Localization of Multi-Dimensional Wigner Distributions

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Abstract

A well known result of P. Flandrin states that a Gaussian uniquely maximizes the integral of the Wigner distribution over every centered disc in the phase plane. While there is no difficulty in generalizing this result to higher-dimensional poly-discs, the generalization to balls is less obvious. In this note we provide such a generalization.

1 Introduction

The Wigner quasi-probability distribution was introduced by Wigner [16] in 1932 in order to study quantum corrections to classical statistical mechanics. Nowadays it lies at the core of the phase-space formulation of quantum mechanics (Weyl correspondence), and has a variety of applications in statistical mechanics, quantum optics, and signal analysis, to name a few. In this note we consider the localization problem of the n-particle Wigner distribution in the $2n$-dimensional phase space. We state our results precisely in Theorem 1 below.

Equip the classical phase space $\mathbb{R}^{2n}$ with coordinates (x, y) with $x, y \in \mathbb{R}^n$. The Wigner quasi-probability distribution on $\mathbb{R}^{2n}$, associated with a wave function $\psi \in L^2(\mathbb{R}^n)$ and its complex conjugate ψ^*, is defined by

$$\mathcal{W}_\psi(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \psi(x + \tau/2)\psi^*(x - \tau/2)e^{-i\tau\cdot y}\, d\tau \tag{1.1}$$

The function $\mathcal{W}_\psi$ possesses many of the properties of a phase space probability distribution (see e.g., [4]); in particular, it is real. However, $\mathcal{W}_\psi$ is not a genuine probability distribution as it can assume negative values.

The localization problem, i.e., estimating the integral of the Wigner distribution over a subregion of the phase space, and the closely related problem of the optimal simultaneous concentration of ψ and its Fourier transform $\widehat{\psi}$, have received much attention in the literature both in quantum mechanics, mathematical time-frequency analysis, and signal processing (see e.g. [1, 2, 3, 4, 5, 6, 9, 10, 12, 13, 11], and the references within). Bounds on the L^p norms were found in [7]. More precisely, the problem of interest for us is:

The Wigner Distribution Localization Problem: *given a measurable set $D \subset \mathbb{R}^{2n}$, find the best possible bounds to the localization function*

$$\mathcal{E}(D) := \sup_{\psi} \int_D \mathcal{W}_\psi \, dxdy, \tag{1.2}$$

where the supremum is taken over all the functions $\psi \in L^2(\mathbb{R}^n)$ with $\|\psi\|_2 = 1$.

The quantity $\mathcal{E}(D)$ is invariant under translations in the phase space, and under the action of the group of linear symplectic transformations (see e.g. [15]). There is no upper bound on $\mathcal{E}(D)$; it can be infinite. Indeed, there is a $\psi \in L^2(\mathbb{R})$ such that $\int |\mathcal{W}_\psi| dxdy = \infty$ [4, sect. 4.6]. An example is $\psi(x) = 1$ if $-\frac{1}{2} < x < \frac{1}{2}$ and $\psi(x) = 0$ otherwise. On the other hand, the L^p norm of $\mathcal{W}_\psi$ is bounded [7] for $p \geq 2$ and we can use this information to show that $\mathcal{E}(D)$ is bounded by powers of the volume $|D|$. E.g., the L^∞ norm is at most π^{-n}, so $\mathcal{E}(D) \leq \pi^{-n}|D|$.

