Symplectic embeddings of the ℓ_p-sum of two discs

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Abstract

In this paper we study symplectic embedding questions for the ℓ_p-sum of two discs in $\mathbb{R}^4$, when $1 \leq p \leq \infty$. In particular, we compute the symplectic inner and outer radii in these cases, and show how different kinds of embedding rigidity and flexibility phenomena appear as a function of the parameter p.

1 Introduction and Results

Since Gromov's seminal work [8], questions about symplectic embeddings have lain at the core of symplectic geometry. These questions are usually very difficult even for a relatively simple class of examples. In fact, only recently has it become possible to specify exactly when a four-dimensional ellipsoid is symplectically embeddable in a ball [17], or in another four-dimensional ellipsoid [14]. For more information on symplectic embeddings we refer the reader e.g., to the recent survey [20].

In [18], using the theory of embedded contact homology, the second named author established sharp obstructions for symplectic embeddings of the product of two Lagrangian discs in $\mathbb{R}^4$ into balls, ellipsoids and symplectic polydiscs. This product configuration appears naturally as the phase space of billiard dynamics in a round disc (see e.g., [2,18]). In this note we extend the above results to the family of ℓ_p-sums of two Lagrangian discs. In particular, we show how different kinds of symplectic embedding rigidity and flexibility phenomena appear as functions of the parameter $1 \leq p \leq \infty$. For comparison, we also consider similar embedding questions in the natural counterpart case of the ℓ_p-sum of two symplectic discs. In order to be more precise and state our results, we first introduce some notations.

Consider $\mathbb{R}^4$ equipped with coordinates (x_1, x_2, y_1, y_2), and with the standard symplectic form $\omega_0 = \sum_{i=1}^{2} dx_i \wedge dy_i$. For two subsets $X_1, X_2 \subset \mathbb{R}^4$, we write

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$X_1 \hookrightarrow X_2$ if there is an embedding $\varphi : X_1 \hookrightarrow X_2$ preserving the symplectic form, i.e., $\varphi^*\omega_0 = \omega_0$. We denote the symplectic inner and outer radii of a set $X \subset \mathbb{R}^4$ by

$$r_S(X) = \sup\left\{r \in \mathbb{R} \mid B^4(r) \hookrightarrow X\right\} \text{ and } R_S(X) = \inf\left\{r \in \mathbb{R} \mid X \hookrightarrow B^4(r)\right\},$$

where $B^4(r) = \{z \in \mathbb{R}^4 \mid \pi\|z\|^2 < r\}$ is the Euclidean ball with Gromov width r (i.e., with radius $\sqrt{r/\pi}$). Moreover, for $1 \leq p < \infty$, we denote by

$$\mathbb{X}_p = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \|\mathbf{x}\|^p + \|\mathbf{y}\|^p < 1\right\}, \tag{1}$$

the ℓ_p-sum of two Lagrangian discs, where here $\|\cdot\|$ denotes the standard Euclidean norm on $\mathbb{R}^2$. If $p = \infty$ we set

$$\mathbb{X}_\infty = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \max(\|\mathbf{x}\|, \|\mathbf{y}\|) < 1\right\}.$$

Our first result concerns the inner and outer radii of $\mathbb{X}_p$. More precisely, for $p \geq 1$, we denote the area of the unit disk in $\mathbb{R}^2$ with respect to the standard ℓ_p-norm by

$$A(p) = 4\int_0^1 (1 - r^p)^{1/p}\, dr = \frac{4 \cdot \Gamma(1 + \frac{1}{p})^2}{\Gamma(1 + \frac{2}{p})}, \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function. Moreover, for $v \in [0, (1/4)^{1/p}]$ we define

$$g_p(v) := 2\int_{\left(\frac{1}{2}-\sqrt{\frac{1}{4}-v^p}\right)^{1/p}}^{\left(\frac{1}{2}+\sqrt{\frac{1}{4}-v^p}\right)^{1/p}} \sqrt{(1 - r^p)^{2/p} - \frac{v^2}{r^2}}\, dr. \tag{3}$$

