On the quantum homology algebra of toric Fano manifolds

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November 25, 2008

Abstract

In this paper we study certain algebraic properties of the small quantum homology algebra for the class of symplectic toric Fano manifolds. In particular, we examine the semisimplicity of this algebra, and the more general property of containing a field as a direct summand. Our main result provides an easily verifiable sufficient condition for these properties which is independent of the symplectic form. Moreover, we answer two questions of Entov and Polterovich negatively by providing examples of toric Fano manifolds with non semisimple quantum homology, and others in which the Calabi quasimorphism is not unique.

1 Introduction.

The quantum homology algebra[1] $QH_*(X, \omega)$ of a symplectic manifold (X, ω) is, roughly speaking, the singular homology of X endowed with a modified algebraic structure, which is a deformation of the ordinary intersection product. It was originally introduced by the string theorists Vafa and Witten [47, 49] in the context of topological quantum field theory. There followed rigorous mathematical constructions by Ruan and Tian [42] in the symplectic setting, and by Kontsevich and Manin [29] in the algebra-geometric setting. We refer the reader to [25, 36] and the references within for detailed expositions to quantum homology.

In this paper we focus on certain algebraic properties of the quantum homology algebra. Before we proceed we fix our notation (see Subsection 3.1 for details). For a $2d$-dimensional symplectic manifold (X, ω), consider the quantum homology $QH_*(X, \omega) := H_*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda^{\downarrow}$ with coefficients in $\Lambda^{\downarrow} := \mathbb{K}^{\downarrow}[q, q^{-1}]$, where $\mathbb{K}^{\downarrow}$ is the field of generalized Laurent series

$$\mathbb{K}^{\downarrow} = \Big\{ \sum_{\lambda \in \mathbb{R}} a_\lambda s^\lambda \mid a_\lambda \in \mathbb{C}, \text{ and } \{\lambda \,|\, a_\lambda \neq 0\} \text{ is discrete and bounded above in } \mathbb{R} \Big\}.$$

A grading on $QH_*(X, \omega)$ is given by $\deg(a \otimes s^\lambda q^j) = \deg(a) + 2j$, and the quantum product

*The first named author was supported by NSF grant DMS-0706976.

[1] Throughout the text we consider exclusively the small quantum (co)homology, thus from now on we omit the adjective "small" when referring to it.

of $a \in H_*(X,\mathbb{Q})$ and $b \in H_*(X,\mathbb{Q})$ is defined by:

$$a * b = \sum_{A \in H_2(X)} (a * b)_A \otimes s^{-\omega(A)} q^{-c_1(A)},$$

where $(a*b)_A \in H_*(X,\mathbb{Q})$ is defined via the Gromov-Witten invariants (see Subsection 3.1). This product gives $QH_*(X,\omega)$ the structure of a commutative, associative algebra with a unit element (see e.g., [36]).

One of the main properties of quantum homology algebra discussed in this paper is the semisimplicity property. Note that there are several slightly different notions of semisimplicity in the context of quantum homology due to Dubrovin [12], Kontsevich and Manin [29], and Abrams [1]. In this paper we follow Abrams' notion of the semisimplicity.

Recall that a finite dimensional commutative algebra over a field is said to be *semisimple* if it decomposes into a direct sum of fields. We say that the quantum homology $QH_*(X)$ is semisimple if its graded part $QH_{2d}(X)$, which is a finite-dimensional $\mathbb{K}^{\downarrow}$-subalgebra, is a semisimple algebra. Our main motivation to study the semisimplicity property of the quantum homology algebra comes from the recent works of Entov and Polterovich [15, 16, 17, 18] on Calabi quasimorphisms and symplectic quasi-states, in which the semisimplicity of the quantum homology plays a key role.

The following theorem has been originally proven in the case of monotone symplectic manifolds in [15] (using a slightly different setting), then generalized by the first named author in [38] to the class of rational strongly semipositive symplectic manifolds that satisfy a technical condition which was eventually removed in [16].

Theorem. *Let (X,ω) be a rational*[2] *strongly semipositive symplectic manifold of dimension $2d$ having semisimple quantum homology. Then X admits a Calabi quasimorphism and a symplectic quasi-state.*

For the definition of Calabi quasimorphisms and symplectic quasi-states, and detailed discussion of their application in symplectic geometry we refer the reader to [15, 18]. Beside demonstrating applications to Hofer's geometry and C^0-symplectic topology, Entov and Polterovich used the above theorem to obtain Lagrangian intersection type results. For example, in [7] they proved (together with Biran) that the Clifford torus in $\mathbb{C}P^n$ is not displaceable by a Hamiltonian isotopy. In a later work [17], they proved the non-displaceability of certain singular Lagrangian submanifolds, a result which is currently out of reach for the conventional Lagrangian Floer homology technique. We refer the reader to [17] for more details in this direction.

Very recently, McDuff pointed out that the semisimplicity assumption in the above theorem can be relaxed to the weaker assumption that $QH_{2d}(X,\omega)$ contains a field as a direct summand. Moreover, she showed that in contrast with semisimplicity, this condition holds true for one point blow-ups of non-uniruled symplectic manifolds such as the standard

[2]It is very plausible that the rationality assumption can be removed due to the recent works of Oh [39], and Usher [45].

symplectic four torus T^4 (see [34] and [16] for details), consequently enlarging the class of manifolds admitting Calabi quasimorphisms and symplectic quasi-states. Thus, in what follows we will study not only the semisimplicity of the quantum homology algebra, but also the more general property of containing a field as a direct summand.

A different motivation to study the semisimplicity of the quantum homology algebra is due to a work of Biran and Cornea. In [6] they showed that in certain cases the semisimplicity of the quantum homology implies restrictions on the existence of certain Lagrangian submanifolds. We refer the reader to [6] Subsection 6.5 for more details.

Finally, a third motivation comes from physics, where in the symplectic toric Fano case the semisimplicity of the quantum homology algebra implies that the corresponding $\mathcal{N} = 2$ Landau-Ginzburg model is massive. The physical interpretation is that the theory has massive vacua and the infrared limit of this model is trivial. See [25] and the references within for precise definition and discussion.

Examples of symplectic manifolds with semisimple quantum homology are $\mathbb{C}P^d$ (see e.g. [15]); complex Grassmannians; and the smooth complex quadric $Q = \{z_0^2 + \cdots + z_d^2 - z_{d+1}^2 = 0\} \subset \mathbb{C}P^{d+1}$ (see [1] for the last two examples). As mentioned above, McDuff [34] (see also [16]) provides a large class of examples of symplectic manifolds whose quantum homology contains a field as a direct summand but is not semisimple, by considering the one point blow-up of a non-uniruled symplectic manifold. Using the Künneth formula for quantum homology, one can show that both semisimplicity and the property of containing a field as a direct summand are preserved when taking products (see [16]).

Another class of examples are toric Fano 2-folds. Recall that up to rescaling the symplectic form by a constant factor there are exactly five symplectic toric Fano 2-folds: $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^2$, and the blowups of $\mathbb{C}P^2$ at $1, 2$ and 3 points. The following theorem is a combination of results from [38] and [16]:

Theorem. *If (X, ω) is a symplectic toric Fano 2-fold then $QH_*(X, \omega)$ is semisimple.*

In view of the above, Entov and Polterovich posed the following question in [16]:

Question: Is it true that the algebra $QH_*(X, \omega)$ is semisimple for any symplectic toric Fano manifold (X, ω)?

It is known (see e.g. [27] Corollary 5.12, and [20] Proposition 7.6) that semisimplicity holds for a general toric symplectic form. For the sake of completeness, we include this statement:

Theorem A. *Let X be a smooth $2d$-dimensional toric Fano variety. Then for a general*[3] *choice of a toric symplectic form ω on X, the quantum homology $QH_*(X, \omega)$ is semisimple.*

However, it turns out that the answer to the question of Entov and Polterovich is negative. The first counter example exists in (real) dimension eight:

[3] The space of toric symplectic forms has natural structure of a topological space, and *general* here means that ω belongs to a certain open dense subset in this space.

Proposition B. *There exists a monotone*[4] *symplectic toric Fano 4-fold* (X, ω) *whose quantum homology algebra* $QH_*(X, \omega)$ *is not semisimple.*

Using Künneth's formula we also produce examples of non-monotone symplectic Fano manifolds (X, ω) with non semisimple quantum cohomology algebras. In particular, there exists a non-monotone Fano 5-fold (X, ω) with a non semisimple $QH_*(X, \omega)$. Note that it would be interesting to construct an example of non-decomposable non-monotone symplectic Fano manifold with this property.

Recall that a toric Fano manifold X may be equipped with a distinguished toric symplectic form ω_0, namely the normalized monotone symplectic form corresponding to $c_1(X)$. This is the unique symplectic form for which the corresponding moment polytope is reflexive (see Section 2). Our second result shows that as far as semisimplicity is concerned, the symplectic form ω_0 is, in a manner of speaking, the worst.

Theorem C. *Let* X *be a toric Fano manifold of (real) dimension* $2d$*, and let* ω *be a toric symplectic form on* X*. If* $QH_*(X, \omega_0)$ *is semisimple then* $QH_*(X, \omega)$ *is semisimple.*

Inspired by McDuff's observation we modify Entov's and Polterovich's question and ask:

Question: Is it true that the algebra $QH_{2d}(X, \omega)$ contains a field as a direct summand for any symplectic toric Fano manifold (X, ω)?

Computer based calculations show that no counter example exists in real dimension less than or equal to 8. We hope to return to the above question in the future. Meanwhile, we prove the following analog of Theorem C:

Theorem D. *Let* X *be a toric Fano manifold of (real) dimension* $2d$*, and let* ω *be a toric symplectic form on* X*. If* $QH_{2d}(X, \omega_0)$ *contains a field as a direct summand, then* $QH_{2d}(X, \omega)$ *contains a field as a direct summand.*

In Subsection 3.3 we show that the property of $QH_{2d}(X, \omega)$ of having a field as a direct summand is equivalent to the existence of a non-degenerate critical point of a certain (combinatorially defined) function W_X, called the Landau-Ginzburg superpotential, assigned naturally to (X, ω). McDuff's observation and Theorem D reduce the question of the existence of Calabi quasimorphisms and symplectic quasi-states on a symplectic toric manifold (X, ω) to the normalized monotone case (X, ω_0), and hence to the problem of analyzing the critical points of a function W_X, depending only on X and not on the symplectic form. This can be done easily in many cases. In particular we construct the following new examples of symplectic manifolds admitting Calabi quasimorphisms and symplectic quasi-states:

Corollary E. Let X be either a toric Fano 3-fold or a toric Fano 4-fold. Then X admits a Calabi quasimorphism and a symplectic quasi-state.

Another byproduct of our method is the following two propositions. The first one, inspired by McDuff [35], answers a question raised by Entov and Polterovich [15] regarding

[4] Recall that (X, ω) is called monotone if $c_1 = \kappa[\omega]$, where $\kappa > 0$, and c_1 is the first Chern class of X.

the uniqueness of the Calabi quasimorphism. We will briefly recall the definition of a Calabi quasimorphism in Section 6. For a detailed discussion see [15, 18].

Corollary F. Let (X, ω) be the blow up of $\mathbb{C}P^2$ at one point equipped with a symplectic form ω. If $\omega(L)/\omega(E) > 3$, where L is the class of a line in $\mathbb{C}P^2$, and E is the class of the exceptional divisor in X, then there are two different Calabi quasimorphisms on (X, ω).

Remark: Other examples of symplectic manifolds for which the Calabi quasimorphism is not unique were constructed by Entov, McDuff, and Polterovich in [19]. We chose to include the above example here due to the simplicity of the argument. Moreover, we remark that Corollary F can be easily extended to other toric Fano manifolds.

Finally, we finish this section with a folklore result, known to experts in the field and proven in full detail by Auroux ([3] Theorem 6.1). The results in [3] are more general ([3] Proposition 6.8), and do not rely on Batyrev's description of the quantum homology algebra. However, since by using Proposition 3.3 the proof of the claim below becomes much simpler, we felt it might be useful to include it here as well.

Corollary G. For a smooth toric Fano manifold X, the critical values of the superpotential W_X are the eigenvalues of the linear operator $QH^0(X, \omega) \to QH^0(X, \omega)$ given by the multiplication by $q^{-1}c_1(X)$.

Structure of the paper: In Section 2 we recall some basic definitions and notations regarding symplectic toric manifolds. In Section 3 we give three equivalent description of the quantum cohomology of toric Fano manifolds. In section 4 we prove Theorems A, C, D, and corollary G. For technical reasons it is more convenient for us to use quantum cohomology instead of homology. In this setting Theorem A becomes Theorem 4.1, and Theorems C and D are combined together to Theorem 4.3. In Section 5 we prove Proposition B and Corollary E, and in Sections 6 we prove Corollary F.

Acknowledgement: We thank D. Auroux, L. Polterovich, P. Seidel, and M. Temkin for insightful comments and discussions. We are grateful to D. McDuff for important remarks and suggestions, and for pointing out some inaccuracies in the first draft of this paper. We also wish to thank the anonymous referees for their comments, which helped us to improve the presentation of the text.

