The Existence of a Billiard Orbit in the Regular Hyperbolic Simplex

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Abstract

In this note we establish the existence of a $n+1$ periodic billiard trajectory inside an n-dimensional regular simplex in the hyperbolic space, which hits the interior of every facet exactly once.

1 Introduction and Result

The billiard dynamical system describes the motion of a massless particle in a domain with a perfectly reflecting boundary (see e.g. [12, 17] for two excellent surveys on the subject). A particular intriguing class of examples, whose dynamics is in general neither integrable nor chaotic, is the class of polygonal billiards. On the one hand, this class serves as a promising model for quantum chaos [4], and on the other it is closely related to geodesic flows on flat surfaces and Teichmüller dynamics [14].

In 1775, J.F. de Tuschis a Fagnano observed that in every acute triangle in the Euclidean plane, the orthic triangle, whose vertices are the feet of the altitudes, represents a periodic billiard trajectory. Nevertheless, the existence of periodic billiard orbits for polygonal billiards turns out to be a challenging question, even for seemingly simple examples like obtuse triangles in the plane (see [10, 11]). A substantial progress in this direction has recently been made by R. Schwartz. In [16] he proved, using computer assisted techniques, that every obtuse triangle with angles not exceeding 100° has a periodic billiard trajectory. Another case where periodic trajectories are known to exist is the case of rational triangles and polygons (see e.g., [5, 13, 14]). We remark that in higher dimensions nearly nothing is known about the existence of periodic orbits.

Recently, using barycenter coordinates, the existence of a Fagnano periodic billiard trajectory inside the regular simplex in the Euclidean space $\mathbb{E}^n$ was established in [3]. Despite the lack of linear structure, in this note we extend the result of [3] to the hyperbolic space $\mathbb{H}^n$. More precisely, we consider hyperbolic regular simplices i.e., the convex hulls of $n+1$ points (vertices) in the hyperbolic space for which all the distances between two distinct vertices are equal. The billiards dynamics inside a hyperbolic simplex is defined in much the same way as in the Euclidean case: the particle moves along geodesic arcs within the simplex, interrupted by elastic collisions against the boundary where the motion undergoes a specular reflection (see Section 2 below for the precise definition).

Our main result in this note is the following:

Theorem 1.1. *Let $\triangle^n$ be a regular n-simplex with a given edge length in the hyperbolic space $\mathbb{H}^n$. Then, there exists an $(n+1)$-periodic billiard trajectory inside $\triangle^n$ which hits the interior of every facet exactly once.*

The proof of Theorem 1.1 follows the steps of [3], where the main new input being the approach by which we overcome several difficulties arising from the lack on linear structure in the hyperbolic space. Moreover, the same approach can be used to obtain the hyperbolic analog of the $2n$-periodic orbit constructed in [3]. The details are spelled out in [2].

Remark I: In the case where $n = 2$, the regular simplex is an equilateral triangle in the hyperbolic plane and the billiard orbit provided by the theorem above coincides with the orthic triangle i.e., the triangle whose vertices are the endpoints of the altitudes, which is the well known Fagnano billiard trajectory. In contrast with the two-dimensional case, for $n > 2$, a direct computation shows that as in the Euclidean case, the trajectory connecting the midpoint of the facets, which in the regular simplex coincides with the trajectory connecting the endpoints of the altitudes, fails to form a billiard orbit. We do not know the precise geometric (or physical) meaning of the bouncing points of the billiard orbit provided by the theorem above.

Remark II: Another natural questions regarding the above mentioned periodic orbits is the question of stability (see e.g. [10]). It is not hard to check that for $n = 2$ the orbit provided by Theorem 1.1 is stable under small perturbations. Though we did not check the details for larger n, we believe that this orbit is stable for any n.

Remark III: On top of the theoretical mathematical interest in studying billiard dynamics in the framework of hyperbolic geometry, it also has several implications to physics. As an example, we mention the remarkable connection between certain polyhedral billiards in the hyperbolic space and solutions to the vacuum Einstein equations in the vicinity of a space-like singularity, which was uncovered in a series of works starting with the pioneering papers of Belinskii, Khalatnikov and Lifshitz (see e.g., [6, 7, 8] and the references therein).

Structure of the paper: In Section 2 we recall some relevant facts from hyperbolic geometry, and introduce some of the technical ingredients needed later in the proof of Theorem 1.1, which in turn is given in Section 3.

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2 Background from Hyperbolic Geometry

In this section we first recall some relevant notions and facts from hyperbolic geometry. For a detailed exposition of the subject, see e.g, the books [1]. Then, in Subsection 2.2, we provide the main ingredients in the proof of Theorem 1.1 above.

The n-dimensional hyperbolic space $\mathbb{H}^n$ is the unique simply connected and complete n-dimensional Riemannian manifold of constant curvature -1. In what follows we shall denote by d the corresponding hyperbolic metric. Among the several models for the hyperbolic space, one can consider the half-space conformal model (also denoted by $\mathbb{H}^n$ to simplify notation) given by the metric space

$$\mathbb{H}^n = \left(\mathbb{E}^n_+, \ ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_1^2} \right),$$

where $\mathbb{E}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{E}^n \,|\, x_1 > 0\}$ is the upper half space of the Euclidean space $\mathbb{E}^n$.

The compactification $\overline{\mathbb{H}}^n = \mathbb{H}^n \cup \partial\mathbb{H}^n$ consists of $\mathbb{H}^n$ together with the set $\partial\mathbb{H}^n = \mathbb{E}^{n-1} \cup \{\infty\}$ of its points at infinity. It is well known that any two points $A, B \in \mathbb{H}^n$ can be joined by a unique geodesic (h-line), we shall denote the geodesics segment connecting them by $[AB]$. A hyperplane in $\mathbb{H}^n$ is a codimension-one totally geodesic subspace of $\mathbb{H}^n$, which divides the hyperbolic space into two half-spaces (see e.g., [1], Chapter 1, §3). Note that any hyperplane is isometric to $\mathbb{H}^{n-1}$. For example, in the half space model mentioned above, the geodesic hyperplanes are $(n-1)$-spheres and $(n-1)$-planes orthogonal to $\partial\mathbb{H}^n$. As in the Euclidean case, a reflection in the hyperbolic space $\mathbb{H}^n$ with respect to a given hyperplane is an isometric involution fixing point-wise the hyperplane and isometrically interchanging the two half-spaces of $\mathbb{H}^n$ associated with it. It is well known that there is a unique reflection in every hyperplane in $\mathbb{H}^n$.