Theorem 1. *For $1 \leq p < \infty$, the symplectic inner radius of $\mathbb{X}_p$ is given by*

$$r_S(\mathbb{X}_p) = \begin{cases} 2\pi\,(1/4)^{1/p}, & \text{if } 1 \leq p \leq 2, \\ A(p), & \text{if } p \geq 2, \end{cases}$$

and the symplectic outer radius by

$$R_S(\mathbb{X}_p) = \begin{cases} A(p), & \text{if } 1 \leq p \leq 2, \\ 2\pi\,(1/4)^{1/p}, & \text{if } 2 \leq p \leq 9/2, \\ 2\pi\left(g_p'\right)^{-1}(-2\pi/3) + 3g_p\left(\left(g_p'\right)^{-1}(-2\pi/3)\right), & \text{if } 9/2 < p < \infty. \end{cases}$$

Remark 2. It is shown in Proposition 18 below that g_p' is injective and its image contains $-2\pi/3$ for $p \geq 9/2$. Thus the expression in the last line of Theorem 1 is well defined. We also recall that in [18] it was proved that $r_S(\mathbb{X}_\infty) = 4$ and $R_S(\mathbb{X}_\infty) = 3\sqrt{3}$. It is clear that $A(p) \to 4$ as $p \to \infty$, and we will see below that $R_S(\mathbb{X}_p) \to 3\sqrt{3}$ as $p \to \infty$. Moreover, it can be shown explicitly that the functions defined by the formulas above are continuous at $p = 2$ and $p = 9/2$. This should indeed be the case, since $r_S(\mathbb{X}_p)$ and $R_S(\mathbb{X}_p)$ are clearly continuous. Finally, we remark that at $p = 9/2$, a certain "phase-transition" occurs between rigidity and flexibility of the embedding $\mathbb{X}_p \hookrightarrow B^4(r)$, as explained in Theorem 10 below.

The proof of Theorem 1 follows the approach of [18], i.e., we use the theory of integrable Hamiltonian systems to describe the domain $\mathbb{X}_p$ as a toric domain, and then use the machinery of ECH capacities to find symplectic embedding obstructions. To find optimal symplectic embeddings on the other hand, we combine results from [3], [5], [11], and [12]. We turn now to explain this in more details.

1.1 The ℓ_p-sum of two Lagrangian discs as a toric domain

Toric domains form a large class of symplectic manifolds that generalizes ellipsoids and polydiscs. For this class, certain symplectic embedding questions are better understood, particularly in dimension four, see, e.g., [4, 5, 13]. For our purposes, a domain $X \subset \mathbb{R}^4$ is toric if it is invariant under the standard action of $\mathbb{T}^2$ on $\mathbb{R}^4 \simeq \mathbb{C}^2$. Here, we allow X to have boundary and corners. Note that a toric domain X is completely determined by its image under the moment map $\mu : \mathbb{C}^2 \simeq \mathbb{R}^4 \to \mathbb{R}^2$, given by $\mu(z_1, z_2) = (\pi|z_1|^2, \pi|z_2|^2)$, where $z_j = x_j + iy_j$ for $j = 1, 2$. For a domain $\Omega \subset \mathbb{R}^2_{\geq 0}$, we denote the corresponding toric domain $\mu^{-1}(\Omega)$ in $\mathbb{R}^4$ by X_Ω.

From now on we assume that $\Omega \subset \mathbb{R}^2_{\geq 0}$ is the region bounded by the coordinate axes, and the graph of a decreasing continuous function $\varphi : [0, a] \to \mathbb{R}_{\geq 0}$ such that $\varphi(a) = 0$.

Definition 3. With the above notations, the set X_Ω is called a *concave* or *convex* toric domain if the function φ is convex or concave, respectively.

Remark 4. In the literature there are more general definitions of convex/concave toric domains including, e.g., rectangles touching the origin. However, we will not need to use them in the context of this paper, c.f. [5, 10].

The main ingredient in the proof of Theorem 1 above is the following result.