For certain D, however, $\mathcal{E}(D)$ is not only finite, it is even less than 1. In [2], Flandrin conjectured this to be true for all convex domains, and he showed that for all centered two-dimensional discs $B^2(r)$ of radius r, the standard normalized Gaussian function $\pi^{-1/4}\exp(-x^2/2)$ is the unique maximizer of (1.2). In particular $\mathcal{E}(B^2(r)) = 1 - e^{-r^2}$ (see [2], cf. [4]). It follows immediately from the definition of the Wigner distribution that Flandrin's proof can be easily generalized to higher dimensional poly-discs because the maximization problem then has a simple product structure. A less obvious case is the $2n$-dimensional Euclidean ball $B^{2n}(r)$. The following is the generalization of Flandrin's result, and our main result:

Theorem 1. *The standard normalized Gaussian $\pi^{-n/4}\exp(-x^2/2)$ in $L_2(\mathbb{R}^n)$ is the unique maximizer of the Wigner distribution localization problem for any $2n$-dimensional Euclidean ball centered at the origin. In particular,*

$$\mathcal{E}\left(B^{2n}(r)\right) = \frac{1}{\pi^n} \int_{B^{2n}(r)} e^{-\sum_{i=1}^n (x_i^2 + y_i^2)} \, dxdy = 1 - \frac{\Gamma(n, r^2)}{(n-1)!}, \tag{1.3}$$

where $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$ is the upper incomplete gamma function.

Remarks: (1.) Owing to the translation covariance of the Wigner distribution, equation (1.3) also applies to a ball of radius r centered anywhere in $\mathbb{R}^{2n}$. It is only necessary to multiply the Gaussian by an appropriate linear form $\exp(a \cdot x)$. Moreover, since the localization function (1.2) is invariant under the action of the group of linear symplectic transformations, Theorem 1 can also be adapted to any image of the Euclidean ball under linear symplectic maps.

(2.) Another generalization is to replace the integral over the ball with the integral over $\mathbb{R}^{2n}$, but with a weight that is a symmetric decreasing function (i.e., a radial and non-increasing function of the radius $\sqrt{x^2 + y^2}$). By the "layer cake representation" [8, sect. 1.13] the standard Gaussian again maximizes uniquely.

2 Proof of Theorem 1

We start with the following preliminaries. Recall that the mixed Wigner distribution of two states $\psi_1, \psi_2 \in L^2(\mathbb{R}^n)$ is defined by

$$\mathcal{W}_{\psi_1,\psi_2}(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \psi_1(x+\tau/2)\psi_2^*(x-\tau/2)e^{-i\tau y}\, d\tau \ . \tag{2.1}$$

Note that in contrast to (1.1), $\mathcal{W}_{\psi_1,\psi_2}$ is not generally real, but, nevertheless, Hermitian i.e., $\mathcal{W}_{\psi_1,\psi_2} = \mathcal{W}^*_{\psi_2,\psi_1}$. Moreover, it is not hard to check that the mixed Wigner distribution is sesquilinear.

Next, let $\mu = (\mu_1, \ldots, \mu_n)$ be a multiindex of non-negative integers, and let $x \in \mathbb{R}^n$. The Hermite functions $H_\mu(x)$ on $\mathbb{R}^n$ are defined [14, 15] to be the product of the normalized one-dimensional Hermite functions, i.e., $H_\mu(x) = \prod_{j=1}^n h_{\mu_j}(x_j)$, where

$$h_k(x) = \pi^{-\frac{1}{4}}\,(k!)^{-\frac{1}{2}}\,2^{-\frac{k}{2}}\,(-1)^k\,e^{x^2/2}\,\frac{d^k}{dx^k}e^{-x^2} \ . \tag{2.2}$$

It is well known that the $\{H_\mu\}$ form a complete orthonormal system for $L^2(\mathbb{R}^n)$, and that

$$\mathbb{H}\,H_\mu = |\mu|\,H_\mu, \tag{2.3}$$

where $|\mu| = \sum_{j=1}^n \mu_j$, and $\mathbb{H}$ is the Schrödinger operator $\mathbb{H} = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2 - \frac{n}{2}$. Here Δ denotes the standard n-dimensional Laplacian. In particular, the sesquilinearity of the Wigner distribution implies that for any $\psi \in L_2(\mathbb{R}^n)$, one has

$$\mathcal{W}_\psi = \sum_\mu \sum_\nu \langle\psi, H_\mu\rangle\,\langle\psi, H_\nu\rangle^*\,\mathcal{W}_{H_\mu,H_\nu} \ . \tag{2.4}$$

The following lemma shows that the integral of the off-diagonal elements of (2.4) over any centered ball $B^{2n}(r)$ vanishes (cf. [5] Section 2.3).