Finally, a set $X \subset \mathbb{H}^n$ is said to be *convex* if for any two points $A, B \in X$ it contains the segment $[AB]$. The convex hull of a set $Y \subset \mathbb{H}^n$ is the intersection of all the convex sets in $\mathbb{H}^n$ containing Y.

2.1 Hyperbolic center of mass

Following [9], we define the notion of center of mass in hyperbolic space. A point mass is an ordered pair (X, x), where its location $X \in \mathbb{H}^n$ is a point of the hyperbolic space and its weight x is a non-negative real number.

Definition 2.1 ([9]). *Given any two point-masses (X, x) and (Y, y), their center of mass, or centroid, $(X, x) * (Y, y)$ is the point mass (Z, z), such that Z is the unique point that lies on the segment $[XY]$ and satisfies*

$$x \sinh d(X, Z) = y \sinh d(Y, Z).$$

Its corresponding mass is given by

$$z = x \cosh d\,(X, Z) + y \cosh d\,(Y, Z).$$

As shown in [9], the operator $*$ is well defined, commutative and associative. This allows, in particular, to define the *centroid* of a finite set of point masses. Moreover, it follows immediately from Definition 2.1 that

$$(X, w_1) *(X, w_2) = (X, w_1 + w_2), \quad (X, w) *(Y, 0) = (X, w),$$

and that for every non-negative real number λ one has,

$$(X, x) *(Y, y) = (Z, z) \Leftrightarrow (X, \lambda x) *(Y, \lambda y) = (Z, \lambda z).$$

Furthermore, since the center of mass is defined solely by means of geodesics and distances along them, it commutes with isometries. More precisely, for every isometry σ of the hyperbolic space $\mathbb{H}^n$ one has

$$(X, x) *(Y, y) = (Z, z) \Leftrightarrow (\sigma X, x) *(\sigma Y, y) = (\sigma Z, z).$$

2.2 Regular simplices in $\mathbb{H}^n$

In this subsection we introduce some facts regarding regular simplices in the hyperbolic space. In particular, we compute the centroid of the regular simplex (with unit mass on the vertices), and prove Proposition 2.3, which plays a key role in the proof of Theorem 1.1. We start with the following:

Definition 2.2. *An n-simplex $\triangle^n$ in $\mathbb{H}^n$ is the convex hull of $n + 1$ points in $\mathbb{H}^n$, called vertices. It is said to be regular if every permutation of its vertices is induced by an isometry of $\mathbb{H}^n$.*

It is well known (see e.g. [1], Chapter 6, §2), that up to isometries of $\mathbb{H}^n$, for every $a \in \mathbb{R}_+$ there is a unique hyperbolic regular n-simplex in $\mathbb{H}^n$ with edge length a. In what follows we shall denoted this simplex by $\triangle^n_a$.

Definition 2.3. *A point $C \in \mathbb{H}^n$ that is equidistant from all the vertices of the simplex $\triangle_a^n$ is called a midpoint of $\triangle_a^n$.*

It is not hard to check that for any $n \in \mathbb{N}$ and $a > 0$, there is a unique midpoint C_a^n of $\triangle_a^n$.

Definition 2.4. *Let $\triangle^n \subset \mathbb{H}^n$ be a regular hyperbolic n-simplex with vertices $\{V_0, \ldots, V_n\}$. For every $0 \le j \le n$, the j-facet of $\triangle^n$, denoted by F_j, is the regular $(n-1)$-simplex given by the convex hull of the vertices $\{V_k\}$, where $0 \le k \le n$, and $k \ne j$. In what follows we shall denote by σ_j the reflection in $\mathbb{H}^n$ with respect to the (unique) hyperplane in which F_j lies.*

In the following proposition we gather some basic properties of the regular simplex in the hyperbolic space (see Figure 1 below).

Proposition 2.1. *Let $\triangle_a^n$ be a hyperbolic regular n-simplex with edge length $a \in \mathbb{R}^+$. Let $\{V_0, \ldots, V_n\}$ be its vertices, C_n its midpoint, and C_{n-1} the midpoint of the facet F_n defined by $\{V_0, \ldots, V_{n-1}\}$. Then for any $n \ge 1$,*

(i) $C_n \in [C_{n-1}V_n]$,

(ii) $\angle C_n C_{n-1} V_j = \angle V_n C_{n-1} V_j = \frac{\pi}{2}$, for every $0 \le j \le n-1$ and $n > 1$,

(iii) $\cosh^2 d(V_j, C_n) = \frac{n \cosh a + 1}{n+1}$, for every $0 \le j \le n$,

(iv) $\cosh^2 d(V_n, C_{n-1}) = \frac{n \cosh^2 a}{(n-1)\cosh a + 1}$.

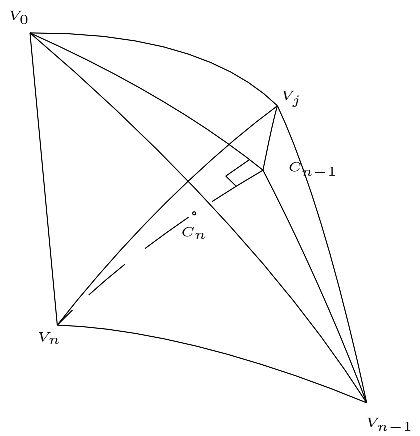

Figure 1: $\angle V_n C_{n-1} V_0$ is a right angle.