Theorem 5. *For $p \in [1, \infty)$, the interior of the Lagrangian product $\mathbb{X}_p$ is symplectomorphic to the interior of the toric domain X_{Ω_p}, where Ω_p is the relatively open set in $\mathbb{R}^2_{\geq 0}$ bounded by the coordinate axes and the curve parametrized by*

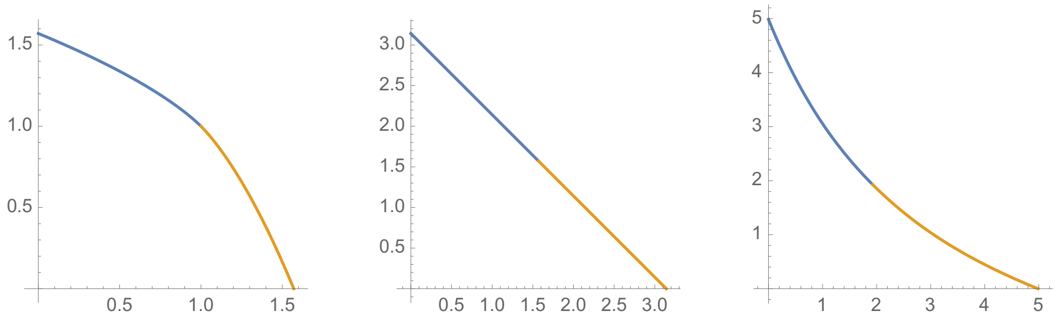
$$\begin{aligned} (2\pi v + g_p(v), g_p(v)), \quad & \textit{for } v \in [0, (1/4)^{1/p}], \\ (g_p(-v), -2\pi v + g_p(-v)), \quad & \textit{for } v \in [-(1/4)^{1/p}, 0], \end{aligned} \tag{4}$$

where $g_p : [0, (1/4)^{1/p}] \to \mathbb{R}$ is the function defined by (3) *above.*

The following analogous result for $\mathbb{X}_\infty$ was shown in [18].

Theorem 6 ([18, Theorem 3])**.** *The domain $\mathbb{X}_\infty$ is symplectomorphic to the toric domain X_{Ω_∞}, where Ω_∞ is the relatively open set in $\mathbb{R}^2_{\geq 0}$ bounded by the coordinate axes and the curve parametrized by*

$$2\left(\sqrt{1 - v^2} + v(\pi - \arccos v), \sqrt{1 - v^2} - v \arccos v\right), \quad \textit{for } v \in [-1, 1]. \tag{5}$$



(a) $p = 1$ (b) $p = 2$ (c) $p = 6$

Figure 1: The set Ω_p for different values of p

Remark 7. Note that the curve (4) is invariant under the reflection about the line $y = x$. Moreover, a simple calculation shows that (4) converges to (5) as $p \to \infty$.

With some additional computational work, we further claim that:

Proposition 8. *The toric domain X_{Ω_p} defined in Theorem 5 above is convex for $p \in [1, 2]$, and concave for $p \in [2, \infty]$.*

Figure 1 shows the set Ω_p for $p = 1, 2, 6$. One can directly check that Ω_2 is a right triangle, which reflects the fact that $\mathbb{X}_2$ is the Euclidean ball.

1.2 The rigidity and flexibility of the embeddings

In this section we discuss certain rigidity and flexibility phenomena of the embeddings described in Theorem 1, and explain the significance of the specific values of p appearing in that theorem. We start with the following notions of symplectic embedding rigidity, which for the purpose of this paper we state only in $\mathbb{R}^4$.

Definition 9. Let X_1 and X_2 be subdomains of $\mathbb{R}^4$.

(a) The symplectic embedding problem $X_1 \overset{?}{\hookrightarrow} X_2$ is said to be *rigid* if

$$X_1 \hookrightarrow rX_2 \iff X_1 \subseteq rX_2.$$

(b) The symplectic embedding problem $X_1 \overset{?}{\hookrightarrow} X_2$ is said to be *torically rigid* if the embedding $\widetilde{X}_1 \overset{?}{\hookrightarrow} \widetilde{X}_2$ is rigid, where $\widetilde{X}_1$ and $\widetilde{X}_2$ are two toric domains whose interiors are symplectomorphic to the interiors of X_1 and X_2, respectively.

(c) The symplectic embedding problem $X_1 \overset{?}{\hookrightarrow} X_2$ is said to be *non-rigid* if it is neither rigid, nor torically rigid.

With this terminology, we can now state the rigidity of the symplectic embeddings from Theorem 1 as follows.

Theorem 10. *Let $B \subset \mathbb{R}^4$ be a Euclidean ball, and let $\mathbb{X}_p$ be the ℓ_p-sum of two Lagrangian discs given by* (1). *Then,*

(a) The symplectic embedding $B \overset{?}{\hookrightarrow} \mathbb{X}_p$ is rigid for $1 \leq p \leq 2$.

