the Ekeland-Hofer-Zehnder capacity $c_{\text{EHZ}}(\mathcal{K} \times \mathcal{T})$, of the Lagrangian product $\mathcal{K} \times \mathcal{T}$, is the length of the shortest periodic \mathcal{T} -billiard trajectory in \mathcal{K} , measured with respect to the support function $h_{\mathcal{T}}$.*

This theorem provides an effective way to estimate (and sometimes compute) the Ekeland-Hofer-Zehnder capacity of Lagrangian product configurations in the phase space. For example, in [8] (see Remark 4.2 therein) we used elementary tools from convex geometry to show that for centrally symmetric convex bodies, the shortest \mathcal{T} -billiard trajectory in \mathcal{K} is a 2-periodic trajectory connecting a tangency point q_0 of \mathcal{K} and a homotetic copy of \mathcal{T}° to $-q_0$ (see Figure 2). This result extends a previous result by Ghomi [31] for Euclidean billiards. In both cases, the main difficulty in the proof is to show that the above mentioned 2-periodic trajectory is indeed the shortest one. With this geometric observation at our disposal, we proved in [8] the following result: denote by $\operatorname{inrad}_{\mathcal{T}}(\mathcal{K}) = \max\{r \mid r\mathcal{T} \subset \mathcal{K}\}$.

Theorem 3.8. *If $\mathcal{K} \subset \mathbb{R}_q^n$, $\mathcal{T} \subset \mathbb{R}_p^n$ are centrally symmetric convex bodies, then*

$$c_{\text{EHZ}}(\mathcal{K} \times \mathcal{T}) = \bar{c}(\mathcal{K} \times \mathcal{T}) = 4 \operatorname{inrad}_{\mathcal{T}^\circ}(\mathcal{K})$$

Note that Theorem 3.8 immediately implies Theorem 3.3 above, which in turn implies Theorem 3.2. Thus, we have shown that Mahler's conjecture follows from a special case of Viterbo's conjecture. In fact, it follows immediately from the proof of Theorem 3.2 that Mahler's conjecture is equivalent to Viterbo's conjecture when the latter is restricted to the Ekeland-Hofer-Zehnder capacity, and to convex domains of the form $\Sigma \times \Sigma^\circ$, where $\Sigma \subset \mathbb{R}_q^n$ is a centrally symmetric convex body. We hope that further pursuing this line of research will lead to a breakthrough in understanding both conjectures.

3.1. Bounds on the length of the shortest billiard trajectory.

Going somehow in the opposite direction, one can also use the theory of symplectic capacities to provide several bounds and inequalities for the length of the shortest

periodic billiard trajectory in a smooth convex body in \mathbb{R}^n . In [7] we prove the following theorem, which for the sake of simplicity we state only for the case of Euclidean billiards (for several other related results see [3, 5, 11, 31, 47, 48, 88]).

Theorem 3.9. *Let $K \subset \mathbb{R}^n$ be a smooth convex body, and let $\xi(K)$ denote the length of the shortest periodic billiard trajectory in K . Then,*

- (i) $\xi(K_1) \leq \xi(K_2)$, for any convex domains $K_1 \subseteq K_2 \subseteq \mathbb{R}^n$ (monotonicity);
- (ii) $\xi(K) \leq C\sqrt{n} \text{Vol}(K)^{\frac{1}{n}}$, for some universal constant $C > 0$;
- (iii) $4\text{inrad}(K) \leq \xi(K) \leq 2(n+1)\text{inrad}(K)$;
- (iv) $\xi(K_1 + K_2) \geq \xi(K_1) + \xi(K_2)$ (Brunn-Minkowski type inequality).

We remark that the inequality $4\text{inrad}(K) \leq \xi(K)$ in (iii) above was proved already in [31], the monotonicity property was well known to experts in the field (although it has not been addressed in the literature to the best of our knowledge), and all the results in Theorem 3.9 were later recovered and generalized by different methods (see [3, 47, 48]). Moreover, in light of the “classical versus quantum” relation between the length spectrum in Riemannian geometry and the Laplace spectrum, via trace formulae and Poisson relations, Theorem 3.9 can be viewed as a classical counterpart of some well-known results for the first Laplace eigenvalue on convex domains. It is interesting to note that, to the best of the author’s knowledge, the exact value of the constant C in part (ii) of Theorem 3.9 is unknown already in the two-dimensional case.