For the proof of Proposition 2.1 we shall need the following lemma:

Lemma 2.1. *For $n \in \mathbb{N}$ and $\zeta > 1$, define $\beta, \gamma, \delta \in \mathbb{R}$ by*

$$\begin{aligned}
\cosh^2 \beta &= \frac{n\zeta + 1}{n+1}, \\
\cosh^2 \gamma &= \frac{(n+1)\,\zeta^2}{n\zeta + 1}, \\
\cosh^2 \delta &= \frac{(n+1)\zeta + 1}{n+2}.
\end{aligned}$$

Note that these definitions make sense since the quantities on the right-hand side are all greater than one. Then the following identity holds:

$$\cosh^2(\gamma - \delta)\cosh^2 \beta = \cosh^2 \delta. \tag{1}$$

The proof of Lemma 2.1 is postponed to the Appendix.

Proof of Proposition 2.1. We argue by induction on the dimension n. For $n = 1$, the simplex is a segment between two points, V_0 and V_1, distance a apart. The point C_1 is their midpoint, and $C_0 = V_0$, since it is the midpoint of a degenerate face containing only one point. It follows immediately that the point C_1 is on the segment $[C_0V_1]$, and that

$$\begin{aligned}
&\cosh^2 d(V_0, C_1) = \cosh^2 d(V_1, C_1) = \cosh^2 \frac{a}{2} = \frac{\cosh a + 1}{2}, \\
&\cosh^2 d(V_1, C_0) = \cosh^2 a.
\end{aligned}$$

Assume now that the proposition holds for $n = k$. Let $\triangle_a^{k+1}$ be a regular $(k+1)$-simplex with vertices $\{V_0, \ldots, V_{k+1}\}$, and let F_{k+1} be its facet given by the convex hull of $\{V_0, \ldots, V_k\}$ (which is a regular k-simplex with side length a). Let g be the geodesic line in $\mathbb{H}^{k+1}$ perpendicular to the facet F_{k+1}, and passing through its midpoint C_k. The hyperplane in which F_{k+1} lies divides $\mathbb{H}^{k+1}$ into two half-spaces, and we denote by $\tilde{g}$ the part of g that lies on the same half-space as $\triangle_a^{k+1}$.

Next, let P, Q be two points on $\tilde{g}$, such that Q is between C_k and P and

$$\cosh^2 d(C_k, P) = \frac{(k+1)\cosh^2 a}{k\cosh a + 1}, \tag{2}$$

$$\cosh^2 d(Q, P) = \frac{(k+1)\cosh a + 1}{k+2}. \tag{3}$$

We remark that for the above to be well defined the expression for the distance between C_k and P must be larger than the expression for the distance between P and Q, and indeed, since $\cosh a > 1$, one has that

$$\frac{(k+1)\cosh a + 1}{k+2} < \frac{(k+1)\cosh a}{k+1} = \frac{(k+1)\cosh^2 a}{k\cosh a + \cosh a} < \frac{(k+1)\cosh^2 a}{k\cosh a + 1},$$

and thus $d(Q,P) < d(C_k,P)$, as the inverse hyperbolic cosine function is positive and monotonically increasing.

From the definition of the points P and Q it follows that for every $0 \leq j \leq k$ one has $\angle PC_kV_j = \angle QC_kV_j = \frac{\pi}{2}$, and thus using the hyperbolic law of cosines we conclude that:

$$\cosh^2 d(P,V_j) = \cosh^2 d(P,C_k) \cosh^2 d(C_k,V_j), \tag{4}$$

$$\cosh^2 d(Q,V_j) = \cosh^2 (d(P,C_k) - d(P,Q)) \cosh^2 d(C_k,V_j). \tag{5}$$

Using the induction hypothesis and (2) above, equality (4) gives

$$\cosh^2 d(P,V_j) = \frac{(k+1)\cosh^2 a}{(k\cosh a + 1)} \cdot \frac{(k\cosh a + 1)}{(k+1)} = \cosh^2 a.$$

This implies that the distance from P to each of the vertices of F_{k+1} equals a. Thus, from the uniqueness (up to isometries) of the regular $(k+1)$-simplex and our choice of the point P we conclude that P must coincide with V_{k+1}.

Next, we simplify the right-hand side of expression (5) using Lemma 2.1, by replacing n by k, ζ by $\cosh a$, β by $d(C_k,V_j)$, γ by $d(P,C_k)$ and δ by $d(P,Q)$. The lemma's premise holds by the induction hypothesis and (2) and (3) above, and we conclude that for every $0 \leq j \leq k$ one has

$$\cosh^2 d(Q,V_j) = \cosh^2 d(Q,P) = \cosh^2 d(Q,V_{k+1}). \tag{6}$$

Thus, the distance between Q and each vertex of F_{k+1} equals the distance between Q and V_{k+1}. Again, from the uniqueness property of the midpoint it follows that Q must coincide with C_{k+1}, and consequently assertion (i) of the proposition holds for $n = k+1$. Moreover, since the geodesic g connecting V_{k+1} and C_k is perpendicular to the facet F_{k+1}, it follows that $\angle C_{k+1}C_kV_j = \angle V_{k+1}C_kV_j = \frac{\pi}{2}$, which proves assertion (ii) for $n = k+1$.

Substituting $P = V_{k+1}$ and $Q = C_{k+1}$ in (2) and (3) for $n = k$ yields

$$\cosh^2 d(C_{k+1},V_{k+1}) = \frac{(k+1)\cosh a + 1}{k+2},$$

$$\cosh^2 d(C_k,V_{k+1}) = \frac{(k+1)\cosh^2 a}{k\cosh a + 1}.$$

The first equality together with (6) above prove assertion (iii) for $n = k+1$, and the second (iv). This completes the proof of Proposition 2.1. $\square$

We now turn to compute the centroid of the hyperbolic regular simplex.