(b) The symplectic embedding $B \overset{?}{\hookrightarrow} \mathbb{X}_p$ is torically rigid for all $p \geq 1$.

(c) The symplectic embedding $\mathbb{X}_p \overset{?}{\hookrightarrow} B$ is rigid for $2 \leq p \leq 9/2$.

(d) The symplectic embedding $\mathbb{X}_p \overset{?}{\hookrightarrow} B$ is torically rigid for $1 \leq p \leq 9/2$.

(e) The symplectic embedding $\mathbb{X}_p \overset{?}{\hookrightarrow} B$ is non-rigid for $p > 9/2$.

Remark 11. From our point of view, the most surprising part of Theorem 10 is the change of behavior of the embedding question $\mathbb{X}_p \overset{?}{\hookrightarrow} B$ at the value $p = 9/2$. Note that there is no change of convexity of the toric image at this value of p, and that there is no a priori reason for this transition from rigidity to flexibility of the embedding. We refer the reader to Lemma 19 and Proposition 20 below for more details on the appearance of the value $p = 9/2$ in this setting.

1.3 The ℓ^p-sum of two symplectic discs

As a natural counterpart of the above results, in this section we discuss the analogues of Theorems 1 and 10 for the ℓ^p-sum of two equal-size symplectic discs in $\mathbb{R}^4$. More precisely, for $p > 0$ let

$$\mathbb{B}_p(\mathbb{C}^2) = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid \pi^{p/2} \left(\|z_1\|^p + \|z_2\|^p\right) < 1\right\}.$$

Note that as $p \to \infty$, the set $\mathbb{B}_p(\mathbb{C}^2)$ converges to the symplectic polydisc

$$\mathbb{B}_\infty(\mathbb{C}^2) := \{(z_1, z_2) \in \mathbb{C}^2 \mid \max\{\pi\|z_1\|^2, \pi\|z_2\|^2\} < 1\}.$$

We remark that in the symplectic literature $\mathbb{B}_\infty(\mathbb{C}^2)$ is usually denoted by $P(1,1)$. It follows directly from the definition that $\mathbb{B}_p(\mathbb{C}^2)$ is a toric domain X_{Λ_p}, where Λ_p is the relatively open set in $\mathbb{R}^2_{\geq 0}$ bounded by the coordinate axes and the curve

$$x^{p/2} + y^{p/2} = 1 \quad \text{(or } \max\{\|x\|, \|y\|\} = 1 \text{ for } p = \infty).$$

Thus, $\mathbb{B}_p(\mathbb{C}^2)$ is concave if $0 < p \leq 2$, and convex if $p \geq 2$. Moreover, it is clear that in this case the notions of rigidity and toric rigidity coincide.

Theorem 12. *Let $B \subset \mathbb{R}^4$ be the Euclidean unit ball. Then*

(a) The symplectic embedding $B \overset{?}{\hookrightarrow} \mathbb{B}_p(\mathbb{C}^2)$ is rigid for all $p \geq 1$, and

$$B(c) \hookrightarrow \mathbb{B}_p(\mathbb{C}^2) \iff c \leq \min\{1, 2^{1-2/p}\}.$$

(b) The symplectic embedding $\mathbb{B}_p(\mathbb{C}^2) \overset{?}{\hookrightarrow} B$ is rigid for $p \geq 2$, and

$$\mathbb{B}_p(\mathbb{C}^2) \hookrightarrow B(c) \iff c \geq 2^{1-2/p}.$$

(c) The symplectic embedding $\mathbb{B}_p(\mathbb{C}^2) \overset{?}{\hookrightarrow} B$ is non-rigid for $1 \leq p < 2$ and

$$\mathbb{B}_p(\mathbb{C}^2) \hookrightarrow B(c) \iff c \geq \left(1 + 2^{\frac{p}{p-2}}\right)^{1-2/p}.$$

In the last case, we can actually prove a little more. Consider the ellipsoid

$$E(a,b) = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid \pi\left(\frac{|z_1|^2}{a} + \frac{|z_2|^2}{b}\right) < 1\right\}.$$