Lemma 2.1. *Let μ,ν be two multi-indices with $\mu \neq \nu$. Then, for every $r \geqslant 0$, one has*

$$\int\limits_{B^{2n}(r)} \mathcal{W}_{H_\mu,H_\nu}\,dxdy = 0 \ . \tag{2.5}$$

Proof of Lemma 2.1. It is well known (see e.g. [6]) that for the one-dimensional Hermite functions $\{h_m\}$, one has:

$$\mathcal{W}_{h_j,h_k}(x_1,y_1) = \begin{cases} \pi^{-1}\,(k!/j!)^{1/2}\,(-1)^k\,(\sqrt{2}z_1)^{j-k}\,e^{-(|z_1|^2)}\,L_k^{j-k}(2|z_1|^2) & \text{if } j \geq k, \\ \pi^{-1}\,(j!/k!)^{1/2}\,(-1)^j\,(\sqrt{2}\,\overline{z}_1)^{k-j}\,e^{-(|z_1|^2)}\,L_j^{k-j}(2|z_1|^2) & \text{if } k \geq j\ . \end{cases} \tag{2.6}$$

Here $z_1 = x_1 + iy_1$, and L_n^α are the Laguerre polynomials defined by

$$L_j^\alpha(x) = \frac{x^{-\alpha}e^x}{j!}\frac{d^j}{dx^j}(e^{-x}x^{j+\alpha}), \tag{2.7}$$

for $j \geq 0$ and $\alpha > -1$. Hence the lemma holds in the 2-dimensional case, i.e., when $n = 1$, because the integral of z^j or $\overline{z}^j$ over any circle centered at the origin equals zero when

$j \neq 0$. The higher-dimensional case follows for the same reason from (2.6), the fact that the Wigner distribution function $\mathcal{W}_{H_\mu,H_\nu}(x,y)$ is the product of $\mathcal{W}_{h_{m_j},h_{n_j}}(x_j,y_j)$, and the rotation invariance of the ball $B^{2n}(r)$. □

An immediate corollary of Lemma 2.1, definition (1.2), and equality (2.4) is

Corollary 2.2. *In the notation above,*

$$\mathcal{E}\left(B^{2n}(r)\right) = \sup_{\mu} \int_{B^{2n}(r)} \mathcal{W}_{H_\mu} dxdy, \tag{2.8}$$

where the supremum is taken over all multi-indices $\mu = (\mu_1, \dots, \mu_n)$ of non-negative integers.

The following lemma is the main ingredient in the proof of Theorem 1.

Lemma 2.3. *For any integer $\lambda \geq 0$ and multi-indices μ_1, μ_2 with $\lambda = |\mu_1| = |\mu_2|$, one has*

$$\int_{B^{2n}(r)} \mathcal{W}_{H_{\mu_1}} dxdy = \int_{B^{2n}(r)} \mathcal{W}_{H_{\mu_2}} dxdy, \text{ for every } r \geqslant 0 \ . \tag{2.9}$$

Postponing the proof of Lemma 2.3, we first conclude the proof of Theorem 1.

Proof of Theorem 1. It follows from Corollary 2.2 and Lemma 2.3 above that

$$\mathcal{E}\left(B^{2n}(r)\right) = \sup_{\lambda} \int_{B^{2n}(r)} \mathcal{W}_{H_{\mu_\lambda}} dxdy, \tag{2.10}$$

where $\mu_\lambda = (\lambda, 0, \dots, 0)$, and λ is a non-negative integer. Moreover, from (2.6) and the definition of the Wigner distribution it follow that:

$$\mathcal{W}_{H_{\mu_\lambda}}(x,y) = \frac{(-1)^\lambda}{\pi^n}\, e^{-\sum_{i=1}^n (x_i^2+y_i^2)}\, L_\lambda\big(2(x_1^2+y_1^2)\big), \tag{2.11}$$

where $L_\lambda(z)$ are the $\alpha = 0$ Laguerre polynomials (2.7). Setting $z_j = x_j + iy_j$, we conclude that

$$\int\limits_{B^{2n}(r)} \mathcal{W}_{H_{\mu_\lambda}} dxdy = \int\limits_{\sum_{j=2}^n |z_j|^2 \leq r^2} e^{-\sum_{j=2}^n |z_j|^2} \Big(\int\limits_{|z_1|^2 \leq r^2 - \sum_{j=2}^n |z_j|^2} \frac{(-1)^\lambda}{\pi^n}\, e^{-|z_1|^2}\, L_\lambda(2|z_1|^2)\, dz_1 \Big) dz_2 \cdots dz_n. \tag{2.12}$$

On the other hand, from Flandrin's result in the 1-dimensional case [2], it follows that

$$\int\limits_{B^2(\alpha)} \mathcal{W}_{h_\lambda}\, dx_1 dy_1 = \int\limits_{|z_1|^2 \leq \alpha^2} (-1)^\lambda\, e^{-|z_1|^2}\, L_\lambda(2|z_1|^2)\, dz_1 \leqslant \int\limits_{|z_1|^2 \leq \alpha^2} e^{-|z_1|^2}\, dz_1, \tag{2.13}$$

for every radius $\alpha \geqslant 0$. An examination of Flandrin's proof reveals that the inequality is strict for $\lambda > 0$. Hence, for every non-negative integer λ one has

$$\int\limits_{B^{2n}(r)} \mathcal{W}_{H_{\mu_\lambda}}\, dxdy \leqslant \pi^{-n} \int\limits_{\sum_{j=1}^n |z_j|^2 \leq r^2} e^{-\sum_{j=1}^n |z_j|^2} dz_1 \cdots dz_n = 1 - \frac{\Gamma(n, r^2)}{(n-1)!} \tag{2.14}$$

with equality only for $\lambda = 0$. The proof of Theorem 1 now follows from (2.11) and (2.14). □

Remark: The integral in (2.10) is not monotone in λ or in r (except for $\lambda = 0$), as might have been thought. See [1, Fig. 2] and [2] for interesting graphs of these integrals as a function of r.

For the proof of Lemma 2.3 we shall need the following preliminaries. For a non-negative integer λ denote

$$\mathcal{H}_\lambda = \mathrm{span}\{H_\mu \ ; \ |\mu| = \lambda\} \subset L^2(\mathbb{R}^n) \ . \tag{2.15}$$

It follows from (2.3) above that the space $\mathcal{H}_\lambda$ consists of the eigenfunctions of the rotation invariant Schrödinger operator $\mathbb{H} = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2 - \frac{n}{2}$ with eigenvalue λ. In particular, it is a finite-dimensional, $O(n)$-invariant subspace of $L^2(\mathbb{R}^n)$ with orthonormal basis $\{H_\mu : |\mu| = \lambda\}$. It follows that for every $\mathcal{R} \in O(n)$, and every $\widetilde{\mu}$ with $|\widetilde{\mu}| = \lambda$, one has:

$$H_{\widetilde{\mu}}(\mathcal{R}x) = \sum_{\nu \,:\, |\nu| = \lambda} c_\nu(\widetilde{\mu}, \mathcal{R}) \, H_\nu(x), \tag{2.16}$$

where the coefficients $c_\nu(\widetilde{\mu}, \mathcal{R})$ satisfy $\sum |c_\nu(\widetilde{\mu}, \mathcal{R})|^2 = 1$.