2 Preliminaries, notation, and conventions.

In this section we recall some algebraic definitions and collect all the facts we need regarding symplectic toric manifolds.

2.1 Algebraic preliminaries

Convention. All rings and algebras in this paper are commutative with a unit element.

2.1.1 Semigroup algebras.

Let G be a commutative semigroup and let R be a ring. The semigroup algebra $R[G]$ is the R-algebra consisting of finite sums of formal monomials x^g, $g \in G$, with coefficients in R, and equipped with the natural algebra operations. For example, if $G = \mathbb{Z}^d$ then $R[G]$ is the algebra of Laurent polynomials $R[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$, and if $G = \mathbb{Z}_+^d$ then $R[G]$ is the polynomial algebra $R[x_1, \dots, x_d]$. In this paper R will usually be either the field $\mathbb{K}$ or the Novikov ring Λ which are introduced in Subsection 3.1.

2.1.2 Semisimple algebras.

Among the many equivalent definitions of semisimplicity we consider the following:

Definition 2.1. Let $\mathbb{F}$ be a field. A finite dimensional $\mathbb{F}$-algebra A is called *semisimple* if it contains no nilpotent elements.

In the language of algebraic geometry (see e.g. [14]), semisimplicity is equivalent to the affine scheme $\mathrm{Spec} A$ being reduced and finite over $\mathrm{Spec}\,\mathbb{F}$, and in particular zero-dimensional. Note that a Noetherian zero-dimensional scheme is reduced if and only if it is regular. If in addition $\mathit{char}\,\mathbb{F} = 0$ this is equivalent to $\mathrm{Spec} A$ being geometrically regular (i.e., $\mathrm{Spec} A \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is smooth). It follows from this geometric description that a finite dimensional algebra A is semisimple if and only if it is a direct sum of field extensions of $\mathbb{F}$. Moreover, if $\mathit{char}\,\mathbb{F} = 0$ then A is semisimple if and only if $A \otimes_{\mathbb{F}} \mathbb{L}$ is semisimple for *any* field extension $\mathbb{L}/\mathbb{F}$.

We say that $\mathbb{F}$-algebra A *contains a field as a direct summand* if it decomposes *as* $\mathbb{F}$-*algebra* into a direct sum $A = \mathbb{L} \oplus A'$, where $\mathbb{L}/\mathbb{F}$ is a field extension. Again, in geometric terms this condition means that the affine scheme $\mathrm{Spec} A$ contains a regular point as an irreducible component.

2.1.3 Non-Archimedean seminorms.

Let $\mathbb{F}$ be a field. A *non-Archimedean norm* is a function $|\cdot| \colon \mathbb{F} \to \mathbb{R}_+$ satisfying the following properties: $|\lambda\mu| = |\lambda||\mu|$, $|\lambda + \mu| \le \max\{|\lambda|, |\mu|\}$, and $|\lambda| = 0$ if and only if $\lambda = 0$. Note that the norm $|\cdot|$ defines a metric on $\mathbb{F}$. A field $\mathbb{F}$ is called *non-Archimedean* if it is equipped with a non-Archimedean norm such that $\mathbb{F}$ is complete (as a metric space). One can define the corresponding *non-Archimedean valuation* $\nu \colon \mathbb{F} \to \mathbb{R} \cup \{-\infty\}$ on $\mathbb{F}$ by setting[5] $\nu(\lambda) := \log|\lambda|$. It satisfies similar properties, i.e. $\nu(\lambda\mu) = \nu(\lambda) + \nu(\mu)$, $\nu(\lambda + \mu) \le \max\{\nu(\lambda), \nu(\mu)\}$, and $\nu(\lambda) = -\infty$ if and only if $\lambda = 0$.

Let $\mathbb{F}$ be a non-Archimedean field, and let A be an $\mathbb{F}$-algebra. A *non-Archimedean seminorm* on A is a function $\|\cdot\| \colon A \to \mathbb{R}_+$ such that $\|fg\| \le \|f\|\|g\|$, $\|f + g\| \le \max\{\|f\|, \|g\|\}$, and $\|\lambda f\| = |\lambda|\|f\|$ for all $\lambda \in \mathbb{F}$, $f, g \in A$. A seminorm is called *norm* if the following holds:

[5] Usually one defines $\nu(\lambda) := -\log|\lambda|$ and $\nu(0) = \infty$, however, we chose the above normalization to make it compatible with [17] and [37].

$\|f\| = 0$ if and only if $f = 0$. It is well known that if $\|\cdot\|$ is a non-Archimedean seminorm and $\|f\| \neq \|g\|$ then $\|f+g\| = \max\{\|f\|, \|g\|\}$. Given a non-Archimedean seminorm $\|\cdot\|$ one can consider the associated *spectral seminorm* $\|\cdot\|_{\mathrm{sp}}$ defined by $\|f\|_{\mathrm{sp}} = \lim_{k\to\infty} \sqrt[k]{\|f^k\|}$. It is easy to check that $\|\cdot\|_{\mathrm{sp}}$ is a non-Archimedean seminorm on A satisfying $\|f^k\|_{\mathrm{sp}} = \|f\|_{\mathrm{sp}}^k$ for all k. Note however, that $\|\cdot\|_{\mathrm{sp}}$ need not be a norm even if $\|\cdot\|$ is.

Lemma 2.2. *Let $(\mathbb{F}, |\cdot|)$ be a non-Archimedean algebraically closed field, and let A be a finite $\mathbb{F}$-algebra equipped with a non-Archimedean norm $\|\cdot\|$. Let $B \subseteq A$ be a local $\mathbb{F}$-subalgebra, $\mathfrak{m}$ its maximal ideal, and $e_B \in B$ its unit element. Then $B = \mathbb{F}e_B \oplus \mathfrak{m}$ as $\mathbb{F}$-modules, and $\|\lambda e_B + g\|_{\mathrm{sp}} = |\lambda|$ for all $\lambda \in \mathbb{F}$ and $g \in \mathfrak{m}$.*

Proof. The field $B/\mathfrak{m}$ is a finite extension of $\mathbb{F}$, thus $B/\mathfrak{m} = \mathbb{F}$ since $\mathbb{F}$ is algebraically closed; the decomposition now follows. Note that B is finite over $\mathbb{F}$ thus any element $g \in \mathfrak{m}$ is nilpotent, hence $\|g\|_{\mathrm{sp}} = 0$. Note that $\|e_B\| \neq 0$ since $\|\cdot\|$ is a norm, hence $\|e_B\|_{\mathrm{sp}} = \lim_{k\to\infty} \sqrt[k]{\|e_B^k\|} = \lim_{k\to\infty} \sqrt[k]{\|e_B\|} = 1$. Thus $\|\lambda e_B\|_{\mathrm{sp}} = |\lambda| > 0 = \|g\|_{\mathrm{sp}}$ for any $0 \neq \lambda \in \mathbb{F}$ and $g \in \mathfrak{m}$, which implies $\|\lambda e_B + g\|_{\mathrm{sp}} = |\lambda|$ for all $\lambda \in \mathbb{F}$ and $g \in \mathfrak{m}$. □

Corollary 2.3. Let $\mathbb{F}$ be an algebraically closed field, A a finite $\mathbb{F}$-algebra, and set $Z = \mathrm{Spec}A$. Consider a function $f \in \mathcal{O}(Z) = A$ and the linear operator $L_f \colon \mathcal{O}(Z) \to \mathcal{O}(Z)$, defined by $L_f(a) := fa$. Then:

(i) $\mathcal{O}(Z) = \oplus_{q\in Z}\mathcal{O}_{Z,q}$, where $\mathcal{O}_{Z,q}$ denotes the local ring of q, i.e. the localization of $\mathcal{O}(Z)$ with respect to the complement of the corresponding maximal ideal $\mathfrak{m}_q \subset \mathcal{O}(Z)$.

(ii) The set of eigenvalues of L_f is[6] $\{f(q)\}_{q\in Z}$.

(iii) If $\mathbb{F}$ is non-Archimedean and A is equipped with a non-Archimedean norm $\|\cdot\|$ then $\|fe_q\|_{\mathrm{sp}} = |f(q)|$ for any $q \in Z$, where e_q denotes the unit element in $\mathcal{O}_{Z,q}$.

Proof. (i) $\dim_{\mathbb{F}} A < \infty$ implies $\dim Z = 0$ and $\mathcal{O}(Z) = \oplus_{q\in Z}\mathcal{O}_{Z,q}$.

(ii) It is sufficient to show that the operator $L_{f|\mathcal{O}_{Z,q}} \colon \mathcal{O}_{Z,q} \to \mathcal{O}_{Z,q}$ has unique eigenvalue $f(q)$. Note that $fe_q = f(q)e_q + g$, where $g \in \mathfrak{m}_q$ is a nilpotent element. Thus $L_{f|\mathcal{O}_{Z,q}} - f(q)Id_{\mathcal{O}_{Z,q}}$ is nilpotent, which implies the statement.

(iii) Note that $fe_q = f(q)e_q + g$, where $g \in \mathfrak{m}_q$; thus $\|fe_q\|_{\mathrm{sp}} = |f(q)|$ by Lemma 2.2.

□

2.2 Symplectic toric manifolds

Notation. Throughout the paper M denotes a lattice, i.e. a free abelian group of finite rank d, and $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ its dual lattice. We use the notation $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ for the corresponding pair of dual vector spaces of dimension d. We

[6] Recall that the value of a function $f \in \mathcal{O}(Z)$ at $q \in Z$ is defined to be the class of f modulo $\mathfrak{m}_q$. Since $\mathbb{F}$ is algebraically closed and $\mathcal{O}(Z)$ is finitely generated $\mathbb{F}$-algebra $f(q) \in \mathcal{O}(Z)/\mathfrak{m}_q = \mathbb{F}$.

shall use the notation T_N and T_M for the algebraic tori $T_N = \operatorname{Spec}\mathbb{F}[M] = N \otimes_{\mathbb{Z}} \mathbb{F}^*$ and $T_M = \operatorname{Spec}\mathbb{F}[N] = M \otimes_{\mathbb{Z}} \mathbb{F}^*$ over the base field $\mathbb{F}$, where $\mathbb{F}^*$ denotes the set of non-zero elements of $\mathbb{F}$.

Let $T = M_{\mathbb{R}}/M = N \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z})$ be the compact torus of dimension d with lattice of characters M and lattice of cocharacters N. A $2d-$dimensional *symplectic toric manifold* is a closed connected symplectic manifold (X, ω) equipped with an effective Hamiltonian T-action, and a moment map $\mu\colon X \to Lie(T)^* = M_{\mathbb{R}}$ generating (locally) the T-action on X. In other words, for any $g \in T$ there is $x \in X$ such that $g(x) \neq x$, and for any $\xi \in Lie(T)$ and $x \in X$ we have: $d_x\mu(\xi) = \omega(X_\xi, \cdot)$, where X_ξ denotes the vector field induced by ξ under the exponential map.

By a well known theorem of Atiyah [2] and Guillemin-Sternberg [24], the image of the moment map $\Delta := \mu(X) \subset M_{\mathbb{R}}$ is the convex hull of the images of the fixed points of the action. It was proved by Delzant [11] that the moment polytope $\Delta \subset M_{\mathbb{R}}$ has the following properties: (i) there are d edges meeting at every vertex v (*simplicity*), (ii) the slopes of all edges are rational (*rationality*), and (iii) for any vertex v the set of primitive integral vectors along the edges containing v is a basis of the lattice M (*smoothness*). Such a polytope is called *a Delzant polytope*. Recall that any Delzant polytope can be (uniquely) described as the intersection of (the minimal set of) closed half-spaces with rational slopes. Namely, there exist $n_1, \dots, n_r \in N = Hom_{\mathbb{Z}}(M, \mathbb{Z})$ and $\lambda_1, \dots, \lambda_r \in \mathbb{R}$, where r is the number of facets (i.e. faces of codimension one) of Δ such that

$$\Delta = \{m \in M_{\mathbb{R}} \mid (m, n_k) \geq \lambda_k \ \text{ for every } k\}. \tag{2.2.1}$$

Moreover, Delzant gave a complete classification of symplectic toric manifolds in terms of the combinatorial data encoded by a Delzant polytope. In [11] he associated to a Delzant polytope $\Delta \subset M_{\mathbb{R}}$ a closed symplectic manifold $(X^{2d}_\Delta, \omega_\Delta)$ together with a Hamiltonian T-action and a moment map $\mu_\Delta\colon X^{2d}_\Delta \to M_{\mathbb{R}}$ such that $\mu(X^{2d}_\Delta) = \Delta$. He showed that $(X^{2d}_\Delta, \omega_\Delta)$ is isomorphic (as Hamiltonian T-space) to (X^{2d}, ω), and proved that two symplectic toric manifolds are (equivariantly) symplectomorphic if and only if their Delzant polytopes differ by a translation and an element of $Aut(M)$.