Proposition 13. *With the above notations,*

$$\mathbb{B}_1(\mathbb{C}^2) \hookrightarrow E(a,b) \iff \min(a,b) \geq 1/2 \text{ and } \max(a,b) \geq 2/3.$$

Remark 14. It is not hard to check that $\mathbb{B}_1(\mathbb{C}^2)$ and $E(1/2, 2/3)$ have the same volume, and thus $\mathbb{B}_1(\mathbb{C}^2) \hookrightarrow E(1/2, 2/3)$ is a volume filling embedding. Proposition 13 can be extended to prove that there exists an embedding

$$\mathbb{B}_p(\mathbb{C}^2) \hookrightarrow E\left(2^{1-2/p}, \left(1 + 2^{\frac{p}{p-2}}\right)^{1-2/p}\right)$$

for all $p \in [1,2)$, but for $p \in (1,2)$ this embedding is not volume filling. Since the proof of the existence of this embedding for $p \in (1,2)$ is very similar to the case $p = 1$, although much more tedious, we omit it here.

Structure of the paper: In Section 2 we use the integrability of the Hamiltonian system associated with $\mathbb{X}_p$ to prove Theorem 5. In Section 3 we recall some relevant definitions and results concerning the ECH capacities, and in particular compute the first two capacities of $\mathbb{X}_p$. Finally, in Section 4 we prove Theorems 1, 10, and 12, as well as Proposition 13.

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2 Integrable systems and toric domains

In this section we prove Theorem 5, which is the main ingredient in the proof of Theorem 1. We start by recalling the classical Arnold-Liouville theorem and the construction of action-angle coordinates [1]. In what follows, if B is an open set in $\mathbb{R}^n$, by abuse of notations we will denote by ω_0 also the standard symplectic form on $B\times\mathbb{T}^n$, i.e., $\omega_0 = \sum_i d\rho_i \wedge d\theta_i$, where $(\rho_1,\dots,\rho_n)$ and $(\theta_1,\dots,\theta_n)$ are the coordinates on $\mathbb{R}^n$ and on the torus $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$, respectively.

Theorem 15 (Arnold-Liouville)**.** *Let (M^{2n},ω) be a symplectic manifold, and let $F = (H^1,\dots,H^n) : M \to \mathbb{R}^n$ be a C^∞-function whose components Poisson commute, i.e., $\{H^i,H^j\} = 0$ for all $1 \le i,j \le n$.*

(a) If $c \in \mathbb{R}^n$ is a regular value of F, i.e., the differentials of $H_1,\dots,H_n$ are independent on $F^{-1}(c)$, and $F^{-1}(c)$ is compact and connected, then $F^{-1}(c) \cong \mathbb{T}^n$.

(b) Let $U \subset M$ be an open set such that $F(U)$ is simply-connected, and does not contain critical values. For $c \in F(U)$, let $\{\gamma_1^c,\dots,\gamma_n^c\}$ be a set of simple closed curves generating $H_1(F^{-1}(c);\mathbb{Z})$ that depend smoothly on c, and suppose that ω has a primitive λ on U. Consider the function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\varphi(c) = \left(\int_{\gamma_1^c} \lambda, \dots, \int_{\gamma_n^c} \lambda\right). \tag{6}$$

Then φ is a diffeomorphism with its image B, and there exists a symplectomorphism $\Phi : (U,\omega) \to (B \times \mathbb{T}^n, \omega_0)$ such that the following diagram commutes.

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & B \times \mathbb{T}^n \\ \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle \pi_1} \\ F(U) & \xrightarrow{\varphi} & B \end{array}$$

where here π_1 denotes the projection onto the first factor.

Remark 16. The original result in [1] states that if $c \in \mathbb{R}^n$ is a regular value of F that satisfies the assumptions in (a), then it always has a neighborhood U satisfying the assumptions in (b). So the conclusion of Theorem 15(b) holds in a neighborhood of c. On the other hand, we remark that if one can extend the family of curves $\gamma_1^c,\dots,\gamma_n^c$, while still maintaining the assumptions in (b), the diffeomorphisms φ and the symplectomorphism Φ can be extended too.