4. The Uniqueness of Hofer’s Metric

One of the most striking facts regarding the group of Hamiltonian diffeomorphisms associated with a symplectic manifold is that it can be equipped with an intrinsic geometry given by a bi-invariant Finsler metric known as Hofer’s metric [40]. In contrast to the case of finite-dimensional Lie groups, the existence of such a metric on an infinite-dimensional group of transformations is highly unusual due to the lack of local compactness. Hofer’s metric is exceptionally important for at least two reasons: first, Hofer showed in [40] that this metric gives rise to an important symplectic capacity known as “displacement energy”, which turns out to have many different applications in symplectic topology and Hamiltonian dynamics (see e.g., [18, 40, 43, 52, 53, 74, 75]). Second, it provides a certain geometric intuition for the understanding of the long-time behaviour of Hamiltonian dynamical systems.

In [26], Eliashberg and Polterovich initiated a discussion on the uniqueness of Hofer’s metric (cf. [25, 75]). They asked whether for a closed symplectic manifold (M, ω) , Hofer’s metric is the only bi-invariant Finsler metric on the group of Hamiltonian diffeomorphisms. In this section we explain (following [72] and [17]) how tools from classical functional analysis and the theory of normed function spaces can be used to positively answer this question, and show that up to equivalence of

metrics, Hofer's metric is unique. For this purpose, we now turn to more precise formulations.

Let (M, ω) be a closed $2n$ -dimensional symplectic manifold, and denote by $C_0^\infty(M)$ the space of smooth functions that are zero-mean normalized with respect to the canonical volume form ω^n . With every smooth time-dependent Hamiltonian function $H : M \times [0, 1] \rightarrow \mathbb{R}$, one associates a vector field X_{H_t} via the equation $i_{X_{H_t}}\omega = -dH_t$, where $H_t(x) = H(t, x)$. The flow of X_{H_t} is denoted by ϕ_H^t and is defined for all $t \in [0, 1]$. The group of Hamiltonian diffeomorphisms consists of all the time-one maps of such Hamiltonian flows, i.e.,

$$\text{Ham}(M, \omega) = \{\phi_H^1 \mid \phi_H^t \text{ is a Hamiltonian flow}\}.$$

When equipped with the standard C^∞ -topology, the group $\text{Ham}(M, \omega)$ is an infinite-dimensional Fréchet Lie group. Its Lie algebra, denoted here by \mathcal{A} , can be naturally identified with the space of normalized smooth functions $C_0^\infty(M)$. Moreover, the adjoint action of $\text{Ham}(M, \omega)$ on \mathcal{A} is the standard action of diffeomorphisms on functions, i.e., $\text{Ad}_\phi f = f \circ \phi^{-1}$, for every $f \in \mathcal{A}$ and $\phi \in \text{Ham}(M, \omega)$. For more details on the group of Hamiltonian diffeomorphisms see e.g., [43, 62, 75].

Next, we define a Finsler pseudo-distance on $\text{Ham}(M, \omega)$. Given any pseudo-norm $\|\cdot\|$ on \mathcal{A} , we define the length of a path $\alpha : [0, 1] \rightarrow \text{Ham}(M, \omega)$ as

$$\text{length}\{\alpha\} = \int_0^1 \|\dot{\alpha}\| dt = \int_0^1 \|H_t\| dt,$$

where $H_t(x) = H(t, x)$ is the unique normalized Hamiltonian function generating the path α . Here H is said to be normalized if $\int_M H_t \omega^n = 0$ for every $t \in [0, 1]$. The distance between two Hamiltonian diffeomorphisms is given by

$$d(\psi, \varphi) := \inf \text{length}\{\alpha\},$$

where the infimum is taken over all Hamiltonian paths α connecting ψ and φ . It is not hard to check that d is non-negative, symmetric, and satisfies the triangle inequality. Moreover, any pseudo-norm on the Lie algebra \mathcal{A} that is invariant under the adjoint action yields a bi-invariant pseudo-distance function on $\text{Ham}(M, \omega)$, i.e., $d(\psi, \phi) = d(\theta\psi, \theta\phi) = d(\psi\theta, \phi\theta)$, for every $\psi, \phi, \theta \in \text{Ham}(M, \omega)$.