Proposition 2.2. *With the above notations, let (Z_n, z_n) be the centroid of n+1 point masses of unit mass placed in the vertices of $\triangle_a^n$. Then,*

$$(Z_n, z_n) = \left(C_n, \sqrt{(n+1)(n\cosh a + 1)}\right) \tag{7}$$

Proof of Proposition 2.2. We start by showing that Z_n coincides with C_n, the midpoint of $\triangle_a^n$. Let τ be a permutation of $\{0, \ldots, n\}$, and let $\tilde{\sigma}$ be an isometry of $\mathbb{H}^n$ such that $\tilde{\sigma} V_j = V_{\tau j}$ for every $0 \leq j \leq n$. Note that

$$(Z_n, z_n) := \mathop{\ast}_{j=0}^{n} (V_j, 1) = \mathop{\ast}_{j=0}^{n} (V_{\tau j}, 1) = \mathop{\ast}_{j=0}^{n} (\tilde{\sigma} V_j, 1) = (\tilde{\sigma} Z_n, z_n).$$

This implies that the point Z_n lies at the same distance from V_j and $V_{\tau j}$ since

$$d\left(Z_n, V_j\right) = d\left(\tilde{\sigma} Z_n, \tilde{\sigma} V_j\right) = d\left(Z_n, V_{\tau j}\right). \tag{8}$$

Since (8) holds for any $0 \leq j \leq n$ and any permutation of the vertices, it follows that Z_n is equidistant from all the vertices of $\triangle_a^n$, and thus coincides with the midpoint of $\triangle_a^n$.

It remains to show that the masses in both sides of (7) are indeed equal. As before, we argue by induction on the dimension n. For $n = 1$ this follows immediatly from Definition 2.1. We assume the proposition holds for $n = k$. From Definition 2.1 it follows that

$$z_{k+1} = \cosh d\left(C_{k+1}, V_{k+1}\right) + z_k \cosh d\left(C_{k+1}, C_k\right).$$

By combining this with the induction hypothesis we conclude that

$$z_{k+1} = \cosh d\left(C_{k+1}, V_{k+1}\right) + \sqrt{(k+1)\left(k \cosh a + 1\right)} \cosh d\left(C_{k+1}, C_k\right).$$

From assertion (ii) of Proposition 2.1 it follows that $\angle C_{k+1} C_k V_0 = \frac{\pi}{2}$, and thus using the hyperbolic law of cosines we obtain

$$z_{k+1} = \cosh d\left(C_{k+1}, V_{k+1}\right) + \sqrt{(k+1)\left(k \cosh a + 1\right)} \cdot \frac{\cosh d\left(C_{k+1}, V_0\right)}{\cosh d\left(C_k, V_0\right)}.$$

Since $\cosh d\left(C_{k+1}, V_0\right) = \cosh d\left(C_{k+1}, V_{k+1}\right)$, we further deduce that

$$z_{k+1} = \cosh d\left(C_{k+1}, V_0\right) \left(1 + \frac{\sqrt{(k+1)\left(k \cosh a + 1\right)}}{\cosh d\left(C_k, V_0\right)}\right).$$

Finally, from assertion (iii) and (iv) of Proposition 2.1 we get

$$\begin{aligned} z_{k+1} &= \sqrt{\frac{(k+1)\cosh a + 1}{k+2}} \left(1 + \frac{\sqrt{(k+1)\left(k \cosh a + 1\right)}}{\sqrt{\frac{k \cosh a + 1}{k+1}}}\right) \\ &= \sqrt{\frac{(k+1)\cosh a + 1}{k+2}} \left(1 + k + 1\right) = \sqrt{(k+2)\left((k+1)\cosh a + 1\right)}. \end{aligned}$$

This completes the proof of Proposition 2.2. $\square$

Recall that σ_j stands for the reflection in $\mathbb{H}^n$ with respect to the hyperplane in which the facet F_j of the simplex $\triangle_a^n$ lies. The following proposition describes the centroid of a vertex of $\triangle_a^n$ and its reflection with respect to the opposite facet in terms of a weigthed center of mass of the other vertices.

Proposition 2.3. *Let $\triangle_a^n$ be a hyperbolic regular n-simplex with edge length $a > 0$ and vertices $\{V_0, \ldots, V_n\}$. Then for every $0 \le j \le n$ one has*

$$(V_j, 1) * (\sigma_j V_j, 1) = \mathop{\ast}_{\substack{k=0\\k\neq j}}^{n} \left(V_k, \frac{2}{n-1+\frac{1}{\cosh a}} \right).$$

Proof of Proposition 2.3. Fix $0 \le j \le n$. Since the facet F_j is an $(n-1)$-regular simplex with edge length a, it follows from Proposition 2.2 that

$$\mathop{\ast}_{\substack{k=0\\k\neq j}}^{n} (V_k, 1) = \left(W_j, \sqrt{n\left((n-1)\cosh a + 1\right)} \right),$$

where W_j is the midpoint of F_j. From Definition (2.1) it follows that one can multiply all the masses in the above equation by the constant $\frac{2}{n-1+\frac{1}{\cosh a}}$, and hence:

$$\mathop{\ast}_{\substack{k=0\\k\neq j}}^{n} \left(V_k, \frac{2}{n-1+\frac{1}{\cosh a}} \right) = \left(W_j, 2\sqrt{\frac{n\cosh^2 a}{(n-1)\cosh a + 1}} \right) \tag{9}$$

On the other hand, the segment $[V_j, \sigma_j V_j]$ is invariant under σ_j and not fully contained by F_j, so must be perpendicular to F_j. Assertion (ii) of Proposition 2.1, implies that this segment must pass through W_j. Moreover, σ_j is an isometry that leaves $W_j \in F_j$ invariant so $d(V_j, W_j) = d(\sigma_j V_j, W_j)$. The point W_j is therefore the midpoint between V_j and $\sigma_j V_j$.