The precise relations between the combinatorial data of the Delzant polytope Δ and the symplectic structure of X are as follows: the faces of Δ of dimension d' are in one-to-one correspondence with the closed connected equivariant submanifolds of X of (real) dimension $2d'$, namely a face α corresponds to the submanifold $\mu^{-1}(\alpha)$. In particular facets of Δ correspond to submanifolds of codimension 2. Let $z_1, \dots, z_r \in H^2(X, \mathbb{Z})$ be the Poincaré dual of the homology classes of $D_1, \dots, D_r$, where D_k is the submanifold corresponding to the facet given by $(m, n_k) = \lambda_k$. Then the cohomology class $[\omega]$ and the first Chern class $c_1(X)$ are given by

$$\frac{1}{2\pi}[\omega] = -\sum_{i=1}^{r} \lambda_k\, z_k, \quad \text{and} \quad c_1(X) = \sum_{i=1}^{r} z_k \tag{2.2.2}$$

In what follows it will be convenient for us to adopt the algebra-geometric point of view on toric varieties which we now turn to describe.

2.3 Algebraic Toric Varieties.

In this subsection we briefly discuss toric varieties from the algebra-geometric point of view. For a complete exposition of the subject see Fulton's book [21] and Danilov's survey [10].

A subset $\sigma \subset N_{\mathbb{R}}$ is called a *rational, polyhedral cone* if σ is a positive span of finitely many vectors $n_i \in N$, i.e. $\sigma = \mathrm{Span}_{\mathbb{R}_+}\{n_1, \ldots, n_k\}$, $n_i \in N$. It is not hard to check that σ is a rational, polyhedral cone if and only if there exist $m_1, \ldots, m_l \in M \subset \mathrm{Hom}(N_{\mathbb{R}}, \mathbb{R})$ such that $\sigma = \cap_{i=1}^{l} m_i^{-1}(\mathbb{R}_+)$. A rational, polyhedral cone σ is called strictly convex if it contains no lines, i.e. $\sigma \cap (-\sigma) = \{0\}$. For a rational, polyhedral cone $\sigma \subset N_{\mathbb{R}}$ we define the *dual cone* $\check{\sigma}$ to be $\check{\sigma} = \{m \in M_{\mathbb{R}} \,|(m,n) \geq 0 \; \forall n \in \sigma\}$, which is again rational and polyhedral. A *face* τ of a rational, polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is defined to be the intersection of σ with a supporting hyperplane, i.e. $\tau = \sigma \cap \mathrm{Ker}(m)$ for some $m \in \check{\sigma}$. It is easy to see that a face of a (strictly convex) rational, polyhedral cone is again a (strictly convex) rational, polyhedral cone. Faces of codimension one are called *facets*.

For a strictly convex, rational, polyhedral cone σ one can assign the commutative semigroup $M \cap \check{\sigma}$. Note that since $\check{\sigma}$ is rational and polyhedral this semigroup is finitely generated, hence the semigroup algebra $\mathbb{F}[M \cap \check{\sigma}]$ is also finitely generated. We define *affine toric variety* X_σ over $\mathbb{F}$ to be $X_\sigma = \mathrm{Spec}\,\mathbb{F}[M \cap \check{\sigma}]$. If $\tau \subseteq \sigma$ is a face then $X_\tau \hookrightarrow X_\sigma$ is an open subvariety. In particular, since σ is strictly convex, the affine toric variety X_σ contains the torus $X_{\{0\}} = \mathrm{Spec}\,\mathbb{F}[M] = N \otimes_{\mathbb{Z}} \mathbb{F}^* = T_N$ as a dense open subset (as before, $\mathbb{F}^*$ denotes the set of non-zero elements of $\mathbb{F}$). Furthermore, the action of the torus on itself extends to the action on X_σ.

Recall that a collection Σ of strictly convex, rational, polyhedral cones in $N_{\mathbb{R}}$ is called a *fan* if the following two conditions hold:

1. If $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ is a face then $\tau \in \Sigma$.
2. If $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau$ is a common face of σ and τ.

A fan Σ is called *complete* if $|\Sigma| := \cup_{\sigma\in\Sigma}\sigma = N_{\mathbb{R}}$. One-dimensional cones in Σ are called *rays*. The set of cones of dimension k in Σ is denoted by Σ^k, and the primitive integral vector along a ray ρ is denoted by n_ρ.

Given a (complete) fan $\Sigma \subset N_{\mathbb{R}}$ one can construct a *(complete) toric variety* $X_\Sigma = \cup_{\sigma\in\Sigma} X_\sigma$ by gluing X_σ and X_τ along $X_{\sigma\cap\tau}$. Recall that X_Σ has only orbifold singularities if and only if all the cones in Σ are simplicial (in this case it is called *quasi-smooth*); and X_Σ is smooth if and only if for any cone $\sigma \in \Sigma$ the set of primitive integral vectors along the rays of σ forms a part of a basis of the lattice N.

The torus T_N acts on X_Σ and decomposes it into a disjoint union of orbits. To a cone $\sigma \in \Sigma$ one can assign an orbit $O_\sigma \subset X_\sigma$, canonically isomorphic to $\mathrm{Spec}\,\mathbb{F}[M \cap \sigma^\perp]$, where $\sigma^\perp = \{m \in M_{\mathbb{R}} \,|(m,n) = 0 \; \forall n \in \sigma\}$. This defines a one-to-one order reversing correspondence between the cones in Σ and the orbits in X_Σ. In particular orbits of codimension one correspond to rays $\rho \in \Sigma$ and we denote their closures by D_ρ. Thus $\{D_\rho\}_{\rho\in\Sigma^1}$ is the set of T_N-equivariant *primitive Weil divisors*[7] on the variety X_Σ. We remark that the set $\{D_\rho\}_{\rho\in\Sigma^1}$ coincides with the set $\{D_i\}_{1\le i\le r}$ in the setting of the previous subsection.

Let Σ be a fan in $N_\mathbb{R}$ and let X_Σ be the corresponding toric variety. Let $\mathcal{L}$ be an algebraic line bundle on X_Σ. By a *trivialization* of $\mathcal{L}$ we mean an isomorphism $\phi\colon \mathcal{L}_{|T_N} \to \mathcal{O}_{T_N}$ considered up to the natural action of $\mathbb{F}^*$. Recall that any algebraic line bundle on a torus is trivial, hence any algebraic line bundle $\mathcal{L}$ on X_Σ can be equipped with a trivialization. To a pair $(\mathcal{L},\phi)$ one can assign a *piecewise linear integral function* F on the fan Σ, i.e. a function F on $|\Sigma|$ such that $F_{|\sigma}$ is linear for any $\sigma\in\Sigma$ and $F(N)\subseteq\mathbb{Z}$. This defines a bijective homomorphism between the group (with respect to the tensor product) of pairs $(\mathcal{L},\phi)$ and the additive group of piecewise linear integral functions F:

$$F \longleftrightarrow \mathcal{O}(-\sum_{\rho\in\Sigma^1} F(n_\rho)D_\rho).$$

Furthermore, a change of the trivialization corresponds to adding a global integral linear function to F. In the language of divisors one can rephrase the above correspondence as follows: *real/rational/integral*[8] *piecewise linear functions* on the fan Σ are in one-to-one correspondence with $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-*Cartier* T_N-*equivariant divisors*. Such divisors are called T-*divisors*.

Let $(\mathcal{L},\phi)$ be a $T-$divisor, and let F be a corresponding function. Then $\mathcal{L}$ is globally generated if and only if F is *convex*, i.e. $F(tn+(1-t)n')\ge tF(n)+(1-t)F(n')$ for all $n,n'\in N_\mathbb{R}$ and $0\le t\le 1$, and $\mathcal{L}$ is ample if and only if F is *strictly convex*, i.e. F is convex, and its maximal linearity domains are cones in Σ.

Remark 2.4. Note that since any section of $\mathcal{L}$ is completely determined by its restriction to the big orbit, the trivialization ϕ identifies the global sections of $\mathcal{O}_{X_\Sigma}(D)$ with functions on T_N. Furthermore, the following holds

$$H^0(X_\Sigma,\mathcal{O}_{X_\Sigma}(D)) \simeq \mathrm{Span}\{x^m\}_{m\in\Delta_F\cap M}\subset\mathcal{O}(T_N), \tag{2.3.3}$$

where

$$\Delta_F=\Delta_{(\mathcal{L},\phi)}=\{m\in M_\mathbb{R}\,|\,(m,n_\rho)\ge F(n_\rho),\ \text{for every }\rho\}. \tag{2.3.4}$$

Note also, that if one changes the trivialization then Δ_F is translated by the corresponding element of M.

If $\mathcal{L}$ is ample then one can reconstruct the fan Σ from the polytope Δ_F. Namely, cones in Σ are in one-to-one order reversing correspondence with the faces of Δ_F. To a face

[7]Recall that if X is a singular variety then one must distinguish between Weil divisors (i.e. formal finite sums of irreducible subvarieties of codimension one) and Cartier divisors (i.e. global sections of the sheaf $\mathcal{K}^*_X/\mathcal{O}^*_X$, or equivalently, invertible subsheaves(=line subbundles) of $\mathcal{K}$, where $\mathcal{K}$ denotes the sheaf of rational functions on X). There is a natural homomorphism $Cartier(X)\to Weil(X)$ and the corresponding homomorphism between the class groups of divisors $Pic(X)\to Cl(X)$, however these maps in general need not be surjective or injective, but for smooth varieties these are isomorphisms. For any toric variety X these maps are injective, since X is normal. If in addition X is quasi-smooth then at least $Pic(X)\otimes_\mathbb{Z}\mathbb{Q}\to Cl(X)\otimes_\mathbb{Z}\mathbb{Q}$ is an isomorphism.

[8]i.e. $F(N)\subseteq\mathbb{R}/\mathbb{Q}/\mathbb{Z}$.

$\gamma \subseteq \Delta_F$ we assign the cone σ being the dual cone to the inner angle of Δ_F at γ (see [10] §5.8). Furthermore, if m is a vertex of Δ_F and $\sigma_m \in \Sigma$ is the corresponding cone, then $F_{|\sigma_m} = m$. Thus F can also be reconstructed from the polytope Δ_F. This gives a bijective correspondence between polytopes of dimension d in $M_{\mathbb{R}}$ and pairs (Σ, F) as above. Moreover, it is known (see the very end of this section and [21] Sections 4.1 and 4.2) that choosing a strictly convex function F on Σ as above is equivalent to introducing a symplectic structure ω on X_Σ (such that the torus action is Hamiltonian) together with a moment map. Under this identification, the polytope Δ_F (2.3.4) coincides with the polytope Δ (2.2.1) of the symplectic manifold (X_Σ, ω) with the corresponding moment map.

Let F be an integral strictly convex piecewise linear function on $|\Sigma|$. Recall that the orbits in $X_\Sigma \subset N_{\mathbb{R}}$ are in one-to-one order reversing correspondence with the cones in Σ, hence they are in one-to-one order preserving correspondence with the faces of Δ_F. Let $\gamma \subset M_{\mathbb{R}}$ be a face of Δ_F, let $\sigma_\gamma \in \Sigma$ be the corresponding cone, and let $V = \overline{O}_{\sigma_\gamma} \subset X_\Sigma$ be the closure of the corresponding orbit. Then V has a structure of a toric variety with respect to the action of the torus $\operatorname{Spec} \mathbb{C}[M \cap \sigma_\gamma^\perp]$, and the restriction $\mathcal{L}_V$ of $\mathcal{L}$ to V is an ample line bundle on V; however, $\mathcal{L}_V$ has no distinguished trivialization. To define a trivialization one must pick an integral point p in the affine space $\operatorname{Span}(\gamma)$ (e.g. a vertex of γ) and this defines an isomorphism between $\mathcal{L}_V$ and the line bundle associated to the polytope $\gamma - p \subset \sigma_\gamma^\perp$.

Let Σ be a fan in $N_{\mathbb{R}}$ and let X_Σ be the corresponding toric variety. By a *log-form* we mean a rational differential 1-form having at worst simple poles along the components of $X_\Sigma \setminus T_N$. Recall that the sheaf $\Omega^1_{X_\Sigma}(\log)$ of log-forms is trivial vector bundle canonically isomorphic to $M \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma}$ (we assign to $m \in M$ the form $\frac{dx^m}{x^m}$). Moreover there exists an exact sequence $0 \to \Omega^1_{X_\Sigma} \to \Omega^1_{X_\Sigma}(\log) \to \oplus_{\rho \in \Sigma^1} \mathcal{O}_{D_\rho} \to 0$, where the last map is the sum of residues. It follows from the exact sequence above that $K_\Sigma = -\sum_{\rho \in \Sigma^1} D_\rho$ is the canonical (Weil) divisor on X_Σ. If canonical divisor is $\mathbb{Q}-$Cartier (e.g. X_Σ is quasi-smooth) then the canonical divisor corresponds to the rational piecewise linear function F_K defined by the following property: $F_K(n_\rho) = 1$ for any $\rho \in \Sigma^1$.

The dual notion to log-form is *log-derivative. Log-vector fields* also form a trivial vector bundle canonically isomorphic to $N \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma}$, namely to any $n \in N$ corresponds the log-derivative ∂_n defined by $\partial_n x^m = (m, n) x^m$. The notion of log-derivative will be useful in this paper to make proofs coordinate free.