From here forth we deal solely with such pseudo-norms and we refer to d as the pseudo-distance generated by the pseudo-norm $\|\cdot\|$.

We remark in passing that a fruitful study of right-invariant Finsler metrics on $\text{Ham}(M, \omega)$, motivated in part by applications to hydrodynamics, was initiated by Arnold [4]. In addition, non-Finslerian bi-invariant metrics on $\text{Ham}(M, \omega)$ have been intensively studied in the realm of symplectic geometry, starting with the works of Viterbo [89], Schwarz [84], and Oh [70], and followed by many others.

Remark 4.1. When one studies geometric properties of the group of Hamiltonian diffeomorphisms, it is convenient to consider smooth paths $[0, 1] \rightarrow \text{Ham}(M, \omega)$, among which those that start at the identity correspond to smooth Hamiltonian

flows. Moreover, for a given Finsler pseudo-metric on $\text{Ham}(M, \omega)$, a natural geometric assumption is that every smooth path $[0, 1] \rightarrow \text{Ham}(M, \omega)$ has finite length. As it turns out, the latter assumption is equivalent to the continuity of the pseudo-norm on \mathcal{A} corresponding to the pseudo-Finsler metric in the C^∞ -topology (see [17]). Thus, in what follows we shall mainly consider such pseudo-norms.

It is highly non-trivial to check whether a distance function on the group of Hamiltonian diffeomorphisms generated by a pseudo-norm is non-degenerate, that is, $d(\text{Id}, \phi) > 0$ for $\phi \neq \text{Id}$. In fact, for closed symplectic manifolds, a bi-invariant pseudo-metric d on $\text{Ham}(M, \omega)$ is either a genuine metric or identically zero. This is an immediate corollary of a well-known theorem by Banyaga [9], which states that $\text{Ham}(M, \omega)$ is a simple group, combined with the fact that the null-set

$$\text{null}(d) = \{\phi \in \text{Ham}(M, \omega) \mid d(\text{Id}, \phi) = 0\}$$

is a normal subgroup of $\text{Ham}(M, \omega)$. A renowned result by Hofer [40] states that the L_∞ -norm on \mathcal{A} gives rise to a genuine distance function on $\text{Ham}(M, \omega)$ known now as Hofer's metric. This was proved by Hofer for the case of \mathbb{R}^{2n} , then generalized by Polterovich [74], and finally proven in full generality by Lalonde and McDuff [53]. In a sharp contrast to the above, Eliashberg and Polterovich showed in [26] that for a closed symplectic manifold (M, ω) one has

Theorem 4.2 (Eliashberg and Polterovich). *For $1 \leq p < \infty$, the pseudo-distances on $\text{Ham}(M, \omega)$ corresponding to the L_p -norms on \mathcal{A} vanish identically.*

The following question was asked in [26] (cf. [25, 75]):

Question 4.3. *What are the $\text{Ham}(M, \omega)$ -invariant norms on \mathcal{A} , and which of them give rise to genuine bi-invariant metrics on $\text{Ham}(M, \omega)$?*

It was observed in [17] that any pseudo-norm $\|\cdot\|$ on the space \mathcal{A} can be turned into a $\text{Ham}(M, \omega)$ -invariant pseudo-norm via a certain invariantization procedure $\|f\| \mapsto \|f\|_{\text{inv}}$. The idea behind this procedure is based on the notion of infimal convolution (or epi-sum), from convex analysis. Recall that the infimal convolution of two functions f and g on \mathbb{R}^n is defined by $(f \square g)(z) = \inf\{f(x) + g(y) \mid z = x + y\}$. This operator has a simple geometric interpretation: the epigraph (i.e., the set of points lying on or above the graph) of the infimal convolution of two functions is the Minkowski sum of the epigraphs of those functions. The invariantization $\|\cdot\|_{\text{inv}}$ of $\|\cdot\|$ is obtained by taking the orbit of $\|\cdot\|$ under the group action, and consider the infimal convolution of the associated family of norms. More precisely, define

$$\|f\|_{\text{inv}} = \inf \left\{ \sum \|\phi_i^* f_i\| \mid f = \sum f_i, \text{ and } \phi_i \in \text{Ham}(M, \omega) \right\}.$$