From Definition 2.1 it follows that

$$(V_j, 1) * (\sigma_j V_j, 1) = \left(W_j, \cosh d(W_j, V_j) + \cosh d(W_j, \sigma_j V_j) \right). \tag{10}$$

From assertion (iii) of Proposition 2.1 it follows that both $\cosh^2 d(W_j, V_j)$ and $\cosh^2 d(W_j, \sigma_j V_j)$ equal $\frac{n\cosh^2 a}{(n-1)\cosh a+1}$, since $d(W_j, V_j)$ and $d(W_j, \sigma_j V_j)$ are distances between a vertex of a regular n-simplex and the midpoint of the facet in front of it. Hence (10) becomes

$$(V_j, 1) * (\sigma_j V_j, 1) = \left(W_j, 2\sqrt{\frac{n\cosh^2 a}{(n-1)\cosh a + 1}} \right). \tag{11}$$

The proof of the proposition now follows from (9) and (11). $\square$

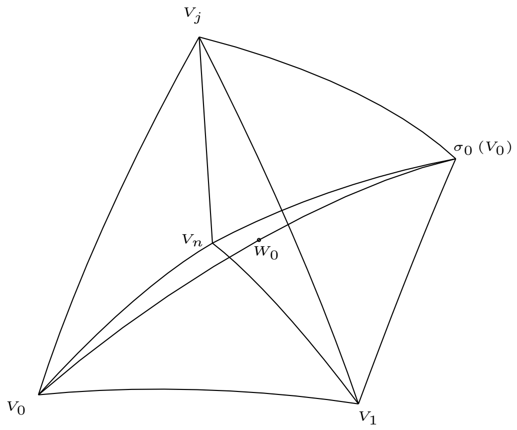

Figure 2: The center of mass of a vertex and its reflection is positioned in the center of mass of the rest of the vertices

We finish this subsection with the following simple observation:

Lemma 2.2. *Let $\triangle_a^n$ be a regular n-simplex with vertices $\{V_0, \ldots, V_n\}$, and $\{w_0, \ldots, w_n\}$ be non-negative real numbers. Let $P = \ast_{j=0}^{n} (V_j, w_j)$ be the centroid of point masses placed in the vertices $\{V_0, \ldots, V_n\}$. Then,*

$$P \in F_k \Leftrightarrow w_k = 0.$$

Proof of Lemma 2.2. The commutativity and associativity of the $\ast$ operator allow one to write

$$P = (V_k, w_k) \ast (Q, m),$$

for some $m \geq 0$, where

$$Q := \mathop{\ast}_{\substack{j=0 \\ j \neq k}}^{n} (V_j, w_j) \in F_k$$

Moreover, $V_k \notin F_k$, and the geodesic through V_k and Q intersects F_k at one point at most, namely Q. Since P lies on the above mentioned geodesic connecting V_k and Q,

$$P \in F_k \Leftrightarrow P \text{ coincides with } Q \Leftrightarrow w_k = 0.$$

□

2.3 Billiards in regular hyperbolic simplices

Billiard dynamics in hyperbolic space, and in particular polygonal and polyhedral billiards, have been extensively studied both in the context of mathematics and physics (see e.g., [7, 15, 18]). As in the Euclidean case, when defining billiard dynamics in the hyperbolic space for a non-smooth domain, one runs into certain technical difficulties in describing the dynamics at the singular parts of the boundary. To avoid these difficulties, in what follows we shall consider only closed billiard orbits in $\triangle_a^n \subset \mathbb{H}^n$ which bounce at the interior of the facets. More precisely,

Definition 2.5. *A closed billiard orbit of period $n+1$ inside $\triangle_a^n \subset \mathbb{H}^n$ is a closed polygonal curve consisting of geodesic segments, and specified by a sequence of points $\{P_j\}_{j=0}^n \in \partial\triangle_a^n$ such that:*

(i) For every $0 \leq j \leq n$, there is a unique $0 \leq k \leq n$ such that $P_j \in F_k$,

(ii) $P_j \in [P_{j-1}, \sigma_j(P_{j+1})]$, where σ_j is the reflection in $\mathbb{H}^n$ with respect to the hyperplane in which the facet F_j lies.

Remark: It is not hard to check that the above definition is equivalent to the definition of billiard trajectories as critical points of the length functional, where trajectories passing through non-smooth parts of the boundary of $\triangle_a^n$ are excluded (see e.g. [12]).

3 Proof of the Main Theorem

Let $\triangle_a^n$ be a regular simplex in the hyperbolic space $\mathbb{H}^n$ with side length a. We turn now to the construction of a billiard trajectory inside $\triangle_a^n$ which bounces at the interior of any facet exactly once. The (bouncing) points of this orbit will be described as locations of centroids of masses positioned at the vertices of the simplex $\triangle_a^n$. For this end, let us first define a finite sequence of real non-negative numbers that will serve as a pool from which these masses will be drawen.

Lemma 3.1. *For every $2 < n \in \mathbb{N}$, and $0 < a \in \mathbb{R}$, there exist $0 < \lambda_{n,a} \in \mathbb{R}$ and a sequence of $n+2$ real numbers $\alpha_0, \ldots, \alpha_{n+1}$, such that*

$$\alpha_0 = \alpha_{n+1} = 0, \ \alpha_1 = \alpha_n = 1, \ \alpha_j > 0 \text{ for every } 1 < j < n, \tag{12}$$

and for every $1 \leq j \leq n$ one has

$$\lambda \alpha_j = \alpha_{j-1} + \alpha_{j+1} + \frac{2}{n - 1 + \frac{1}{\cosh a}} \tag{13}$$

The proof of Lemma 3.1 is postponed to the Appendix.

In what follows, to ease notations, let us extend the vertices $\{V_j\}_{j=1}^n$ to all $j \in \mathbb{Z}$ by cyclicly repeating them in both directions, thus creating an $(n+1)$-periodic sequence $\{V_j\}_{j\in\mathbb{Z}}$ such that $V_j = V_k$ if $j \equiv k \mod n+1$.

With Lemma 3.1 at our disposal, we now define the bouncing points of the billiard trajectory as locations of centroids of point masses placed in the vertices of the simplex of $\triangle_a^n$:

Definition 3.1. *With the above notations, for every* $0 \leq j \leq n$*, let*

$$(P_j, m_j) := \mathop{\ast}_{k=0}^{n} (V_{k+j}, \alpha_k)\,, \tag{14}$$

where $\{\alpha_k\}_{k=0}^n$ *is a sequence which satisfies properties* (12) *and* (13)*, whose existence is ensured by Lemma 3.1 above.*

Finally, we are now in position to prove our main result.