Let $\Delta \subset M_{\mathbb{R}}$ be a polytope containing 0 in its interior. The *dual polytope* $\Delta^* \subset N_{\mathbb{R}}$ is defined to be

$$\Delta^* = \{n \in N_{\mathbb{R}} \mid (m, n) \geq -1, \text{ for every } m \in \Delta\}.$$

The polytope $\Delta \subset M_{\mathbb{R}}$ is called *reflexive* if (i) 0 is contained in its interior, and (ii) both Δ and Δ^* are integral polytopes. Note that if Δ is reflexive then 0 is the only integral point in its interior. It is not hard to check (cf. [5]) that Δ is reflexive if and only if its dual Δ^* is reflexive.

A complete algebraic variety is called *Fano* if its anti-canonical class is Cartier and ample. Recall that if X_Σ is Fano and $K = -\sum D_\rho = -\sum F_K(n_\rho) D_\rho$ is the standard canonical T-divisor then $\Delta_{-F_K} = \Delta_{F_{-K}}$ is reflexive. Moreover, if Δ is reflexive then there exists a unique toric Fano variety X_Σ such that $\Delta = \Delta_{F_K}$, where $K = -\sum D_\rho$, and F_K is as above.

Let X_Σ be a toric Fano variety, $\Delta = \Delta_{F_{-K}}$ be the reflexive polytope assigned to the anticanonical divisor $-K = \sum D_\rho$, and Δ^* be the dual reflexive polytope. Consider the dual toric Fano variety $X^*_\Sigma = X_{\Sigma^*}$ assigned to the polytope Δ^*. Then the fan Σ coincides with the fan over the faces of Δ^*, and the fan Σ^* is the fan over the faces of Δ.

Let now $X = X_\Sigma$ and $X^* = X^*_\Sigma$ be a pair of dual toric Fano varieties, and assume that X is smooth. Then any maximal cone in Σ is simplicial, and is generated by a basis of N; hence the facets of the dual polytope Δ^* are basic simplexes. Thus the irreducible components of the complement of the big orbit in X^* are isomorphic to $\mathbb{P}^{d-1}$. Furthermore, the restriction of the anticanonical linear system $\mathcal{O}_{X^*}(-K_{X^*})$ to such a component is isomorphic to the anti-tautological line bundle $\mathcal{O}_{\mathbb{P}^{d-1}}(1)$.

Remark 2.5. The following facts will be useful: (i) (see [21] section 3.2) the Euler characteristic of a quasi-smooth complete toric variety is equal to $|\Sigma^d|$, and (ii) (Kushnirenko's theorem, a particular case of Bernstein's theorem - see [21] section 5.3) if D is an ample T-divisor on a toric variety X_Σ, and $\Delta \subset M_\mathbb{R}$ is the corresponding polytope, then the intersection number D^d is given by $D^d = d!\mathrm{Volume}(\Delta)$, where the volume is relative to the lattice M.

Assume now that $\mathbb{F} = \mathbb{C}$. Given an ample T-divisor $(\mathcal{L}, \phi)$ on a toric variety X_Σ, one can assign to it a symplectic form $\omega_{\mathcal{L},\phi}$ in the following way: first note that ϕ defines a distinguished (up-to the action of a symmetric group and up-to a common multiplicative factor) basis in $H^0(X_\Sigma, \mathcal{L}^{\otimes r})$ for any r. Let $X_\Sigma \hookrightarrow \mathbb{P} = \mathbb{P}(H^0(X_\Sigma, \mathcal{L}^{\otimes r})^*)$ be the natural embedding (where r is assumed to be large enough). Recall that projective spaces have canonical symplectic structures provided by the Fubini-Study forms. Now we simply pull back the Fubini-Study symplectic form of volume 1 from $\mathbb{P}$ to X_Σ, and since it is invariant under the action of the symmetric group, we get a well defined symplectic form on X_Σ. To make this construction independent of r and to make the moment polytope compatible with $\Delta_{(\mathcal{L},\phi)}$ all we have to do is to multiply the form by $\frac{2\pi}{r}$. We denote this *normalized symplectic* form by $\omega_{\mathcal{L},\phi}$ or ω_F if F is the strictly convex piecewise linear function associated to $(\mathcal{L}, \phi)$. Thus $(\mathcal{L}, \phi)$ defines the structure of a symplectic toric manifold on X_Σ. Furthermore, the action of the compact torus $T = N \otimes_\mathbb{Z} S^1 \subset N \otimes_\mathbb{Z} \mathbb{C}^* = T_N$ is Hamiltonian. Such a manifold admits a moment map $\mu_{\omega_F} \colon X \to Lie(T)^* = M_\mathbb{R}$. In our case μ_{ω_F} is defined by

$$\mu_{\omega_F}(p) = \frac{\sum_{m \in \Delta_F} |x^m(p)|^2\, m}{\sum_{m \in \Delta_F} |x^m(p)|^2},$$

and its image is the polytope $\Delta_F = \Delta_{(\mathcal{L},\phi)}$ (cf. [21] sections 4.1 and 4.2).

3 The Quantum Cohomology

3.1 Symplectic Definition

We start with a symplectic definition of the quantum (co)homology of a $2d$-dimensional symplectic manifold (X, ω), using Gromov-Witten invariants (see [36]). For simplicity, throughout the text we assume that (X, ω) is semipositive manifold (see e.g. [36], Subsection 6.4). The class of symplectic toric Fano manifolds is a particular example.

By abuse of notation, we write $\omega(A)$ and $c_1(A)$ for the results of evaluation of the cohomology classes $[\omega]$ and c_1 on $A \in H_2(X;\mathbb{Z})$. Here $c_1 \in H^2(X;\mathbb{Z})$ denotes the first Chern class of X. We denote by $\mathbb{K}^{\downarrow}$ the field of generalized Laurent series over $\mathbb{C}$:

$$\mathbb{K}^{\downarrow} = \Big\{ \sum_{\lambda \in \mathbb{R}} a_\lambda s^\lambda \mid a_\lambda \in \mathbb{C}, \text{ and } \{\lambda \,|\, a_\lambda \neq 0\} \text{ is discrete and bounded above in } \mathbb{R} \Big\} \quad (3.1.1)$$

Similarly, we define $\mathbb{K}^{\uparrow}$ to be the field of generalized Laurent series where the set $\{\lambda \,|\, a_\lambda \neq 0\}$ is discrete and bounded from below in $\mathbb{R}$. In the definition of the quantum homology we shall use the Novikov ring $\Lambda^{\downarrow} := \mathbb{K}^{\downarrow}[q, q^{-1}]$, and in the definition of the quantum cohomology we use the "dual" ring $\Lambda^{\uparrow} := \mathbb{K}^{\uparrow}[q, q^{-1}]$. By setting $\deg(s) = 0$ and $\deg(q) = 2$ we introduce the structure of graded rings on $\Lambda^{\downarrow}$ and $\Lambda^{\uparrow}$.

As a graded module the quantum (co)homology algebra of (X, ω) is given by:

$$QH_*(X, \omega) = H_*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda^{\downarrow}, \quad QH^*(X, \omega) = H^*(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda^{\uparrow}.$$

The grading on $QH_*(X, \omega)$ (respectively on $QH^*(X, \omega)$) is set by $\deg(a \otimes s^\lambda q^j) = \deg(a) + 2j$, where $\deg(a)$ is the degree of the class a in the (co)homology of (X, ω). For $a \in H_i(X, \mathbb{Q})$ and $b \in H_j(X, \mathbb{Q})$, define $(a \otimes 1) * (b \otimes 1) \in QH_{i+j-2d}(X, \omega)$ by

$$(a \otimes 1) * (b \otimes 1) = \sum_{A \in H_2(X)} (a * b)_A \otimes s^{-\omega(A)} q^{-c_1(A)},$$

where $(a * b)_A \in H_{i+j-2d+2c_1(A)}(M, \mathbb{Q})$ is defined by the requirement that

$$(a * b)_A \circ c = GW_A(a, b, c), \quad \text{for all } c \in H_*(X, \mathbb{Q}).$$

Here $\circ$ is the usual intersection index and $GW_A(a, b, c)$ denotes the Gromov-Witten invariant that, roughly speaking, counts the number of pseudo-holomorphic spheres representing the class A and intersecting a triple of generic representatives of the classes $a, b, c \in H_*(X, \mathbb{Q})$ (see e.g. [36], [41], or [42] for the precise definition). The product $*$ is extended to the whole $QH_*(X, \omega)$ by linearity over $\Lambda^{\downarrow}$. The quantum multiplication is a "quntisation" of the classical cap-product in singular homology. It is commutative, associative (see [36, 41, 42, 31, 49]). Note that the fundamental class $[X]$ is the unit element with respect to the quantum multiplication $*$, and that $QH_*(X, \omega)$ is a finite-rank module over $\Lambda^{\downarrow}$. Moreover, if $a, b \in QH_*(X, \omega)$ are graded elements, then $\deg((a \otimes 1) * (b \otimes 1)) = \deg(a) + \deg(b) - 2d$.

Although the above definition is more geometric, in what follows we shall mainly use the quantum cohomology due to some technicalities. The quantum product in this case is defined as follows: for $\alpha, \beta \in H^*(X, \mathbb{Q})$ with Poincaré duals $a = \mathrm{PD}(\alpha), b = \mathrm{PD}(\beta)$ we have $(\alpha \otimes 1) * (\beta \otimes 1) = \mathrm{PD}_q^{-1}(a * b)$ where the quantum Poincaré duality map $\mathrm{PD}_q \colon QH^*(X, \omega) \to QH_{2d-*}(X, \omega)$ is given by $\mathrm{PD}_q(\gamma \otimes s^\lambda q^j) = \mathrm{PD}(\gamma) \otimes s^{-\lambda} q^{-j}$; hence

$$(\alpha \otimes 1) * (\beta \otimes 1) = \sum_{A \in H_2(X)} \mathrm{PD}^{-1}((a * b)_A) \otimes s^{\omega(A)} q^{c_1(A)}.$$

As mentioned in the introduction, our main object of study is the subalgebra $QH_{2d}(X, \omega)$. It is not hard to check that it is a commutative algebra of finite rank over $\mathbb{K}^\downarrow$. Quantum Poincaré duality induces an isomorphism between the quantum homology and cohomology (see [36] remark 11.1.16). This isomorphism is compatible with the natural isomorphism $\mathbb{K}^\downarrow \to \mathbb{K}^\uparrow$ taking s to s^{-1}, and it identifies $QH_{2d}(X, \omega)$ with $QH^0(X, \omega)$. Hence, in what follows we will work with the algebra $QH^0(X, \omega)$ over the field $\mathbb{K}^\uparrow$ instead of the algebra $QH_{2d}(X, \omega)$ over $\mathbb{K}^\downarrow$.

Convention. From this point on we set $\mathbb{K} := \mathbb{K}^\uparrow$ and use the Novikov ring $\Lambda := \mathbb{K}[q, q^{-1}]$.

Remark 3.1. Note that the field $\mathbb{K}$ is a non-Archimedean field with respect to the non-Archimedean norm $\left|\sum a_\lambda s^\lambda\right| := 10^{-\inf\{\lambda \,|\, a_\lambda \neq 0\}}$. Furthermore, by [13] Corollary 18.3.2 it is Henselian (relative to its algebraic closure), hence algebraically closed by [13] Corollary 17.3.3. Note also that the map $\|\cdot\| \colon QH^*(X, \omega) \to \mathbb{R}_+$ defined by $\|\sum_{\lambda, j} a_{\lambda j} s^\lambda q^j\| = 10^{-\inf\{\lambda \,|\, \exists\, a_{\lambda j} \neq 0\}}$, where $a_{\lambda j} \in H^*(X, \mathbb{C})$, is a non-Archimedean norm on the quantum cohomology algebras $QH^*(X, \omega)$ and $QH^0(X, \omega)$.

3.2 Batyrev's Description of the Quantum Cohomology

In [4], Batyrev proposed a combinatorial description of the quantum cohomology algebra of toric Fano manifolds, using a "quantum" version of the "classical" Stanley-Reisner ideal. This was later proved by Givental in [22, 23]. For a different approach to the proof we refer the reader to McDuff-Tolman [37] and Cieliebak-Salamon [8].

Before describing Batyrev's work let us first briefly recall the definition of the classical cohomology of toric Fano manifolds. The complete details can be found in [10] §10,11,12, and [21] section 3.2 and Chapter 5.

Let Σ be a simplicial fan, and let X_Σ be the corresponding toric variety over $\mathbb{C}$. It is known that any cohomology class has an equivariant representative. Moreover, $H^{2k}(X_\Sigma, \mathbb{Q})$ is generated as a vector space by the classes[9] of the closures of k-dimensional orbits. Note that any such closure V is an intersection of some equivariant divisors D_ρ with appropriate multiplicity that depends on the singularity of the X_Σ along V. To be more precise, if $V = \overline{O_\sigma}$, $\sigma \in \Sigma^k$, and $\rho_1, \ldots, \rho_k$ are the rays of σ then $V = \mathrm{mult}(\sigma) \prod_{i=1}^k D_{\rho_i}$, where $\mathrm{mult}(\sigma)$ denotes the covolume of the sublattice spanned by $n_{\rho_1}, \ldots, n_{\rho_k}$ in the lattice $\mathrm{Span}(\sigma) \cap N$. Thus we have a surjective homomorphism of algebras $\psi \colon \mathbb{Q}[z_\rho]_{\rho \in \Sigma^1} \to H^{2*}(X_\Sigma, \mathbb{Q})$, where $\mathbb{Q}[z_\rho]_{\rho \in \Sigma^1}$ is the polynomial algebra in free variables z_ρ indexed by the rays $\rho \in \Sigma^1$.