We remark that in the above definition of $\|f\|_{\text{inv}}$ the sum $\sum f_i$ is assumed to be finite. Note that $\|\cdot\|_{\text{inv}} \leq \|\cdot\|$. Thus, if $\|\cdot\|$ is continuous in the C^∞ -topology, then so is $\|\cdot\|_{\text{inv}}$. Moreover, if $\|\cdot\|'$ is a $\text{Ham}(M, \omega)$ -invariant pseudo-norm, then:

$$\|\cdot\|' \leq \|\cdot\| \implies \|\cdot\|' \leq \|\cdot\|_{\text{inv}}.$$

In particular, the above invariantization procedure provides a plethora of $\text{Ham}(M, \omega)$ -invariant genuine norms on \mathcal{A} , e.g., by applying it to the $\|\cdot\|_{C^k}$ -norms.

In [72] we made a first step toward answering Question 4.3 using tools from the theory of normed spaces and functional analysis. More precisely, regarding the first part of Question 4.3, we proved

Theorem 4.4 (Ostrover and Wagner). *Let $\|\cdot\|$ be a $\text{Ham}(M, \omega)$ -invariant norm on \mathcal{A} such that $\|\cdot\| \leq C\|\cdot\|_\infty$ for some constant C . Then $\|\cdot\|$ is invariant under all measure preserving diffeomorphisms of M .*

In other words, any $\text{Ham}(M, \omega)$ -invariant norm on \mathcal{A} that is bounded above by the L_∞ -norm, must also be invariant under the much larger group of measure preserving diffeomorphisms. Theorem 4.4 plays an important role in the proof of the following result, which gives a partial answer to the second part of Question 4.3.

Theorem 4.5 (Ostrover and Wagner). *Let $\|\cdot\|$ be a $\text{Ham}(M, \omega)$ -invariant norm on \mathcal{A} such that $\|\cdot\| \leq C\|\cdot\|_\infty$ for some constant C , but the two norms are not equivalent.¹ Then the associated pseudo-distance d on $\text{Ham}(M, \omega)$ vanishes identically.*

Although Theorem 4.5 gives a partial answer to the second part of Question 4.3, *prima facie*, there might be $\text{Ham}(M, \omega)$ -invariant norms on \mathcal{A} which are either strictly bigger than the L_∞ -norm, or incomparable to it. In a joint work with L. Buhovsky [17] we showed that under the natural continuity assumption mentioned in Remark 4.1 above, this cannot happen. Hence, up to equivalence of metrics, Hofer's metric is unique. More precisely,

Theorem 4.6 (Buhovsky and Ostrover). *Let (M, ω) be a closed symplectic manifold. Any C^∞ -continuous $\text{Ham}(M, \omega)$ -invariant pseudo-norm $\|\cdot\|$ on \mathcal{A} is dominated from above by the L_∞ -norm i.e., $\|\cdot\| \leq C\|\cdot\|_\infty$ for some constant C .*

Combining Theorem 4.6 and Theorem 4.5 above, we obtain:

Corollary 4.7. *For a closed symplectic manifold (M, ω) , any bi-invariant Finsler pseudo-metric on $\text{Ham}(M, \omega)$, obtained by a pseudo-norm $\|\cdot\|$ on \mathcal{A} that is continuous in the C^∞ -topology, is either identically zero, or equivalent to Hofer's metric. In particular, any non-degenerate bi-invariant Finsler metric on $\text{Ham}(M, \omega)$ which is generated by a norm that is continuous in the C^∞ -topology gives rise to the same topology on $\text{Ham}(M, \omega)$ as the one induced by Hofer's metric.*