Proof of Theorem 1.1. We will show that the sequence of points $\{P_j\}_{j=0}^n$, defined in equation (14) form a periodic billiard trajectory in $\triangle_a^n$.

First, note that in the definition of the point P_j, the mass positioned in the vertex V_j is $\alpha_0 = 0$. Thus, from Lemma 2.2 it follows that P_j belongs to the facet F_j. On the other hand, the rest of the masses that appear in the definition of the point P_j are strictly positive. Hence, applying Lemma 2.2 once again, this time to the facet F_j (considered as regular $(n-1)$-simplex), we obtain that the point P_j does not belong to any other facet of the simplex, as required by property (i) of Definition 2.5.

It remains to show that the sequence $\{P_j\}_{j=0}^n$ satisfies property (ii) of Definition 2.5, i.e., that $P_j \in [P_{j-1}, \sigma_j P_{j+1}]$, for every $0 \leq j \leq n$ (see Figure 3 below). In fact we will show, by means of center of mass arguments, that

$$(P_j, \lambda m_j) = (P_{j-1}, m_{j-1}) \ast (\sigma_j P_{j+1}, m_{j+1})\,, \tag{15}$$

where $\lambda > 0$ is the positive constant ensured by Lemma 3.1, and thus $P_j \in [P_{j-1}, \sigma_j P_{j+1}]$ as required.

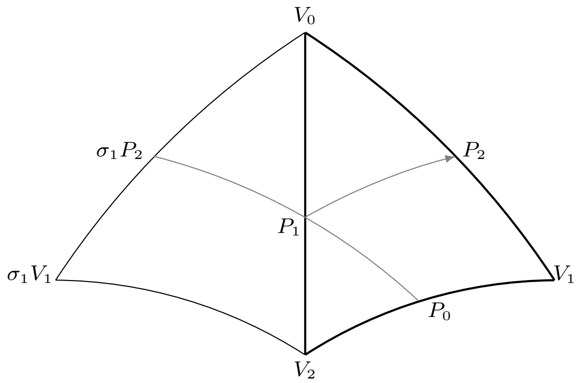

Figure 3: Property (ii) of Definition 2.5 for $n = 2, j = 1$: $P_1 \in [P_0, \sigma_1 P_2]$

To this end, we start with the following computation. Let $0 \leq j \leq n$. From the properties of the center of mass (Definition 2.1), the choice of the sequence $\{\alpha_k\}_{k=0}^{n+1}$ (Definition 3.1), and the definition of the bouncing points $\{P_j\}_{j=0}^{n}$ (Definition 3.1) it follows that

$$\begin{aligned}
(\sigma_j(P_{j+1}), m_{j+1}) &= \mathop{\ast}_{k=0}^{n} (\sigma_j(V_{k+j+1}), \alpha_k) \\
&= \mathop{\ast}_{k=0}^{n-1} (\sigma_j(V_{k+j+1}), \alpha_k) \ast (\sigma_j(V_{j+n+1}), \alpha_n) \\
&= \mathop{\ast}_{k=0}^{n-1} (\sigma_j(V_{k+j+1}), \alpha_k) \ast (\sigma_j(V_{j+n+1}), 1) \\
&= \mathop{\ast}_{k=0}^{n-1} (\sigma_j(V_{k+j+1}), \alpha_k) \ast (\sigma_j V_j, 1) \\
&= \mathop{\ast}_{k=1}^{n} (\sigma_j(V_{k+j}), \alpha_{k-1}) \ast (\sigma_j V_j, 1) \\
&= \mathop{\ast}_{k=1}^{n} (V_{k+j}, \alpha_{k-1}) \ast (\sigma_j V_j, 1),
\end{aligned}$$

where the last equality holds since V_{k+j} does not equal V_j for $1 \leq k \leq n$, and is therfore a vertex of F_j which is invariant under the reflection σ_j.

On the other hand,

$$\begin{aligned}
(P_{j-1}, m_{j-1}) &= \mathop{\ast}_{k=0}^{n} (V_{k+j-1}, \alpha_k) \\
&= \mathop{\ast}_{k=2}^{n} (V_{k+j-1}, \alpha_k) \ast (V_{j-1}, \alpha_0) \ast (V_j, \alpha_1) \\
&= \mathop{\ast}_{k=2}^{n} (V_{k+j-1}, \alpha_k) \ast (V_{n+j}, \alpha_{n+1}) \ast (V_j, \alpha_1) \\
&= \mathop{\ast}_{k=2}^{n+1} (V_{k+j-1}, \alpha_k) \ast (V_j, \alpha_1) \\
&= \mathop{\ast}_{k=1}^{n} (V_{k+j}, \alpha_{k+1}) \ast (V_j, 1).
\end{aligned}$$

Using once again the properties of the centroid, a direct calculation of the center of mass of the two point masses above yields that the right-hand side of relation (15) equals

$$\left(\mathop{\ast}_{k=1}^{n} (V_{k+j}, \alpha_{k-1})\right) \ast (\sigma_j V_j, 1) \ast \left(\mathop{\ast}_{k=1}^{n} (V_{k+j}, \alpha_{k+1})\right) \ast (V_j, 1) =$$
$$\left(\mathop{\ast}_{k=1}^{n} (V_{k+j}, \alpha_{k-1} + \alpha_{k+1})\right) \ast (V_j, 1) \ast (\sigma_j V_j, 1).$$

Using Proposition 2.3 above, one can replace $(V_j, 1) \ast (\sigma_j V_j, 1)$ by an expression free of the reflection σ_j and thus obtain:

$$
\begin{aligned}
&(\sigma_j P_{j+1}, m_{j+1}) \ast (P_{j-1}, m_{j-1}) \\
&= \Big(\mathop{\ast}_{k=1}^{n} (V_{k+j}, \alpha_{k-1} + \alpha_{k+1}) \Big) \ast \left(\mathop{\ast}_{k=1}^{n} \left(V_{k+j}, \frac{2}{n-1+\frac{1}{\cosh a}} \right) \right) \\
&= \mathop{\ast}_{k=1}^{n} \left(V_{k+j}, \alpha_{k-1} + \alpha_{k+1} + \frac{2}{n-1+\frac{1}{\cosh a}} \right).
\end{aligned}
$$