We note the following useful fact: In order to identify which coefficients $c_\nu(\widetilde{\mu}, \mathcal{R})$ are non-zero, it is only necessary to check the leading powers on the two sides of (2.16). That is, the left side of (2.16) defines a polynomial of degree λ in the indeterminates $x_1, \dots, x_n$. The highest degree terms are the monomials $x_1^{\mu_1} \cdots x_n^{\mu_n}$ with $\sum_{j=1}^n \mu_j = \lambda$, but there are also monomials of degree lower than λ. In order to show that a given H_ν appears with a non-zero coefficient in the decomposition (2.16), it is only necessary to show that there is a highest degree monomial $x_1^{\nu_1} \cdots x_1^{\nu_n}$ in the decomposition. It is *not* necessary to check the lower degree monomials; they will appear automatically because we know that the decomposition contains only Hermite functions of degree λ and no others.

Proof of Lemma 2.3*:* Fix a non-negative integer λ, and $r \geqslant 0$. We consider the maximum problem

$$\max_{\mu \,:\, |\mu| = \lambda} \int\limits_{B^{2n}(r)} \mathcal{W}_{H_\mu} dxdy, \tag{2.17}$$

and denote by $\widetilde{\mu}$ one of its maximizers.

From the sesquilinearity property of the Wigner distribution and Lemma 2.1, we conclude that for every $\mathcal{R} \in O(n)$ one has:

$$\int\limits_{B^{2n}(r)} \mathcal{W}_{H_{\widetilde{\mu}}(\mathcal{R}x)} \, dxdy = \sum_\nu |c_\nu(\widetilde{\mu}, \mathcal{R})|^2 \int\limits_{B^{2n}(r)} \mathcal{W}_{H_\nu} \, dxdy \ . \tag{2.18}$$

Since $H_{\widetilde{\mu}}$ is a maximizer, this implies that for any ν_0 with $c_{\nu_0}(\widetilde{\mu}, \mathcal{R}) \neq 0$ one has

$$\int\limits_{B^{2n}(r)} \mathcal{W}_{H_{\widetilde{\mu}}} \, dxdy = \int\limits_{B^{2n}(r)} \mathcal{W}_{H_{\widetilde{\mu}}(\mathcal{R}x)} \, dxdy = \int\limits_{B^{2n}(r)} \mathcal{W}_{H_{\nu_0}} \, dxdy \ , \tag{2.19}$$

i.e., H_{ν_0} is also a maximizer. The lemma will be proved if we can show that, starting from any $\widetilde{\mu}$, we can, by a succession of rotations and intermediate indices, finally reach any given ν.

The proof will proceed in two steps. The first is to go from $\widetilde{\mu}$, by a succession of two-dimensional rotations, to $(\lambda, 0, 0, \ldots, 0)$ with $\lambda = \sum_{j=1}^{n} \widetilde{\mu}_j$.

First, we show that there is a rotation $\mathcal{R}' \in O(n)$ with

$$c_{\widetilde{\mu}'}(\widetilde{\mu}, \mathcal{R}') \neq 0, \text{ where } \widetilde{\mu}' := ((\widetilde{\mu}_1 + \widetilde{\mu}_2), 0, \widetilde{\mu}_3, \ldots, \widetilde{\mu}_n) \ . \tag{2.20}$$

Thus, $\widetilde{\mu}'$ is also a maximizer. In a similar fashion, we can go from $\widetilde{\mu}'$ to $\widetilde{\mu}''$, where $\widetilde{\mu}'' := ((\widetilde{\mu}_1 + \widetilde{\mu}_2 + \widetilde{\mu}_3), 0, 0, \widetilde{\mu}_4, \ldots, \widetilde{\mu}_n)$. Proceeding inductively, we finally arrive at the conclusion that $(\lambda, 0, \ldots, 0)$ is a maximizer.