2.1 The toric picture of the Lagrangian ℓ_p-sum

Fix $p \in [1, \infty)$. For $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \oplus \mathbb{R}^2$, a natural defining Hamiltonian function for the Lagrangian p-sum $\mathbb{X}_p$ (1) is the function $H_p : \mathbb{R}^4 \to \mathbb{R}$ given by

$$H_p(\mathbf{x}, \mathbf{y}) := \|\mathbf{x}\|^p + \|\mathbf{y}\|^p.$$

Note that $\partial \mathbb{X}_p = H_p^{-1}(1)$, and while for $p \geq 2$ the function H_p is C^1, it is not differentiable for $1 < p < 2$. Thus, in order to use the Arnold-Liouville theorem stated above, we first approximate H_p by a sequence of smooth functions. We write H_p as $H_p = H_p^2 \circ H_p^1$, where $H_p^1 : \mathbb{R}^4 \to \mathbb{R}^2$ and $H_p^2 : \mathbb{R}^2 \to \mathbb{R}$ are given by

$$H_p^1(\mathbf{x}, \mathbf{y}) = (\|\mathbf{x}\|^2, \|\mathbf{y}\|^2), \quad \text{and} \quad H_p^2(s, t) = s^{p/2} + t^{p/2}.$$

Note that $H_p^1 \in C^\infty(\mathbb{R}^4)$, and that H_p^2 is smooth away from the coordinate axes. We approximate H_p by a family of smooth Hamiltonian functions

$$H_p^\varepsilon(\mathbf{x}, \mathbf{y}) := H_p^{2,\varepsilon} \circ H_p^1(\mathbf{x}, \mathbf{y}),$$

where the function $H_p^{2,\varepsilon} : \mathbb{R}^2_{\geq 0} \setminus \{0\} \to \mathbb{R}$ is defined as follows. For $\varepsilon > 0$ small, let

$$f^\varepsilon(t) := \begin{cases} \alpha^\varepsilon(t), & \text{for } 0 \leq t \leq \varepsilon, \\ (1 - t^{p/2})^{2/p}, & \text{for } \varepsilon \leq t \leq 1 - \varepsilon, \\ \beta^\varepsilon(t), & \text{for } 1 - \varepsilon < t < 1, \end{cases}$$

where α^ε and β^ε are smooth decreasing functions with $\alpha^\varepsilon(0) = 1$, $\beta^\varepsilon(1) = 0$, $(\alpha^\varepsilon)'(0) < 0$, and such that

$$\varepsilon < \widetilde{\varepsilon} \Rightarrow \alpha^\varepsilon \geq \alpha^{\widetilde{\varepsilon}} \text{ and } \beta^\varepsilon \geq \beta^{\widetilde{\varepsilon}},$$

and such that the function $t \mapsto t f^\varepsilon(t)$ is strictly monotone for all $t \in [0, \varepsilon] \cup [\varepsilon, 1 - \varepsilon]$. In particular, $t \mapsto t f^\varepsilon(t)$ has a unique critical point at $t = 1/2^{2/p}$, where the function $t f^\varepsilon(t)$ attains its global maximum. For $(s, t) \in \mathbb{R}^2_{\geq 0} \setminus \{0\}$, we now define

$$H_p^{2,\varepsilon}(s, t) = \lambda^{-p/2},$$

where λ is the unique number in $\mathbb{R}_{>0}$ such that $\lambda s = f^\varepsilon(\lambda t)$. Note that one has $H_p^{2,\varepsilon} \in C^\infty(\mathbb{R}^2_{\geq 0} \setminus \{0\})$. Finally, we set $H_p^\varepsilon = H_p^{2,\varepsilon} \circ H_p^1 \in C^\infty(\mathbb{R}^4 \setminus \{0\})$.

Next, we denote by $V : \mathbb{R}^4 \to \mathbb{R}$ the standard "angular momentum" given by

$$V(\mathbf{x}, \mathbf{y}) := y_1 x_2 - y_2 x_1.$$

The following propositions shows, roughly speaking, that the dynamical system associated with $\mathbb{X}_p$ is "integrable" in the sense of Theorem 15 above. More precisely,

Proposition 17. *Let $F^\varepsilon = (H_p^\varepsilon, J) : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^2$. Then*

(a) $\{H_p^\varepsilon, V\} = 0$.

(b) The image of F^ε consists of all points $(h, v) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \setminus \{(0,0)\}$ such that $|v| \leq \left(\frac{h}{2}\right)^{2/p}$, with equality occurring if and only if (h, v) is a critical value.