Since equation (13) is satisfied for every $1 \le k \le n$, we conclude that:

$$(\sigma_j P_{j+1}, m_{j+1}) \ast (P_{j-1}, m_{j-1}) = \mathop{\ast}_{k=1}^{n} (V_{k+j}, \lambda \alpha_k).$$

The expression on the right-hand side is very similar to the definition of $(P_0, \lambda m_0)$, the only difference being in the range of k's – it does not include 0. However, since $0 = \alpha_0 = \lambda \alpha_0$, extending the range of k to include 0 has no effect on the value of the expression and hence:

$$(\sigma_j P_{j+1}, m_{j+1}) \ast (P_{j-1}, m_{j-1}) = \mathop{\ast}_{k=0}^{n} (V_{k+j}, \lambda \alpha_k) = (P_0, \lambda m_0)$$

This completes the proof of Theorem 1.1. □

APPENDIX

Proof of Lemma 2.1. A direct computation shows that

$$
\begin{aligned}
\sinh^2 \gamma &= \cosh^2 \gamma - 1 = \frac{(n+1)\,\zeta^2}{n\zeta + 1} - 1 = \frac{(n\zeta + \zeta + 1)\,(\zeta - 1)}{n\zeta + 1}, \\
\sinh^2 \delta &= \cosh^2 \delta - 1 = \frac{(n+1)\,\zeta + 1}{n+2} - 1 = \frac{(n+1)\,(\zeta - 1)}{n+2}.
\end{aligned}
$$

Combining this with a well known hyperbolic trigonometric identity gives

$$
\begin{aligned}
&\cosh^2(\gamma - \delta) = \Big(\cosh(\gamma)\cosh(\delta) - \sinh(\gamma)\sinh(\delta) \Big)^2 \\
&= \cosh^2\gamma \cosh^2\delta + \sinh^2\gamma \sinh^2\delta - 2\sqrt{\cosh^2\gamma \cosh^2\delta \sinh^2\gamma \sinh^2\delta} \\
&= \frac{(n+1)\,\zeta^2}{(n\zeta+1)} \cdot \frac{(n\zeta+\zeta+1)}{(n+2)} + \frac{(n\zeta+\zeta+1)\,(\zeta-1)}{(n\zeta+1)} \cdot \frac{(n+1)\,(\zeta-1)}{(n+2)} \\
&\quad - 2\sqrt{\frac{(n+1)\,\zeta^2}{(n\zeta+1)} \cdot \frac{(n\zeta+\zeta+1)}{(n+2)} \cdot \frac{(n\zeta+\zeta+1)\,(\zeta-1)}{(n\zeta+1)} \cdot \frac{(n+1)\,(\zeta-1)}{(n+2)}} \\
&= \frac{(n+1)\,(n\zeta+\zeta+1)}{(n\zeta+1)\,(n+2)} \Big(\zeta^2 + (\zeta-1)^2 - 2\zeta\,(\zeta-1) \Big) \\
&= \frac{(n+1)\,(n\zeta+\zeta+1)}{(n\zeta+1)\,(n+2)}.
\end{aligned}
$$

Thus, we conclude that

$$\cosh^2\beta\cosh^2(\gamma-\delta)=\frac{(n\zeta+1)}{(n+1)}\frac{(n+1)(n\zeta+\zeta+1)}{(n\zeta+1)(n+2)}=\frac{(n+1)\zeta+1}{n+2}=\cosh^2\delta$$

This completes the proof of Lemma 2.1. $\square$

Proof of Lemma 3.1. Define $h:[0,\infty)\to\mathbb{R}_+$ by

$$h(x):=\frac{x^{\frac{n-1}{2}}+x^{-\frac{n-1}{2}}}{x^{\frac{n+1}{2}}+x^{-\frac{n+1}{2}}}$$

It can be directly checked that h is differentiable on $[0,\infty)$, and that:

$$h'(x)=\left(x-x^{-1}\right)\cdot\frac{-\sum\limits_{j=1}^{n}x^{n-2j}-nx^{-1}}{\left(x^{\frac{n+1}{2}}+x^{-\frac{n+1}{2}}\right)^2},$$

for $x>1$ and that $h'_+(0)=1$. Next, define $g:[1,\infty)\to\mathbb{R}$ by

$$g(y):=h\left(y+\sqrt{y^2-1}\right)-1+(y-1)\left(n-1+\frac{1}{\cosh a}\right).$$

The function g is also differentiablel, and for $y>1$ one has

$$g'(y)=h'\left(y+\sqrt{y^2-1}\right)\cdot\left(1+\frac{y}{\sqrt{y^2-1}}\right)+\left(n-1+\frac{1}{\cosh a}\right).$$

Set $\xi_y=y+\sqrt{y^2-1}$, and note that $\xi_y^{-1}=y-\sqrt{y^2-1}$. The derivative $g'(y)$ can be now expressed by means of ξ_y as follows:

$$\begin{aligned}g'(y)&=h'(\xi_y)\cdot\left(1+\frac{\xi_y+\xi_y^{-1}}{\xi_y-\xi_y^{-1}}\right)+\left(n-1+\frac{1}{\cosh a}\right)\\&=\left(\xi_y-\xi_y^{-1}\right)\cdot\frac{\left(-\sum\limits_{j=1}^{n}\xi_y^{n-2j}-n\xi_y^{-1}\right)}{\left(\xi_y^{\frac{n+1}{2}}+\xi_y^{-\frac{n+1}{2}}\right)^2}\cdot\left(1+\frac{\xi_y+\xi_y^{-1}}{\xi_y-\xi_y^{-1}}\right)+\left(n-1+\frac{1}{\cosh a}\right).\end{aligned}$$