[9]Below we shall not destinguish between the closure of the orbit and its class in the cohomology.

Let $x^m \in \mathbb{C}[M]$ be a rational function on X_Σ. Then $div(x^m) = \sum_{\rho\in\Sigma^1}(m, n_\rho)D_\rho$. Thus $\sum_{\rho\in\Sigma^1}(m, n_\rho)z_\rho \in \mathrm{Ker}(\psi)$ for any $m \in M$. We denote by $P(X_\Sigma) \subset \mathbb{Q}[z_\rho]_{\rho\in\Sigma^1}$ the ideal generated by $\sum_{\rho\in\Sigma^1}(m, n_\rho)z_\rho$, $m \in M$. Note that if $\rho_1, \ldots, \rho_k$ do not generate a cone in Σ then $\cap_{i=1}^k D_{\rho_i} = \emptyset$, and thus $\prod_{i=1}^k z_{\rho_i} \in \mathrm{Ker}(\psi)$. We denote by $SR(X_\Sigma) \subset \mathbb{Q}[z_\rho]_{\rho\in\Sigma^1}$ the Stanley-Reisner ideal, i.e. the ideal generated by $\prod_{i=1}^k z_{\rho_i}$ where $\rho_1, \ldots, \rho_k$ do not generate a cone in Σ. It is well known that $\mathrm{Ker}(\psi) = P(X_\Sigma) + SR(X_\Sigma)$. Hence

$$H^{2*}(X_\Sigma, \mathbb{Q}) = \frac{\mathbb{Q}[z_\rho]_{\rho\in\Sigma^1}}{P(X_\Sigma) + SR(X_\Sigma)}.$$

We turn now to Batyrev's description of the quantum cohomology. We say that the set of rays $\rho_1, \ldots, \rho_k$ is a *primitive collection* if $\rho_1, \ldots, \rho_k$ do not generate a cone in Σ while any proper subset does generate a cone in Σ. Note that the set of monomials $\prod_{i=1}^k z_{\rho_i}$ assigned to primitive collections forms a minimal set of generators of $SR(X_\Sigma)$. The quantum version of the Stanley-Reisner ideal $QSR(X_\Sigma)$ is generated by the quantization of the minimal set of generators above.

More precisely, assume that we are given a smooth Fano toric variety X_Σ, and a piecewise linear strictly convex function F on $|\Sigma|$ defining an ample $\mathbb{R}$-divisor on X_Σ. Let C be a primitive collection of rays. Then $\sum_{\rho\in C} n_\rho$ belongs to a cone $\sigma_C \in \Sigma$, and we assume that σ_C is the minimal cone containing it. It is not hard to check that σ_C does not contain ρ for all $\rho \in C$ (cf [4]). Since X_Σ is smooth, $\sum_{\rho\in C} n_\rho = \sum_{\rho\subseteq\sigma_C} a_\rho n_\rho$, where a_ρ are strictly positive integers. We define the quantization of the generator $\prod_{\rho\in C} z_\rho$ to be

$$\prod_{\rho\in C} q^{-1}s^{F(n_\rho)}z_\rho - \prod_{\rho\subseteq\sigma_C}(q^{-1}s^{F(n_\rho)}z_\rho)^{a_\rho}$$

The quantum version of $SR(X_\Sigma)$ is the ideal $QSR(X_\Sigma, F) \subset \Lambda[z_\rho]_{\rho\in\Sigma^1}$ generated by the quantization of the minimal set of generators. We define Batyrev's quantum cohomology to be

$$QH_B^*(X_\Sigma, F; \Lambda) := \frac{\Lambda[z_\rho]_{\rho\in\Sigma^1}}{P(X_\Sigma) + QSR(X_\Sigma, F)},$$

and

$$QH_B^*(X_\Sigma, F; \mathbb{K}) := QH_B^*(X_\Sigma, F; \Lambda) \otimes_\Lambda \Lambda \,/\, \langle q - 1\rangle.$$

Recall that (X, ω) and (X_Σ, F) represents the same symplectic toric Fano manifold as explained in Subsection 2.3 above. For a proof of the following result using notation and conventions similar to ours[10] see Proposition 5.2 in [37]

Theorem 3.2. *For a symplectic toric Fano manifold $(X, \omega) = (X_\Sigma, F)$ there is a ring isomorphism*

$$QH_B^*(X_\Sigma, F; \Lambda) \simeq QH^*(X, \omega), \tag{3.2.2}$$

given by the map which sends z_ρ to the Poincaré dual of the homology class of D_ρ.

We remark that the identification (3.2.2) may fail without the Fano assumption (see [9] example 11.2.5.2 and [37]).

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[10] If we order the rays in Σ, or, equivalently, the facets of the polytope Δ_F, then the dictionary between our notation and the notation in [37] is as follows: $n_{\rho_i} \leftrightsquigarrow -\eta_i$, $z_{\rho_i} \leftrightsquigarrow y_i$, $F(n_{\rho_i}) \leftrightsquigarrow -\eta_i(D_i)$, $\mu \leftrightsquigarrow \Phi$, $D_{\rho_i} \leftrightsquigarrow \Phi^{-1}(D_i)$, $s \leftrightsquigarrow t$ and $a_{\rho_i} \leftrightsquigarrow c_i$.

3.3 The Landau-Ginzburg Superpotential

Here we present an analytic description of the quantum cohomology algebra for symplectic toric Fano varieties which arose from the study of the corresponding Landau-Ginzburg model in Physics [32, 47, 26]. We follow the works of Batyrev [4], Givental [22], Hori-Vafa [26], Fukaya-Oh-Ohta-Ono [20]. For a symplectic toric Fano manifold X we introduce a superpotential considered as a section of the anti-canonical line bundle on the dual toric Fano variety X^*_Σ. We construct a map from the quantum cohomology algebra of X to the algebra of functions on the Jacobian scheme of the superpotential, and prove that it is an isomorphism.

Let X_Σ be a smooth Fano toric variety, and let F be a piecewise linear strictly convex function on $|\Sigma|$ defining an ample $\mathbb{R}$-divisor on X_Σ. Consider the Landau-Ginzburg superpotential

$$W_{F,\Sigma} := \sum_{\rho\in\Sigma^1} s^{-F(n_\rho)} x^{n_\rho}$$

defined on the torus $\mathrm{Spec}\mathbb{K}[N] = T_M$. This function can be considered also as a section of the anti-canonical line bundle on the dual toric Fano variety X^*_Σ over the field $\mathbb{K}$ (see Remark 2.4). One can assign to it the Jacobian ring $\mathbb{K}[N]/J_{W_{F,\Sigma}}$, where $J_{W_{F,\Sigma}}$ denotes the Jacobian ideal, i.e. the ideal generated by all partial derivatives of $W_{F,\Sigma}$, or, equivalently, by all log-derivatives[11] of the superpotential: $\{\partial_m W_{F,\Sigma}\}_{m\in M}$.

Proposition 3.3. *If (X_Σ, F) is a rational smooth symplectic toric Fano variety, and $W_{F,\Sigma}$ as above then*

$$QH^*(X,\omega) \cong QH^*_B(X_\Sigma, F; \Lambda) \cong \Lambda[N]/J_{W_{F,\Sigma}},$$

and in particular

$$QH^0(X,\omega) \cong QH^*_B(X_\Sigma, F; \mathbb{K}) \cong \mathbb{K}[N]/J_{W_{F,\Sigma}}.$$

For the proof of Proposition 3.3 we shall need the following lemma.

Lemma 3.4. *Let $X = X_\Sigma$ be a smooth toric Fano variety over the base field $\mathbb{K}$, F be a piecewise linear strictly convex function on $|\Sigma|$, and $W = W_{F,\Sigma}$ be the corresponding Landau-Ginzburg superpotential, or more generally, a section $\sum_{\rho\in\Sigma^1} b_\rho x^{n_\rho}$ of the anticanonical bundle on X^* with all $b_\rho \neq 0$. Let $Z_W \subset X^* = X^*_\Sigma$ be the subscheme defined by the ideal sheaf $\mathcal{J}_W \subset \mathcal{O}_{X^*}$, where $\mathcal{J}_W(-K_{X^*}) \subset \mathcal{O}_{X^*}(-K_{X^*})$ is generated by all log-derivatives of W. Then Z_W is a projective subscheme of the big orbit $T_M \subset X^*$ of degree $|\Sigma^d|$. In particular it is zero dimensional, $\mathcal{O}(Z_W) = \mathbb{K}[N]/J_W$, and* $\dim_{\mathbb{C}} \mathcal{O}(Z_W) = |\Sigma^d|$.

Proof of Lemma 3.4. Since X_Σ is smooth each irreducible component of $X^*_\Sigma \setminus T_M$ is isomorphic to $\mathbb{P}^{d-1}$. Recall that such components are in one-to-one correspondence with the rays of the dual fan Σ^*, or equivalently with the maximal cones in Σ. Furthermore, if $\sigma \in \Sigma^d$ is a cone and $D^*_\sigma \simeq \mathbb{P}^{d-1}$ is the corresponding component then the restriction of

[11] Recall that for each $m \in M$ one can associate a log-derivative ∂_m defined by $\partial_m(x^n) = (m,n)x^n$. If we fix a pair of dual basis of M and N then ∂_{m_i} and ∂_i differ by the invertible function x^{n_i} on the torus.

the anticanonical linear system to such a component $\mathcal{O}_{X^*}(-K_{X^*}) \otimes \mathcal{O}_{D^*_\sigma}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{d-1}}(1)$, and the homogeneous coordinates on D^*_σ are naturally parameterized by the rays $\rho \subset \sigma$. We denote these coordinates by y_ρ.

We consider W and its log-derivatives as sections of $\mathcal{O}_{X^*}(-K_{X^*})$. Then, $\partial_m W = \sum_{\rho\in\Sigma^1}(m, n_\rho)b_\rho x^{n_\rho}$ and its restriction to D^*_σ is given by $\sum_{\rho\subset\sigma}(m, n_\rho)b_\rho y_\rho$. Clearly the set of these equations for $m \in M$ has no common roots, hence $Z_W \subset T_M$. But $Z_W \subset X^*_\Sigma$ is closed, hence a projective scheme. Thus Z_W is zero dimensional.

By definition Z_W is the scheme-theoretic intersection of d sections of $\mathcal{O}_{X^*}(-K_{X^*})$, hence by Kushnirenko's theorem

$$\deg Z_W = (-K_{X^*})^d = d!Volume(\Delta^*) = d! \sum_{\sigma\in\Sigma^d} Volume(\Delta^* \cap \sigma) = |\Sigma^d|,$$

since $\Delta^* \cap \sigma$ is a primitive simplex for any $\sigma \in \Sigma^d$. $\square$

Proof of Proposition 3.3. Consider the natural homomorphism

$$\psi\colon \Lambda[z_\rho]_{\rho\in\Sigma^1} \to \Lambda[N], \quad \text{defined by } \psi(z_\rho) = qs^{-F(n_\rho)}x^{n_\rho}. \tag{3.3.3}$$

Since X_Σ is smooth and projective (hence complete) the fan Σ is complete, and any $n \in N$ is an integral linear combination of vectors n_ρ, $\rho \in \Sigma^1$. Thus, ψ is surjective.

Next we claim that the quantum Stanley-Reisner ideal $QSR(X_\Sigma, F)$ lies in the kernel of ψ. Indeed, let C be a primitive collection and

$$\prod_{\rho\in C} q^{-1}s^{F(n_\rho)}z_\rho - \prod_{\rho\subseteq\sigma_C}(q^{-1}s^{F(n_\rho)}z_\rho)^{a_\rho},$$

be the corresponding quantum generator. It follows from the definition of ψ that:

$$\psi\Big(\prod_{\rho\in C} q^{-1}s^{F(n_\rho)}z_\rho - \prod_{\rho\subseteq\sigma_C}(q^{-1}s^{F(n_\rho)}z_\rho)^{a_\rho}\Big) = x^{\sum_{\rho\in C} n_\rho} - x^{\sum_{\rho\subseteq\sigma_C} a_\rho n_\rho} = 0.$$

Moreover, ψ sends the ideal $P(X_\Sigma)$ into $J_{W_{F,\Sigma}}$. Indeed, let $\sum_{\rho\in\Sigma^1}(m, n_\rho)z_\rho,\ m \in M$ be a generator of $P(X_\Sigma)$. Then:

$$\psi(\sum_{\rho\in\Sigma^1}(m, n_\rho)z_\rho) = q\sum_{\rho\in\Sigma^1}(m, n_\rho)s^{-F(n_\rho)}x^{n_\rho} = q\partial_m W_{F,\Sigma} \in J_{W_{F,\Sigma}}.$$

Thus, ψ defines a *surjective* homomorphism $QH^*_B(X_\Sigma, F; \Lambda) \to \Lambda[N]/J_{W_{F,\Sigma}}$.