(c) If $|v| < \left(\frac{h}{2}\right)^{2/p}$, then $(F^\varepsilon)^{-1}(h, v)$ is compact and connected.

Proof. (a) Let (r, θ) be the polar coordinates for $\mathbf{y}$, and let (p_r, p_θ) be the associated coordinates for $\mathbf{x}$. In particular,

$$\|\mathbf{y}\|^2 = r^2, \qquad \|\mathbf{x}\|^2 = p_r^2 + \frac{p_\theta^2}{r^2}, \qquad \mathbf{y} \times \mathbf{x} = p_\theta,$$

and

$$H_p^\varepsilon(\mathbf{x}, \mathbf{y}) = H_p^{2,\varepsilon}\left(\|x\|^2, \|y\|^2\right) = H_p^{2,\varepsilon}\left(p_r^2 + \frac{p_\theta^2}{r^2}, r^2\right), \quad \text{and} \quad V(\mathbf{x}, \mathbf{y}) \ = p_\theta.$$

Consequently one has $\{H_p^\varepsilon, V\} = 0$.

(b) Suppose that $H_p^\varepsilon(\mathbf{x}, \mathbf{y}) = h$, and $V(\mathbf{x}, \mathbf{y}) = v$. It follows from the definition of the function H_p^ε that

$$h^{-2/p}\|\mathbf{x}\|^2 = f^\varepsilon(h^{-2/p}\|\mathbf{y}\|^2). \tag{7}$$

By the assumptions on f^ε, the maximum of the function $t \mapsto t f^\varepsilon(t)$ is $1/2^{4/p}$, which is attained at $t = 1/2^{2/p}$. So it follows from (7) that

$$v^2 \leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2 = h^{2/p}\|\mathbf{y}\|^2 f^\varepsilon(h^{-2/p}\|\mathbf{y}\|^2) \ \ \leq \left(\tfrac{h}{2}\right)^{4/p}.$$

Moreover, the extremal values of v are attained if and only if $\mathbf{x}$ is orthogonal to $\mathbf{y}$, and $h^{-2/p}\|\mathbf{y}\|^2$ is the point of maximum of $t \mapsto t f^\varepsilon(t)$. We next compute the gradients ∇H_p^ε and ∇V. Since $H_p^{2,\varepsilon}$ is homogeneous, it follows that

$$\nabla H_p^\varepsilon(\mathbf{x}, \mathbf{y}) = c\left(\mathbf{x}, -(f^\varepsilon)'(h^{-2/p}\|\mathbf{y}\|^2)\mathbf{y}\right),$$

for some $c > 0$. On the other hand, an easy calculation gives

$$\nabla V(\mathbf{x}, \mathbf{y}) = (J_0\mathbf{y}, -J_0\mathbf{x}), \text{ where } J_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It is clear that ∇H_p^ε and ∇V never vanish. Thus, $(\mathbf{x}, \mathbf{y})$ is a critical point of F^ε if and only if $\nabla H_p^\varepsilon(\mathbf{x}, \mathbf{y})$ and $\nabla V(\mathbf{x}, \mathbf{y})$ are parallel. This happens if and only if

$$\mathbf{x} = \pm\sqrt{-(f^\varepsilon)'(h^{-2/p}\|\mathbf{y}\|^2)}J_0\mathbf{y}. \tag{8}$$

Using (8) we conclude that (7) is equivalent to requiring that $\mathbf{x}$ is orthogonal to $\mathbf{y}$, and

$$h^{2/p} f^\varepsilon(h^{-2/p}\|\mathbf{y}\|^2) = -(f^\varepsilon)'(h^{-2/p}\|\mathbf{y}\|^2)\|\mathbf{y}\|^2,$$

i.e., $h^{-2/p}\|\mathbf{y}\|^2$ is a critical point of $tf^\varepsilon(t)$. By assumption this function has a single critical point. So (8) holds if and only if $\mathbf{x}$ is orthogonal to $\mathbf{y}$ and $h^{-2/p}\|\mathbf{y}\|^2$ is the point of maximum of $tf^\varepsilon(t)$. Therefore $|v| = \left(\frac{h}{2}\right)^{2/p}$ if and only if (h, v) is a critical value as required.