Thus, we conclude that

$$g'(y)=\frac{-\sum\limits_{j=1}^{n}\xi_y^{n-2j}-n\xi_y^{-1}}{\left(\xi_y^{\frac{n+1}{2}}+\xi_y^{-\frac{n+1}{2}}\right)^2}\cdot 2\xi_y+\left(n-1+\frac{1}{\cosh a}\right).\tag{16}$$

The function $g(y)$ is smooth and defined for $y = 1$, and hence

$$g'(1) = \lim_{y \to 1^+} g'(y) = 2\frac{-\sum_{j=1}^{n} 1 - n1^{-1}}{\left(1^{\frac{n+1}{2}} + 1^{-\frac{n+1}{2}}\right)^2} + n - 1 + \frac{1}{\cosh a} =$$

$$= 2\frac{-n - n}{(1+1)^2} + n - 1 + \frac{1}{\cosh a} = \frac{1}{\cosh a} - 1 < 0.$$

This implies in particular that there exists $\epsilon > 0$ for which $g(1+\epsilon) < g(1) = 0$. On the other hand, it is not hard to verify that for large enough values of y, one has $g(y) > 0$. The intermediate value theorem implies that there exists $1 + \epsilon < y_0$ such that $g(y_0) = 0$. Next define:

$$\lambda := 2y_0 > 2,$$
$$\xi := \xi_{y_0} > 1,$$
$$b := \frac{2}{\left(n - 1 + \frac{1}{\cosh a}\right)}.$$

Note that with these notations, the equation

$$0 = g(y_0) = h\left(y_0 + \sqrt{y_0^2 - 1}\right) - 1 + (y_0 - 1)\left(n - 1 + \frac{1}{\cosh a}\right),$$

can be rewritten as

$$h(\xi) = 1 - \left(\frac{\lambda}{2} - 1\right)\frac{2}{b} = 1 - \frac{\lambda - 2}{b}.$$

We are now in a position to provide an explicit formula for a sequence $\{\alpha_j\}_{j=0}^{n+1}$ that satisfies the required conditions of the lemma. Let,

$$\alpha_j := \frac{b}{\lambda - 2}\left(1 - \frac{\xi^{j - \frac{n+1}{2}} + \xi^{\frac{n+1}{2} - j}}{\xi^{\frac{n+1}{2}} + \xi^{-\frac{n+1}{2}}}\right).$$

Note that by definition

$$\alpha_0 = \alpha_{n+1} = \frac{b}{\lambda - 2}\left(1 - \frac{\xi^{-\frac{n+1}{2}} + \xi^{\frac{n+1}{2}}}{\xi^{\frac{n+1}{2}} + \xi^{-\frac{n+1}{2}}}\right) = 0,$$

and

$$\alpha_1 = \alpha_n = \frac{b}{\lambda - 2}\left(1 - \frac{\xi^{1 - \frac{n+1}{2}} + \xi^{\frac{n+1}{2} - 1}}{\xi^{\frac{n+1}{2}} + \xi^{-\frac{n+1}{2}}}\right) = \frac{b}{\lambda - 2}\Big(1 - h(\xi)\Big)$$

$$= \frac{b}{\lambda - 2} \cdot \frac{\lambda - 2}{b} = 1.$$

Next, let us verify that the sequence α_j satisfies (13) for $1 \le j \le n$. Indeed,

$$\begin{aligned}\alpha_{j-1} + \alpha_{j+1} = {} & \frac{b}{\lambda - 2}\left(1 - \frac{\xi^{j-1-\frac{n+1}{2}} + \xi^{\frac{n+1}{2}-j+1}}{\xi^{\frac{n+1}{2}} + \xi^{-\frac{n+1}{2}}}\right) \\ & + \frac{b}{\lambda - 2}\left(1 - \frac{\xi^{j+1-\frac{n+1}{2}} + \xi^{\frac{n+1}{2}-j-1}}{\xi^{\frac{n+1}{2}} + \xi^{-\frac{n+1}{2}}}\right),\end{aligned}$$

which can be simplified to:

$$\alpha_{j-1} + \alpha_{j+1} = \frac{b}{\lambda - 2}\left(\lambda - \frac{\left(\xi^{j-\frac{n+1}{2}} + \xi^{\frac{n+1}{2}-j}\right)}{\left(\xi^{\frac{n+1}{2}} + \xi^{-\frac{n+1}{2}}\right)}\left(\xi + \xi^{-1}\right)\right) - b.$$

Now, by substituting λ for $\xi + \xi^{-1}$, we obtain, as required, that:

$$\alpha_{j-1} + \alpha_{j+1} = \lambda\frac{b}{\lambda - 2}\left(1 - \frac{\xi^{j-\frac{n+1}{2}} + \xi^{\frac{n+1}{2}-j}}{\xi^{\frac{n+1}{2}} + \xi^{-\frac{n+1}{2}}}\right) - b = \lambda\alpha_j - b.$$

Finally, to complete the proof of the lemma, it remains to show that $\alpha_j > 0$ for every $0 < j < n+1$. For this end, rewrite α_j as:

$$\alpha_j = \frac{b}{\lambda - 2} - \frac{b}{(\lambda - 2)\left(\xi^{\frac{n+1}{2}} + \xi^{-\frac{n+1}{2}}\right)}\left(\xi^{j-\frac{n+1}{2}} + \xi^{\frac{n+1}{2}-j}\right). \qquad (17)$$

Note that the expression $\xi^t + \xi^{-t}$ is monotonically increasing with respect to $|t|$, and hence the expression $(\xi^{j-\frac{n+1}{2}} + \xi^{\frac{n+1}{2}-j})$ is monotonically increasing with respect to $\left|j - \frac{n+1}{2}\right|$. Moreover, recall that $\lambda > 2$ and $b > 0$. Therefore, the expression on the right hand side of (17) achieves a global maxima at $j = \frac{n+1}{2}$, and strictly decreases as $\left|j - \frac{n+1}{2}\right|$ increases, that is – as j approaches 0 on one side, or $n+1$ on the other. Since it was already shown that $\alpha_0 = \alpha_{n+1} = 0$, we conclude that $\alpha_j > 0$ for all $0 < j < n+1$. This completes the proof of Lemma 3.1. □

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