In the rest of this section we briefly describe the strategy of the proof of Theorem 4.6 in the two-dimensional case. For the proof of the general case see [17]. We start with two straightforward reduction steps. First, for technical reasons, we shall consider pseudo-norms on the space $C^\infty(M)$, instead of the space \mathcal{A} . The original claim will follow, since any $\text{Ham}(M, \omega)$ invariant pseudo-norm $\|\cdot\|$ on \mathcal{A} can be naturally extended to an invariant pseudo-norm $\|\cdot\|'$ on $C^\infty(M)$ by

$$\|f\|' := \|f - M_f\|, \text{ where } M_f = \frac{1}{\text{Vol}(M)} \int_M f \omega^n.$$

¹Two norms are said to be equivalent if $\frac{1}{C} \|\cdot\|_1 \leq \|\cdot\|_2 \leq C \|\cdot\|_1$ for some constant $C > 0$.

Note that if $\|\cdot\|$ is continuous in the C^∞ -topology, then so is $\|\cdot\|'$, and that the two norms coincide on the space \mathcal{A} . Second, by using a standard partition of unity argument, we can reduce the proof of Theorem 4.6 to a “local result”, i.e., it is sufficient to prove the theorem for $\text{Ham}_c(W, \omega)$ -invariant pseudo-norms on the space of compactly supported smooth functions $C_c^\infty(W)$, where $W = (-L, L)^2$ is an open square in \mathbb{R}^2 (see [17] for the details).

The next step, which is one of the key ideas of the proof, is to define the “largest possible” $\text{Ham}_c(W, \omega)$ -invariant norm on the space of compactly supported smooth functions $C_c^\infty(W)$. To this end, we fix a (non-empty) finite collection of functions $\mathcal{F} \subset C_c^\infty(W)$, and define:

$$\mathcal{L}_{\mathcal{F}} := \left\{ \sum_{i,k} c_{i,k} \Phi_{i,k}^* f_i \mid c_{i,k} \in \mathbb{R}, \Phi_{i,k} \in \text{Ham}_c(W, \omega), \right. \\ \left. f_i \in \mathcal{F}, \text{ and } \#\{(i,k) \mid c_{i,k} \neq 0\} < \infty \right\}.$$

We equip the space $\mathcal{L}_{\mathcal{F}}$ with the norm

$$\|f\|_{\mathcal{L}_{\mathcal{F}}} = \inf \sum |c_{i,k}|,$$

where the infimum is taken over all the representations $f = \sum c_{i,k} \Phi_{i,k}^* f_i$ as above.

Definition 4.8. For any compactly supported function $f \in C_c^\infty(W)$, let

$$\|f\|_{\mathcal{F}, \max} = \inf \left\{ \liminf_{i \rightarrow \infty} \|f_i\|_{\mathcal{L}_{\mathcal{F}}} \right\},$$

where the infimum is taken over all subsequences $\{f_i\}$ in $\mathcal{L}_{\mathcal{F}}$ which converge to f in the C^∞ -topology. As usual, the infimum of the empty set is set to be $+\infty$.

The main feature of the norm $\|\cdot\|_{\mathcal{F}, \max}$ is that it dominates from above any other $\text{Ham}_c(W, \omega)$ -invariant pseudo-norm that is continuous in the C^∞ -topology.

Lemma 4.9. *Let $\mathcal{F} \subset C_c^\infty(W)$ be a non-empty finite collection of smooth compactly supported functions in W . Then any $\text{Ham}_c(W, \omega)$ -invariant pseudo-norm $\|\cdot\|$ on $C_c^\infty(W)$ that is continuous in the C^∞ -topology satisfies*

$$\|\cdot\| \leq C \|\cdot\|_{\mathcal{F}, \max},$$

for some absolute constant C .