A rotation $\mathcal{R}'$ that accomplishes the first step to $\widetilde{\mu}'$ is simply $\mathcal{R}' : x_1 \to (x_1 + x_2)/\sqrt{2},\ x_2 \to (x_1 - x_2)/\sqrt{2},\ x_j \to x_j$ for $j > 2$. The monomial $x_1^{\widetilde{\mu}_1} x_2^{\widetilde{\mu}_2}$ becomes $\frac{1}{2}(x_1 + x_2)^{\widetilde{\mu}_1}(x_1 - x_2)^{\widetilde{\mu}_2}$ and this obviously contains the monomial $x_1^{(\widetilde{\mu}_1 + \widetilde{\mu}_2)}$ with a non-zero coefficient.

The second step is to go in the other direction, from $(\lambda, 0, \ldots, 0)$ to $(\nu_1, \nu_2, \ldots, \nu_n)$ when $\sum_{j=1}^{n} \nu_j = \lambda$. As before, we do this with a sequence of two-dimensional rotations, the first of which takes us from $(\lambda, 0, \ldots, 0)$ to $(\lambda - \nu_2, \nu_2, 0, \ldots, 0)$. From thence we go to $(\lambda - \nu_2 - \nu_3, \nu_2, \nu_3, 0, \ldots, 0)$, and so forth. This can be accomplished with the same rotation as before, namely $\mathcal{R}' : x_1 \to (x_1 + x_2)/\sqrt{2},\ x_2 \to (x_1 - x_2)/\sqrt{2},\ x_j \to x_j$ for $j > 2$.

$\square$

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References

[1] Bracken, A. J., Doebner, H.-D., Wood, J. G. *Bounds on integrals of the Wigner function,* Phys. Rev. Lett. **83** (1999), 3758-3761.

[2] Flandrin, P. *Maximum signal energy concentration in a time-frequency domain,* Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing (ICASSP '88), **4** (1988), pp.2176-2179, NY, USA.

[3] Gröchenig, K. *Foundations of time-frequency analysis,* Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001.

[4] Janssen, A. J. E. M. *Positivity and spread of bilinear time-frequency distributions,* in The Wigner distribution, 1-58, Elsevier, Amsterdam, 1997.

[5] Janssen, A. J. E. M. *Positivity of Weighted Wigner Distributions,* SIAM J. Math. Anal. **12** no. 5, (1981), 1-58.

[6] Leonhardt, U. *Measuring the quantum state of light,* Cambridge Studies in Modern Optics 22, Cambridge University Press, Cambridge, 1997.

[7] Lieb, E.H. *Integral Bounds for Radar Ambiguity Functions and Wigner Distributions*, J. Math. Phys. **31** (1990), 594-599.

[8] Lieb, E.H. and Loss, M. *Analysis*, Amer. Math. Soc., Providence, R.I., 1997.

[9] Oonincx, P. J., ter Morsche, H.G., *The fractional Fourier transform and its application to energy localization problems,* EURASIP J. Appl. Signal Process. 2003, no. 12, 1257-1264.

[10] Pool, J.C.T. *Mathematical Aspects of the Weyl Correspondence*, J. Math. Phys. **7**, (1966), 66-76.

[11] Ramanathan, J., Topiwala, P. *Time-frequency localization via the Weyl correspondence,* SIAM J. Math. Anal. **24** (1993), 1378-1393.

[12] Slepian, D. *Some comments on Fourier analysis, uncertainty and modeling,* SIAM Rev. **25** (1983), no. 3, 379-393.

[13] Slepian, D., Pollak, H. O. *Prolate spheroidal wave functions, Fourier analysis and uncertainty. I,* Bell System Tech. J., **40** (1961), 43-63.

[14] Thangavelu, S. *Lectures on Hermite and Laguerre expansions,* Mathematical Notes, **42**, Princeton University Press, Princeton, NJ, 1993.

[15] Wallach, N. R. *Symplectic geometry and Fourier analysis. With an appendix on quantum mechanics by Robert Hermann,* Lie Groups: History, Frontiers and Applications, Vol. V. Math. Sci. Press, Brookline, Mass., 1977.

[16] Wigner, E.P. *On the quantum correction for thermodynamic equilibrium,* Phys. Rev. **40** (1932), 749-759.

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