Note that both algebras $QH^*_B(X_\Sigma, F; \Lambda)$ and $\Lambda[N]/J_{W_{F,\Sigma}}$ are *free* modules over Λ, and thus to complete the proof all we need to do is to compare the ranks. On the one side,

$$\mathrm{rank}_\Lambda QH^*_B(X_\Sigma, F; \Lambda) = \dim_{\mathbb{K}} H^*(X_\Sigma, \mathbb{K}) = \chi(X_\Sigma) = |\Sigma^d|;$$

and on the other side, the rank of $\Lambda[N]/J_{W_{F,\Sigma}}$ over Λ is equal to $\dim_{\mathbb{K}} \mathbb{K}[N]/J_{W_{F,\Sigma}}$, which by Lemma 3.4 equals $|\Sigma^d|$. The proof is now complete. $\square$

Lemma 3.5. *Let X, X^*, and $W = \sum_{\rho\in\Sigma^1} b_\rho x^{n_\rho}$ be as in Lemma 3.4. Then the support of Z_W coincides with the set of critical points of the function $\sum_{\rho\in\Sigma^1} b_\rho x^{n_\rho}$ on the torus T_M. Moreover, a critical point p is non-degenerate if and only if the scheme Z_W is reduced at p.*

Proof. We already proved in Lemma 3.4 that Z_W is a zero-dimensional subscheme of the torus T_M. Thus $p \in Z_W \subset T_M$ if and only if all log-derivatives of W vanish at p if and only if p is a critical point of W. Note that p is a non-degenerate critical point of W if and only if the Hessian is non-degenerate at p, or equivalently, if and only if the differentials of the log-derivatives of W generate the cotangent space $T_p^*T_M$. It remains to show that the latter condition is equivalent to the following: the log-derivatives of W generate the maximal ideal of $p \in T_M$ locally, i.e. $\mathfrak{m}_p = J_{W,p} = J_W\mathcal{O}_{T_M,p}$, where $\mathfrak{m}_p \subset \mathcal{O}_{T_M,p}$ denotes the maximal ideal. Clearly if $\mathfrak{m}_p = J_{W,p}$ then the differentials of the log-derivatives generate[12] $T_p^*T_M = \mathfrak{m}_p/\mathfrak{m}_p^2$. To prove the opposite direction we will need Nakayama's lemma. Indeed, if the differentials of the log-derivatives of W generate $T_p^*T_M$ then $\mathfrak{m}_p = J_{W,p} + \mathfrak{m}_p^2$, thus $\mathfrak{m}_p \cdot (\mathfrak{m}_p/J_{W,p}) = \mathfrak{m}_p/J_{W,p}$, hence, by Nakayama's lemma, $\mathfrak{m}_p/J_{W,p} = 0$, or equivalently $\mathfrak{m}_p = J_{W,p}$. $\square$

Corollary 3.6. For $X = X_\Sigma$, X^*, $W = \sum_{\rho\in\Sigma^1} b_\rho x^{n_\rho}$, and $Z_W \subset X^* = X^*_\Sigma$ as in the lemma the following hold:

(i) $\mathcal{O}(Z_W)$ is semisimple if and only if W has only non-degenerate critical points.

(ii) $\mathcal{O}(Z_W)$ contains a field as a direct summand if and only if W has a non-degenerate critical point.

4 Proof of The Main Results

In this section we prove our main results. We start with Theorem A which follows from the quantum Poincaré duality described in Subsection 3.1 and the following theorem:

Theorem 4.1. *Let X_Σ be a smooth toric Fano variety. Then for a generic choice of a toric symplectic form ω on X_Σ the quantum cohomology $QH^0(X_\Sigma,\omega)$ is semisimple.*

The proof follows the arguments in [27] Corollary 5.12, and [20] Proposition 7.6.

Proof of Theorem 4.1. Let $X^* = X^*_\Sigma$ be the dual Fano toric variety and let $\mathcal{O}_{X^*}(-K_{X^*})$ be the anti-canonical linear system. Following Remark 2.4 we consider the subspace of sections $\mathrm{Span}\{x^{n_\rho}\}_{\rho\in\Sigma^1} \subset H^0(X^*, \mathcal{O}_{X^*}(-K_{X^*}))$. It has codimension one since X_Σ is Fano and smooth, moreover $H^0(X^*, \mathcal{O}_{X^*}(-K_{X^*}))$ is generated by $\mathrm{Span}\{x^{n_\rho}\}$ and the section x^0.

Consider a strictly convex piecewise linear function F and the associated potential $W_{F,\Sigma} = \sum_{\rho\in\Sigma^1} s^{-F(n_\rho)}x^{n_\rho}$. Let $Z_{W_{F,\Sigma}}$ be the subscheme of X^* defined by the log-derivatives

[12] Recall that if $f \in \mathcal{O}_{T_M,p}$ then d_pf is nothing but the class of $f - f(p)$ modulo $\mathfrak{m}_p^2$.

of $W_{F,\Sigma}$ as in Lemma 3.4. Then $QH^0(X_\Sigma, \omega)$ is semisimple if and only if the scheme $Z_{W_{F,\Sigma}}$ is reduced by Corollary 3.6 and Proposition 3.3.

Recall that $\mathcal{O}_{X^*}(-K_{X^*})$ is ample, furthermore it is easy to see that for any $p \in T_M \subset X^*$ the differentials of the global sections of $\mathcal{O}_{X^*}(-K_{X^*})$ generate the cotangent space at p. Thus for a general choice of $W \in H^0(X^*, \mathcal{O}_{X^*}(-K_{X^*}))$ the critical points of W are non-degenerate, hence Z_W is reduced by Lemma 3.5. Moreover the same is true for a general section $W \in \mathrm{Span}\{x^{n_\rho}\}$ since log-derivatives of x^0 are zeroes. Thus there exists a non-zero polynomial[13] $P \in \mathbb{C}[B_\rho]_{\rho\in\Sigma^1}$ with the following property: If $W = \sum b_\rho x^{n_\rho}$ and $P(b_\rho) \neq 0$ then Z_W is reduced.

Let now ω be any toric symplectic form on X_Σ, and let F be a corresponding piecewise linear function on $|\Sigma|$. Note that by varying ω we vary $F(n_\rho)$, and any simultaneous small variation of $F(n_\rho)$ is realized by a toric symplectic form. Indeed, the fan Σ is simplicial thus any simultaneous variation of $F(n_\rho)$ is realized by a piecewise linear function, and since F is strictly convex any small variation gets rise to a strictly convex function. Thus for a general variation ω' of ω all the monomials of P will have different degrees in s, hence $P(s^{-F'(n_\rho)}) \neq 0$, and we are done. □

By a similar argument one can prove the following lemma:

Lemma 4.2. *Let $X = X_\Sigma$ be a (smooth) toric Fano variety, and let X^* be the dual toric Fano variety over the field $\mathbb{K}$. Let $V \subset H^0(X^*, \mathcal{O}_{X^*}(-K^*))$ be a locally closed subvariety defined over $\mathbb{C}$. Assume that $\sum s^{-F(n_\rho)} x^{n_\rho} \in V$ for some strictly convex piecewise linear function F on the fan Σ. Then there exists a rational strictly convex piecewise linear function F' on the fan Σ such that $\sum s^{-F'(n_\rho)} x^{n_\rho} \in V$.*

Proof. The variety $\overline{V}$ is given by a system of polynomial equations $P_1(B_\rho) = ... = P_k(B_\rho) = 0$ for some $P_1, \ldots, P_k \in \mathbb{C}[B_\rho]_{\rho\in\Sigma^1}$, and $V \subseteq \overline{V}$ is open.

Consider a collection of real numbers $(F_\rho)_{\rho\in\Sigma^1}$. Then $P_i(s^{-F_\rho})$ is a formal finite sum of (real) monomials with coefficients in $\mathbb{C}$. Assume now that $P_i(s^{-F_\rho}) = 0$. Then there exists a system L_i of *linear* equations with *integral* coefficients such that $(F_\rho)_{\rho\in\Sigma^1}$ is a solution of L_i, and $P_i(s^{-F'_\rho}) = 0$ for *any* solution $(F'_\rho)_{\rho\in\Sigma^1}$ of the system L_i.

Since $\sum s^{-F(n_\rho)} x^{n_\rho} \in V$ there exists a system $L = \cup L_i$ of linear equations with integral coefficients such that $(F(n_\rho))_{\rho\in\Sigma^1}$ is a solution of L and for any solution $(F'_\rho)_{\rho\in\Sigma^1}$ the following holds: $\sum s^{-F'_\rho} x^{n_\rho} \in \overline{V}$. Thus there exists a *rational* solution of system L obtained from the given one by a small perturbation. Similarly to the proof of Theorem 4.1, any such solution is of the form $(F'(n_\rho))_{\rho\in\Sigma^1}$ where F' is a rational strictly convex piecewise linear function on the fan Σ. Thus $P_i(s^{-F'(n_\rho)}) = 0$ for all i, hence $\sum s^{-F'(n_\rho)} x^{n_\rho} \in V$. □

Recall that when X_Σ is a Fano toric manifold then there exists a distinguished toric symplectic form ω_0 on X_Σ with moment map μ_0, namely the symplectic form corresponding

[13] Here B_ρ are formal variables.

to $c_1(X_\Sigma)$, i.e. to the piecewise linear function F_0 satisfying $F_0(n_\rho) = -1$ for all $\rho \in \Sigma^1$. It is the unique symplectic form for which the corresponding moment polytope is reflexive.

Using the quantum Poincaré duality once again, Theorems C and D follow from:

Theorem 4.3. *Let X_Σ be a smooth toric Fano manifold, and let ω be a toric symplectic form on X_Σ. Then*

(i) If $QH^0(X_\Sigma, \omega_0)$ is semisimple then $QH^0(X_\Sigma, \omega)$ is semisimple.

(ii) If $QH^0(X_\Sigma, \omega_0)$ contains a field as a direct summand then so does $QH^0(X_\Sigma, \omega)$.

Proof of Theorem 4.3: Let F and F_0 be the piecewise linear strictly convex functions corresponding to ω and ω_0, and let W and W_0 be the Landau-Ginzburg superpotentials assigned to F and F_0. From Proposition 3.3 it follows that $QH^0(X_\Sigma, \omega) \cong \mathcal{O}(Z_W) = \mathbb{K}[N]/J_W$ and $QH^0(X_\Sigma, \omega_0) \cong \mathcal{O}(Z_{W_0}) = \mathbb{K}[N]/J_{W_0}$, where $W = W_{F,\Sigma}$ and $W_0 = W_{F_0,\Sigma}$. Note that the loci of sections $W' \in H^0(X^*_\Sigma, \mathcal{O}(-K_{X^*_\Sigma}))$ for which $Z_{W'}$ is zero dimensional and is not reduced/does not contain a reduced point are locally closed and defined over $\mathbb{C}$. Thus, by Lemma 4.2, it is sufficient to prove the theorem only for *rational* symplectic forms ω. Furthermore, note that $s^a \mapsto s^{ak}$ is an automorphism of the field $\mathbb{K}$, hence without loss of generality we may assume that ω is integral. Thus $W, W_0 \in \mathbb{C}[s^{\pm 1}][N]$.

Next, let $Y = \mathrm{Spec}\mathbb{C}[s^{\pm 1}][N]/J_W$ and $Y_0 = \mathrm{Spec}\mathbb{C}[s^{\pm 1}][N]/J_{W_0}$, and consider the natural projections to $\mathrm{Spec}\mathbb{C}[s^{\pm 1}] = \mathbb{C}^*$. Note that the fibers of Y and Y_0 over $s = 1$ are canonically isomorphic since $W\big|_{s=1} = W_0\big|_{s=1}$. We denote these fibers by Y_c ("c" stands for closed). By Lemma 3.4

$$\dim_{\mathbb{C}}(\mathcal{O}(Y_c)) = \dim_{\mathbb{K}} \mathbb{K}[N]/J_{W_0} = |\Sigma^d| < \infty,$$

and $Y_0 = \mathrm{Spec}\mathbb{C}[s^{\pm 1}] \times Y_c$ since $W_0 = s \sum_{\rho \in \Sigma^1} x^{n_\rho}$ and s is invertible in $\mathbb{C}[s^{\pm 1}]$.