Proof of Lemma 4.9. Since the collection \mathcal{F} is finite, set $C = \max\{\|g\|; g \in \mathcal{F}\}$. For any $f = \sum c_{i,k} \Phi_{i,k}^* f_i \in \mathcal{L}_{\mathcal{F}}$, one has

$$\|f\| \leq \sum |c_{i,k}| \|\Phi_{i,k}^* f_i\| \leq C \sum |c_{i,k}|. \quad (1)$$

By the definition of $\|\cdot\|_{\mathcal{L}_{\mathcal{F}}}$, this immediately implies that $\|f\| \leq C \|f\|_{\mathcal{L}_{\mathcal{F}}}$. The lemma now follows by combining (1), the definition of $\|\cdot\|_{\mathcal{F}, \max}$, and the fact that the pseudo-norm $\|\cdot\|$ is assumed to be continuous in the C^∞ -topology. \square

The next step, which is the main part of the proof, is to show that for a suitable collection of functions $\mathcal{F} \subset C_c^\infty(W)$, the norm $\|\cdot\|_{\mathcal{F}, \max}$ is in turn bounded from above by the L_∞ -norm. In light of the above, this would complete the proof of Theorem 4.6 in the two-dimensional case.

There are two independent components in the proof of this claim. First, we show that one can decompose any $f \in C_c^\infty(W^2)$ with $\|f\|_\infty \leq 1$ into a finite combination $f = \sum_{j=1}^{N_0} \epsilon_j \Psi_j^* g_j$. Here, $\epsilon_j \in \{-1, 1\}$, $\Psi_j \in \text{Ham}_c(W^2, \omega)$, and g_j are smooth rotation-invariant functions whose L_∞ -norm is bounded by an absolute constant, and which satisfy certain other technical conditions (see Proposition 3.5 in [17] for the precise statement). In what follows we call such functions “simple functions”. We emphasize that N_0 is a constant independent of f . Thus, we can restrict ourselves to the case where f is a “simple function”. In the second part of the proof, we construct an explicit collection $\mathcal{F} = \{f_0, f_1, f_2\}$, where $f_i \in C_c^\infty(W^2)$, and $i = 0, 1, 2$. Using an averaging procedure (see the proof of Theorem 3.4 in [17]), one can show that every “simple function” $f \in C_c^\infty(W^2)$ can be approximated arbitrarily well in the C^∞ -topology by a sum of the form

$$\sum_{i,k} \alpha_{i,k} \tilde{\Psi}_{i,k}^* f_k, \text{ where } \tilde{\Psi}_{i,k} \in \text{Ham}_c(W^2, \omega), \ k \in \{0, 1, 2\},$$

and such that $\sum |\alpha_{i,k}| \leq C \|f\|_\infty$ for some absolute constant C . Combining this with the above definition of $\|\cdot\|_{\mathcal{F}, \max}$, we conclude that $\|f\|_{\mathcal{F}, \max} \leq C \|f\|_\infty$ for every $f \in C_c^\infty(W^2)$. Together with Lemma 4.9, this completes the proof of Theorem 4.6 in the 2-dimensional case.

5. Some Open Questions and Speculations

Do symplectic capacities coincide on the class of convex domains? As mentioned above, since the time of Gromov’s original work, a variety of symplectic capacities have been constructed and the relations between them often lead to the discovery of surprising connections between symplectic geometry and Hamiltonian dynamics. In the two-dimensional case, Siburg [85] showed that any symplectic capacity of a compact connected domain with smooth boundary $\Omega \subset \mathbb{R}^2$ equals its Lebesgue measure. In higher dimensions symplectic capacities do not coincide in general. A theorem by Hermann [37] states that for any $n \geq 2$ there is a bounded star-shaped domain $S \subset \mathbb{R}^{2n}$ with cylindrical capacity $\bar{c}(S) \geq 1$, and arbitrarily small Gromov radius $\underline{c}(S)$. Still, for large classes of sets in \mathbb{R}^{2n} , including ellipsoids, polydiscs and convex Reinhardt domains, all symplectic capacities coincide [37]. In [88] Viterbo showed that for any bounded convex subset Σ of \mathbb{R}^{2n} one has $\bar{c}(\Sigma) \leq 4n^2 \underline{c}(\Sigma)$. Moreover, one has (see [37, 41, 88]) the following:

Conjecture 5.1. *For any convex domain Σ in \mathbb{R}^{2n} one has $\underline{c}(\Sigma) = \bar{c}(\Sigma)$.*

This conjecture is particularly challenging due to the scarcity of examples of convex domains in which capacities have been computed. Moreover, note that

Conjecture 5.1 is stronger than Viterbo's conjecture (Conjecture 2.1 above), as the latter holds trivially for the Gromov radius.

A somewhat more modest question in this direction is whether Conjecture 5.1 holds asymptotically, i.e., whether there is an absolute constant A such that for any convex domain $K \subset \mathbb{R}^{2n}$ one has $\bar{c}(K) \leq A \underline{c}(K)$. It would be interesting to explore whether methods from asymptotic geometric analysis can be used to answer this question.

Are Hanner polytopes in fact symplectic balls in disguise? Recall that Mahler's conjecture states that the minimum possible Mahler volume is attained by a hypercube. It is interesting to note that the corresponding product configuration, when looked at through symplectic glasses, is in fact a Euclidean ball in disguise. More precisely, it was proved in §7.9 of [82] (cf. Corollary 4.2 in [56]) that the interior of the product of a hypercube $Q \subset \mathbb{R}_q^n$ and its dual body, the cross-polytope $Q^\circ \subset \mathbb{R}_p^n$, is symplectomorphic to the interior of a Euclidean ball $B^{2n}(r) \subset \mathbb{R}_q^n \times \mathbb{R}_p^n$ with the same volume. On the other hand, as mentioned in Section 3 above, if Mahler's conjecture holds, then there are other minimizers for the Mahler volume aside of the hypercube. For example, consider the class of Hanner polytopes. A d -dimensional centrally symmetric polytope P is a Hanner polytope if either P is one-dimensional (i.e., a symmetric interval), or P is the free sum or direct product of two (lower dimensional) Hanner polytopes P_1 and P_2 . Recall that the free sum of two polytopes, $P_1 \subset \mathbb{R}^n$, $P_2 \subset \mathbb{R}^m$ is a $n + m$ polytope defined by $P_1 \oplus P_2 = \text{Conv}(\{P_1 \times \{0\}\} \cup \{\{0\} \times P_2\}) \subset \mathbb{R}^{n+m}$. It is not hard to check (see e.g. [80]) that the volume product of the cube is the same as that of Hanner polytopes. Thus every Hanner polytope is also a candidate for a minimizer of the volume product among symmetric convex bodies. In light of the above mentioned result from [82], a natural question is the following:

Question 5.2. *Is every Hanner polytope a symplectic image of a Euclidean ball?*

More precisely, is the interior of every Hanner polytope symplectomorphic to the interior of a Euclidean ball with the same volume? A negative answer to this question would give a counterexample to Conjecture 5.1 above, since it would show that the Gromov radius must be different from the Ekeland-Hofer-Zehnder capacity.

Symplectic embeddings of Lagrangian products. Since Gromov's work [34], questions about symplectic embeddings have lain at the heart of symplectic geometry (see e.g., [12, 13, 35, 45, 56, 60, 61, 63, 82, 83]). These questions are usually notoriously difficult, and up to date most results concern only the embeddings of balls, ellipsoids and polydiscs. Note that even for this simple class of examples, only recently has it become possible to specify exactly when a four-dimensional ellipsoid is embeddable in a ball (McDuff and Schlenk [63]), or in another four-dimensional ellipsoid (McDuff [60]). For some other related works we refer the reader to [15, 19, 22, 30, 38, 39, 71].

Since symplectic capacities can naturally be used to detect symplectic embedding obstructions, and in light of the results mentioned in Section 3 (in particular,

Theorem 3.8), it is only natural to try to extend the above list of currently-known examples, and study symplectic embeddings of convex “Lagrangian products” in the classical phase space. The main advantage of this class of bodies is that the action spectrum can be computed via billiard dynamics. This property would presumably make it easier to compute or estimate the Ekeland-Hofer capacities [24], or Hutchings’ embedded contact homology capacities [44, 45], in this setting. A natural first step in this direction would be to consider the embedding of the Lagrangian product of two balls into a Euclidean ball. More precisely, to study the function $\sigma : \mathbb{N} \rightarrow \mathbb{R}$ defined by

$$\sigma(n) = \inf\{a \mid B_q^n(1) \times B_p^n(1) \xrightarrow{\text{symp}} B^{2n}(a)\}.$$

To the best of the author’s knowledge, the value of $\sigma(n)$ is unknown already for the case $n = 2$.

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