Consider now the algebras of functions $\mathcal{O}((Y_0)_\eta)$ and $\mathcal{O}(Y_\eta)$ on the generic fibers of $Y_0 \to \mathrm{Spec}\mathbb{C}[s^{\pm 1}]$ and $Y \to \mathrm{Spec}\mathbb{C}[s^{\pm 1}]$, i.e.

$$\mathcal{O}((Y_0)_\eta) := \mathcal{O}(Y_0) \otimes_{\mathbb{C}[s^{\pm 1}]} \mathbb{C}(s) \simeq \mathcal{O}(Y_c) \otimes_{\mathbb{C}} \mathbb{C}(s), \text{ and } \mathcal{O}(Y_\eta) := \mathcal{O}(Y) \otimes_{\mathbb{C}[s^{\pm 1}]} \mathbb{C}(s).$$

Note that $\dim_{\mathbb{C}(s)}(\mathcal{O}(Y_\eta)) = \dim_{\mathbb{C}}(\mathcal{O}(Y_c)) = |\Sigma^d| < \infty$ by Lemma 3.4. Note also that $QH^0(X_\Sigma, \omega_0) = \mathcal{O}((Y_0)_\eta) \otimes_{\mathbb{C}(s)} \mathbb{K}$, and $QH^0(X_\Sigma, \omega) = \mathcal{O}(Y_\eta) \otimes_{\mathbb{C}(s)} \mathbb{K}$. Thus $QH^0(X_\Sigma, \omega_0)$ is semisimple over $\mathbb{K}$ (contains a field as a direct summand) if and only if $\mathcal{O}((Y_0)_\eta)$ is semisimple over $\mathbb{C}(s)$ (contains a field as a direct summand) if and only if $\mathcal{O}(Y_c)$ is semisimple over $\mathbb{C}$ (contains a field as a direct summand), and $QH^0(X_\Sigma, \omega)$ is semisimple over $\mathbb{K}$ (contains a field as a direct summand) if and only if $\mathcal{O}(Y_\eta)$ is semisimple over $\mathbb{C}(s)$ (contains a field as a direct summand).

To summarize: all we want to prove is that (i) if $\mathcal{O}(Y_c)$ is semisimple over $\mathbb{C}$ then $\mathcal{O}(Y_\eta)$ is semisimple over $\mathbb{C}(s)$, and (ii) if $\mathcal{O}(Y_c)$ contains a field as a direct summand then $\mathcal{O}(Y_\eta)$ contains a field as a direct summand; or geometrically (i) if Y_c is reduced then Y_η is reduced, and (ii) if Y_c contains a reduced point then Y_η contains a reduced point.

We remark that since we are interested only in the fibers over the generic point and over $s = 1$, we can replace $\mathbb{C}[s^{\pm 1}]$ by its localization at $s = 1$ denoted by R. By abuse of notation $Y \times_{\mathrm{Spec}\mathbb{C}[s^{\pm 1}]} \mathrm{Spec}R$ will still be denoted by Y. Note that in this notation $\mathcal{O}(Y_\eta) = \mathcal{O}(Y) \otimes_R \mathbb{C}(s)$. To complete the proof, we shall need the following two observations:

Claim 4.4. The map $Y \to \mathrm{Spec}R$ is flat and finite.

Lemma 4.5. *Let Y be flat finite scheme over* $\mathrm{Spec}R$*, and let Y_c and Y_η be its fibers over the closed and generic points of* $\mathrm{Spec}R$*. Then*

(i) If Y_c is reduced then Y_η is reduced.

(ii) If Y_c contains a reduced point then Y_η contains a reduced point.

The theorem now follows. □

Proof of Claim 4.4. First let us show flatness. It is sufficient to check flatness locally on Y. Let us fix a closed point $p \in Y \subset \mathrm{Spec}\, R[N]$, hence $p \in Y_c$. We denote $\mathrm{Spec}\, R[N]$ by T. Let $m_1, \ldots, m_d$ be a basis of $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and let ∂_i be the log derivations defined by m_i. Then J_W is generated by $\{\partial_i W\}_{i=1}^d$. We claim that the sequence $\partial_1 W, \ldots, \partial_d W, s-1$ is a sequence of parameters in the maximal ideal $\mathfrak{m}_p \subset \mathcal{O}_{T,p}$. Indeed $\dim_p T = d + 1$ and $\dim \mathrm{Spec}(\mathcal{O}_{T,p}/(\partial_1 W, \ldots, \partial_d W, s-1)) = 0$, since $\dim_{\mathbb{C}}(\mathcal{O}_{T,p}/(\partial_1 W, \ldots, \partial_d W, s-1)) = \dim_{\mathbb{C}} \mathcal{O}_{Y_c,p} \leq \dim_{\mathbb{C}} \mathcal{O}(Y_c) < \infty$. Note that T is regular, hence Cohen Macaulay, thus $\partial_1 W, \ldots, \partial_d W, s-1$ is an $\mathcal{O}_{T,p}-$sequence by [33] Theorem 17.4 (iii). Then the local algebra $\mathcal{O}_{Y,p} = \mathcal{O}_{T,p}/(\partial_1 W, \ldots, \partial_d W)$ is Cohen-Macaulay by [33] Theorem 17.3 (ii), and $\dim_p Y = 1$. Flatness at p now follows from [33] Theorem 23.1, indeed Y is Cohen-Macaulay at p of dimension 1, $\mathrm{Spec}R$ is regular of dimension 1, and the fiber over $s = 1$ has dimension 0.

Remark that flatness of Y over $\mathrm{Spec}R$ implies (and in fact is equivalent to) the following equivalent properties: $s - 1$ is not a zero divisor in $\mathcal{O}(Y)$, and the natural map $\mathcal{O}(Y) \to \mathcal{O}(Y_\eta)$ is an embedding. In what follows we shall use these properties many times.

Next we turn to show that $\mathcal{O}(Y)$ is finite R-module. Let $g_1, \ldots, g_l$, $l = |\Sigma^d|$, be a basis of $\mathcal{O}(Y_c)$ and let $f_1, \ldots, f_l$ be its lifting to $\mathcal{O}(Y) \subset \mathcal{O}(Y_\eta)$. We claim that $f_1, \ldots, f_l$ freely generate $\mathcal{O}(Y)$ as an R-module. Let $\lambda_i(s) \in \mathbb{C}(s)$ be elements such that $0 = \sum \lambda_i(s) f_i \in \mathcal{O}(Y_\eta)$. If not all $\lambda_i(s)$ are equal to zero, then there exists k such that $\mu_i(s) := (s-1)^k \lambda_i(s) \in R$ for all i and $\mu_i(1) \neq 0$ for some i. Then $\sum \mu_i(s) f_i = 0$ hence $\sum \mu_i(1) g_i = 0$ which is a contradiction. Thus $f_1, \ldots, f_l \in \mathcal{O}(Y_\eta)$ are linearly independent, and since $\dim_{\mathbb{C}(s)} \mathcal{O}(Y_\eta) = \dim_{\mathbb{C}} \mathcal{O}(Y_c) = |\Sigma^d| = l$, they form a basis of $\mathcal{O}(Y_\eta)$ over $\mathbb{C}(s)$. It remains to show that $f_1, \ldots, f_l$ generate $\mathcal{O}(Y)$ as R-module. Let $0 \neq f \in \mathcal{O}(Y) \subset \mathcal{O}(Y_\eta)$ be any element then $f = \sum \lambda_i(s) f_i$ for some $\lambda_i(s) \in \mathbb{C}(s)$. As before, if not all the coefficients $\lambda_i(s) \in R$ then there exists $k > 0$ such that $\mu_i(s) := (s-1)^k \lambda_i(s) \in R$ for all i and $\mu_i(1) \neq 0$ for at least one i. Thus $\sum \mu_i(1) g_i \neq 0$ is the class of $(s-1)^k f$ in $\mathcal{O}(Y_c)$ which is zero. This is a contradiction, hence $\mathcal{O}(Y)$ is a flat finite R-module. □

Proof of Lemma 4.5. First, note that the natural map $\mathcal{O}(Y) \to \mathcal{O}(Y_\eta)$ is an embedding since $\mathcal{O}(Y)$ is flat over R. Second, recall that a flat finite module over a local ring is free, thus $\mathcal{O}(Y) \simeq R^l$ as an R-module, and $\mathcal{O}(Y_\eta) = \mathcal{O}(Y) \otimes_R \mathbb{C}(s) \simeq \mathbb{C}(s)^l$ as a $\mathbb{C}(s)$-module (vector space); hence for any $0 \neq f \in \mathcal{O}(Y_\eta)$ there exists a *minimal* integer k such that $(s-1)^k f \in \mathcal{O}(Y)$.

(i): Assume by contradiction that Y_η is not reduced. Then there exists a nilpotent element $0 \neq f \in \mathcal{O}(Y_\eta)$. Let k be the minimal integer such that $(s-1)^k f \in \mathcal{O}(Y)$. Then $0 \neq (s-1)^k f$ is a nilpotent and its class in $\mathcal{O}(Y_c)$ is not zero. Thus, we constructed a non-zero nilpotent in $\mathcal{O}(Y_c)$, which is a contradiction.

(ii): Recall that if $Z = \mathrm{Spec} A$, and A is a finite dimensional algebra over a field then $A = \mathcal{O}(Z) = \oplus_{q \in Z} \mathcal{O}_{Z,q}$ as algebras, where $\mathcal{O}_{Z,q}$ is the localization of $\mathcal{O}(Z)$ at q (Chinese remainder theorem). Furthermore any element in the maximal ideal $\mathfrak{m}_{Z,q} \subset \mathcal{O}_{Z,q}$ is nilpotent. Thus $\mathcal{O}(Y_c) = \oplus_{q \in Y_c} \mathcal{O}_{Y_c,q}$ as algebras, and $\mathcal{O}(Y_\eta) = \oplus_{\epsilon \in Y_\eta} \mathcal{O}_{Y_\eta,\epsilon}$ as algebras (hence as $\mathcal{O}(Y)$-modules).

Assume that $q \in Y_c$ is a reduced point. Then $q \in Y$ is a closed point and $\mathcal{O}_{Y,q} \to \mathcal{O}_{Y_c,q} = \mathbb{C}$ is a surjective homomorphism from a local ring with kernel generated by $s-1$, hence $\mathfrak{m}_{Y,q} = (s-1)\mathcal{O}_{Y,q}$. Tensoring $\mathcal{O}(Y_\eta) = \oplus_{\epsilon \in Y_\eta} \mathcal{O}_{Y_\eta,\epsilon}$ with $\mathcal{O}_{Y,q}$ over $\mathcal{O}(Y)$ we obtain the following decomposition: $\mathcal{O}(Y_\eta) \otimes_{\mathcal{O}(Y)} \mathcal{O}_{Y,q} = \oplus_{\epsilon \in Y_\eta} (\mathcal{O}_{Y_\eta,\epsilon} \otimes_{\mathcal{O}(Y)} \mathcal{O}_{Y,q})$. To finish the proof it is sufficient to show that (a) $\mathcal{O}(Y_\eta) \otimes_{\mathcal{O}(Y)} \mathcal{O}_{Y,q}$ is a field, and (b) $\mathcal{O}_{Y_\eta,\epsilon} \otimes_{\mathcal{O}(Y)} \mathcal{O}_{Y,q}$ is either zero or $\mathcal{O}_{Y_\eta,\epsilon}$.

For (a), note that by Nakayama's lemma $\cap_{k \in \mathbb{N}} \mathfrak{m}_{Y,q}^k = 0$. Thus, any element in $\mathcal{O}_{Y,q}$ is of the form $u(s-1)^k$ for some integer $k \geq 0$ and some invertible element $u \in \mathcal{O}_{Y,q}$. Next, note that $s-1 \in \mathcal{O}_{Y,q}$ is not a nilpotent element since otherwise it would be a zero divisor in $\mathcal{O}(Y)$, and this contradicts the flatness of Y. Thus $\mathcal{O}_{Y,q}$ is an integral domain (in fact it is a DVR) with field of fractions $(\mathcal{O}_{Y,q})_{s-1}$ (localization of $\mathcal{O}_{Y,q}$ with respect to $s-1$). Hence $\mathcal{O}(Y_\eta) \otimes_{\mathcal{O}(Y)} \mathcal{O}_{Y,q} = \mathbb{C}(s) \otimes_R \mathcal{O}_{Y,q} = (\mathcal{O}_{Y,q})_{s-1}$ is a field.

For (b), let $\mathfrak{m} \subset \mathcal{O}(Y) \subset \mathcal{O}(Y_\eta)$ be the maximal ideal of $q \in Y$ and let $\mathfrak{n} \subset \mathcal{O}(Y_\eta)$ be the maximal ideal of ϵ. If q belongs to the closure of ϵ then $\mathcal{O}(Y) \setminus \mathfrak{m} \subseteq \mathcal{O}(Y_\eta) \setminus \mathfrak{n}$, hence $\mathcal{O}_{Y_\eta,\epsilon} \otimes_{\mathcal{O}(Y)} \mathcal{O}_{Y,q} = \mathcal{O}_{Y_\eta,\epsilon}$. If q does not belong to the closure of ϵ then there exists $f \in \mathcal{O}(Y) \setminus \mathfrak{m}$ such that $f \in \mathfrak{n}$. Thus $1 \otimes f \in \mathcal{O}_{Y_\eta,\epsilon} \otimes_{\mathcal{O}(Y)} \mathcal{O}_{Y,q}$ must be invertible and nilpotent at the same time (any element in $\mathfrak{n}\mathcal{O}_{Y_\eta,\epsilon}$ is nilpotent!), hence $\mathcal{O}_{Y_\eta,\epsilon} \otimes_{\mathcal{O}(Y)} \mathcal{O}_{Y,q} = 0$ and we are done. □

4.1 The Critical Values of the Superpotential

Let (X_Σ, F) be a smooth toric Fano variety equipped with a strictly convex piecewise linear function F, or equivalently, with a symplectic form ω and a moment map μ. Recall that $c_1(X_\Sigma)$ in Batyrev's description of the (quantum) cohomology is given by $\sum_{\rho \in \Sigma^1} z_\rho$. Thus, using (3.3.3) to identify Batyrev's description with the Landau-Ginzburg model, one obtains the following formula: $c_1(X_\Sigma) = \sum_{\rho \in \Sigma^1} q s^{F(n_\rho)} x^{n_\rho} = qW$, $W = W_{F,\Sigma}$; hence

$$q^{-1} c_1(X_\Sigma) = W \in \mathbb{K}[N]/J_W = QH^0(X_\Sigma, \omega).$$

Thus the set of critical values of the superpotential W is equal to the set of eigenvalues of multiplication by $q^{-1}c_1(X_\Sigma)$ on $QH^0(X_\Sigma,\omega)$ by Corollary 2.3; which proves Corollary G.

5 Examples and Counter-Examples

In this section we prove Proposition B and Corollary E. We first provide an example of a polytope Δ such that the quantum homology subalgebra $QH_8(X_\Delta,\omega_0)$ of the corresponding (complex) 4-dimensional symplectic toric Fano manifold X_Δ is not semisimple. Here ω_0 is the distinguished (normalized) monotone symplectic form on X_Δ.

We start by making the identification

$$Lie(T)^* = M_\mathbb{R} \simeq \mathbb{R}^d,\ Lie(T) = N_\mathbb{R} \simeq (\mathbb{R}^d)^* \simeq \mathbb{R}^d \tag{5.1}$$

For technical reasons, it would be easier for us to describe the vertices of the dual polytope Δ^*, that are the inward-pointing normals to the facets of Δ. Let

$$\Delta^* = \mathrm{Conv}\{e_1, e_2, e_3, e_4, -e_1+e_4, -e_2+e_4, e_2-e_4, -e_2, -e_4, -e_3-e_4\},$$

where $\{e_1,e_2,e_3,e_4\}$ is the standard basis of $\mathbb{R}^4$. A straightforward computation, whose details we omit (see remark below), shows that Δ is a Fano Delzant polytope. We denote by (X_Δ,ω_0) the corresponding symplectic toric Fano manifold equipped with the canonical symplectic form ω_0.

Remark 5.1. Toric Fano 4-folds are completely classified (see e.g., [4, 43]). We refer the reader to the software package "PALP" [30] with which all the combinatorial data of the 124 Toric Fano 4-dimensional polytopes can be explicitly computed. The above example Δ is the unique reflexive 4-dimensional polytope with 10 vertices, 24 Facets, 11 integer points, and 59 dual integer points (the "PALP" search command is: "class.x -di x -He EH:M11V10N59F24L1000"), and it is listed among the 124 examples in the webpage `http://hep.itp.tuwien.ac.at/~kreuzer/CY/math/0702890/`. In Batyrev's classification [4], X_Δ appears under the notation U_8 as example number 116 in section 4.

The corresponding Landau-Ginzburg super potential $W\colon \mathbb{C}[x_1^\pm,x_2^\pm,x_3^\pm,x_4^\pm] \to \mathbb{C}$ is:

$$W = x_1+x_2+x_3+x_4+\frac{1}{x_2}+\frac{1}{x_4}+\frac{1}{x_3x_4}+\frac{x_4}{x_1}+\frac{x_4}{x_2}+\frac{x_2}{x_4}$$

The partial derivatives are:

$$W_{x_1} = 1-\frac{x_4}{x_1^2},\ W_{x_2} = 1-\frac{x_4+1}{x_2^2}+\frac{1}{x_4},\ W_{x_3} = 1-\frac{1}{x_4x_3^2},\ W_{x_4} = 1+\frac{1}{x_1}+\frac{1}{x_2}-\frac{x_2x_3+x_3+1}{x_3x_4^2}$$

It is easy to check that $z_0 = (-1,-1,-1,1)$ is a critical point of W. On the other hand, the Hessian of W at the point z_0

$$\mathrm{Hess}(W_{X_\Delta}(z_0)) = \begin{pmatrix} -2 & 0 & 0 & -1 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & -2 & 1 \\ -1 & -2 & 1 & -2 \end{pmatrix}$$

has rank 3. Hence z_0 is degenerate, and $QH^0(X_\Delta, \omega_0)$ is not semisimple by Proposition 3.3 and Corollary 3.6 (i). Proposition B now follows from the quantum Poincaré duality.

Remark 5.2. To construct a non-monotone symplectic manifolds with non semisimple quantum homology, one can just consider the product $X_\Delta \times \mathbb{P}^1$ equipped with the symplectic form $\omega_0 \otimes \alpha\omega_{\mathbb{P}^1}$, where $\alpha \gg 1$ and $\omega_{\mathbb{P}^1}$ is the standard symplectic form on $\mathbb{P}^1$.

We now turn to sketch of proof of Corollary E. Note that the combination of McDuff's observation, Theorem D, and Corollary 3.6 (ii) reduces the question of the existence of a Calabi quasimorphism and symplectic quasi-states on a symplectic toric manifold (X, ω) to the problem of finding a non-degenerate critical point of the Landau-Ginzburg superpotential corresponding to (X, ω_0), where ω_0 is the canonical symplectic form on X.

For monotone toric Fano 3-folds and 4-folds, we used a computer to verify that each of the corresponding superpotentials has at least one non-degenerate critical point. We refer the reader to the webpage `http://www-math.mit.edu/~ostrover/Toric-Fano.html` for the Mathematica scripts and commentary.

6 Calabi quasimorphisms

The group-theoretic notion *quasimorphism* was originally introduced with connection to bounded cohomology theory and since then became an important tool in geometry, topology and dynamics (see e.g. [28]). In the context of symplectic geometry, Entov and Polterovich constructed certain homogeneous quasimorphisms, called "Calabi quasimorphism", and showed several applications to Hofer's geometry, C^0-symplectic topology, and Lagrangian intersection theory (see e.g. [15, 18]).

Recall that a real-valued function Π on a group G is called a *homogeneous quasimorphism* if there is a universal constant $C > 0$ such that for every $g_1, g_2 \in G$:

$$|\Pi(g_1 g_2) - \Pi(g_1) - \Pi(g_2)| \leq C, \text{ and } \Pi(g^k) = k\Pi(g) \text{ for every } k \in \mathbb{Z} \text{ and } g \in G.$$

In [15], Entov and Polterovich constructed quasimorphisms on the universal cover of the group of Hamiltonian diffeomorphisms for certain symplectic manifolds. A key ingredient in their construction are the spectral invariants, which were defined by Schwarz [44] in the aspherical case, and by Oh [40] for general symplectic manifolds (see also Usher [45]). These invariants are given by a map $c\colon QH^*(X) \times \widetilde{Ham}(X) \to \mathbb{R}$. We refer the reader to [40] and [36] for the precise definition of the spectral invariants and their properties.

Let (X, ω) be a semi-positive symplectic manifold. Following [15], for an idempotent element $e \in QH^0(X, \omega)$ we define $\mathcal{Q}_e\colon \widetilde{Ham}(X) \to \mathbb{R}$ by $\mathcal{Q}_e = \overline{c}(e, \cdot)$, where $\overline{c}(e, \widetilde{\phi}) = \liminf_{k \to \infty} \frac{c(e, \widetilde{\phi}^k)}{k}$ for all $\widetilde{\phi} \in \widetilde{Ham}(X)$. Entov and Polterovich showed that if $eQH^0(X, \omega)$ is a field then $\mathcal{Q}_e$ is a homogenous quasimorphism (see [15] for the monotone case and [38, 18] for the general case). Moreover, $\mathcal{Q}_e$ satisfies the so called *Calabi property*, which means, roughly speaking, that "locally" it coincides with the Calabi homomorphism (see [15] for the precise definition and proof). A natural question raised in [15] asking whether such a quasimorphism is unique.

Our goal in this section is to prove Corollary F which shows that the answer to the question above is negative. For this we will need some preparation. We start with the following general property of the spectral invariants (see [40, 15, 36]): for every $0 \neq a \in QH^*(X,\omega)$ and $\gamma \in \pi_1(Ham(X)) \subset \widetilde{Ham}(X)$ the following holds: $c(a,\gamma) = c(a*\mathcal{S}(\gamma), \mathbb{1}) = \log \|a * \mathcal{S}(\gamma)\|$, where $\mathcal{S}(\gamma) \in QH^0(X,\omega)$ is the Seidel element of γ (see e.g. [36] for the definition), and $\|\cdot\|$ is the non-Archimedean norm discussed in Remark 3.1. Thus, for every idempotent $e \in QH^0(X,\omega)$ and $\gamma \in \pi_1(Ham(X))$, we have

$$\mathcal{Q}_e(\gamma) = \log \|e * \mathcal{S}(\gamma)\|_{\mathrm{sp}}, \tag{6.1}$$

where $\|\cdot\|_{\mathrm{sp}}$ is the corresponding non-Archimedean spectral seminorm (cf. subsection 2.1.3).

Let now (X_Σ, ω) be a symplectic toric Fano manifold, and F be a corresponding strictly convex piecewise linear function on $|\Sigma|$. Consider the homomorphisms $\iota\colon N \to \mathbb{K}[N]/J_W$, $W = W_{F,\Sigma}$, given by the composition

$$N = \pi_1(T_N) \longrightarrow \pi_1(Ham(X_\Sigma)) \xrightarrow{\mathcal{S}} QH^0(X_\Sigma,\omega) \simeq \mathbb{K}[N]/J_W,$$

where $\mathcal{S}$ is the Seidel map (see e.g [36]). By translating a result of McDuff and Tolman (see [37] Theorem 1.10 and Section 5.1, and [36] page 441) to the Landau-Ginzburg model using (3.3.3), one obtains an explicit formula for ι, namely $\iota(n) = x^n$. To any critical point $p \in Z_W$ one can assign the unit element $e_p \in \mathcal{O}_{Z_W,p}$, which is an idempotent in $\mathcal{O}(Z_W) \simeq QH^0(X_\Sigma,\omega)$. Furthermore, $e_p\mathcal{O}(Z_W) = \mathcal{O}_{Z_W,p}$ is a field if and only if p is a non-degenerate critical point of W. Thus it is sufficient to find two non-degenerate critical points of the superpotential $p, p' \in Z_W$ and $n \in N$ such that $|x^n(p)| \neq |x^n(p')|$, thanks to Corollary 2.3 and (6.1).

Let (X_Σ, F) be the blow up of $\mathbb{P}^2$ at one point equipped with a strictly convex piecewise linear function F, or equivalently, with a symplectic form ω and a moment map μ. After adding a global linear function to F (this operation changes μ, but does not change ω) we may assume that $F(1,0) = 0$, $F(0,1) = 0$, $F(0,-1) = \beta - \alpha$, and $F(-1,-1) = -\alpha$, where $\alpha > \beta > 0$. It is easy to check that $QH^0(X_\Sigma,\omega_0)$ is semisimple, since the superpotential W_0 has only non-degenerate critical points. Thus $QH^0(X_\Sigma,\omega)$ is semisimple by Theorem C, and W has only non-degenerate critical points.

Recall that the fan Σ has four rays generated by $(1,0),(0,1),(0,-1),(-1,-1)$. Set $x_1 = x^{(1,0)}$ and $x_2 = x^{(0,1)}$. Then $W = x_1 + x_2 + s^{\alpha-\beta}x_2^{-1} + s^\alpha x_1^{-1}x_2^{-1}$, and the scheme Z_W of its critical points is given by $x_1 - s^\alpha x_1^{-1}x_2^{-1} = x_2 - s^{\alpha-\beta}x_2^{-1} - s^\alpha x_1^{-1}x_2^{-1} = 0$, or equivalently, $x_1^4 + s^\beta x_1^3 - s^{(\beta+\alpha)} = 0$ and $x_2 = s^\alpha x_1^{-2}$. Assume for simplicity that $\alpha, \beta \in \mathbb{Q}$. The following argument is based on the notions of Newton diagrams/polygons (we refer the reader to [48] Chapter 4, §3 for details): The Newton diagram of $x_1^4 + s^\beta x_1^3 - s^{(\beta+\alpha)} = 0$ has two faces if and only if $\alpha > 3\beta$; otherwise it has unique face. It is classically known that the solutions of such an equation correspond to the faces of the Newton diagram;

each solution can be written as a Puiseux series[14] in s with non-Archimedean valuation determined by the slope of the corresponding face; and the number of solutions (counted with multiplicities) corresponding to a given face is equal to the change of x_1 along the face. Thus if $\alpha > 3\beta$ and $n = (1,0)$ then there exist non-degenerate critical points $p, p' \in Z_W$ such that $|x^n(p)| = 10^{-\beta} \neq 10^{-\alpha/3} = |x^n(p')|$. Note that $\omega(L)/\omega(E) = \alpha/\beta$. Note also that the assumption $\alpha, \beta \in \mathbb{Q}$ is not necessary since the algorithm for finding the solutions in [48] Chapter 4, §3 Theorem 3.1 works for $\alpha, \beta \in \mathbb{R}$ as well; but the solutions themselves belong to the larger field $\mathbb{K}$. Corollary F now follows.

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