(c) First observe that $(F^\varepsilon)^{-1}(h, v)$ is a closed set in $\mathbb{R}^2$ which is contained in a bounded set, and so is compact. Next, let $(\mathbf{x}, \mathbf{y}) \in (F^\varepsilon)^{-1}(h, v)$. First suppose that $v = 0$. So $\mathbf{x}$ and $\mathbf{y}$ are parallel, and at least one of the two is nonzero. Through scaling, one can connect $(\mathbf{x}, \mathbf{y})$ to $(h^{1/p}\mathbf{z}, 0)$, where $\mathbf{z} = \mathbf{x}/\|\mathbf{x}\|$ or $\mathbf{z} = \mathbf{y}/\|\mathbf{y}\|$. Similarly, if $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \in (F^\varepsilon)^{-1}(h, 0)$, it can also be connected to a point of the form $(h^{1/p}\widetilde{\mathbf{z}}, 0) \in \mathbb{R}^2 \times \mathbb{R}^2$ where $\|\widetilde{\mathbf{z}}\| = 1$. Now through a rotation in $\mathbb{R}^2 \times \{0\}$ we can connect $(h^{1/p}\mathbf{z}, 0)$ to $(h^{1/p}\widetilde{\mathbf{z}}, 0)$ while staying in $(F^\varepsilon)^{-1}(h, 0)$. Next suppose that $v \neq 0$, and let $(\mathbf{x}, \mathbf{y}) \in (F^\varepsilon)^{-1}(h, v)$. Using polar coordinates $(p_r, p_\theta, r, \theta)$, it follows from (7) that

$$p_r^2 = h^{2/p} f^\varepsilon(h^{-2/p} r^2) - \frac{v^2}{r^2}. \tag{9}$$

Moreover, any quadruple $(p_r, p_\theta, r, \theta)$ represents a point in $(F^\varepsilon)^{-1}(h, v)$ if $v = p_\theta$ and (r, p_r) satisfy (9). Now if $(p_r, p_\theta, r, \theta)$ and $(\widetilde{p}_r, \widetilde{p}_\theta, \widetilde{r}, \widetilde{\theta})$ are the polar coordinates of two points in $(F^\varepsilon)^{-1}(h, v)$, we construct a path between them as follows. First, note that $v = p_\theta = \widetilde{p}_\theta$. Let $\{\theta(\tau)\}_{\tau \in [0,1]} \subset \mathbb{R}/2\pi\mathbb{Z}$ be a path connecting θ to $\widetilde{\theta}$. We now define a path $\{r(t)\}_{t \in [0,1]} \in \mathbb{R}_{>0}$ as follows. Let $\mathrm{sign} : \mathbb{R} \to \{-1, 0, 1\}$ denote the sign function, and let r_{max} be the largest solution of the equation

$$h^{2/p} r^2 f^\varepsilon(h^{-2/p} r^2) = v^2. \tag{10}$$

For $t \in [0, 1/2]$, let $r(t)$ be the affine path connecting r to r_{max}, and for $t \in [1/2, 1]$, let $r(t)$ be the affine path connecting r_{max} to $\widetilde{r}$. By definition $r(t)$ is continuous. Next define

$$p_r(t) = \begin{cases} \mathrm{sign}(p_r)\sqrt{h^{2/p} f^\varepsilon(h^{-2/p} r(t)^2) - \dfrac{v^2}{r(t)^2}}, & \text{if } t \in [0, 1/2], \\ \mathrm{sign}(\widetilde{p}_r)\sqrt{h^{2/p} f^\varepsilon(h^{-2/p} r(t)^2) - \dfrac{v^2}{r(t)^2}}, & \text{if } t \in [1/2, 1]. \end{cases}$$

The function $p_r(t)$ is well defined and continuous at $t = 1/2$ because $p_r(1/2) = 0$ with either definition. Thus, $(p_r(t), \theta, r(t), \theta(t))$ represents a path connecting $(p_r, p_\theta, r, \theta)$ to $(\widetilde{p}_r, \widetilde{p}_\theta, \widetilde{r}, \widetilde{\theta})$, which shows that $(F^\varepsilon)^{-1}(h, v)$ is connected.

This completes the proof of Proposition 17. $\square$

Equipped with Theorem 15 and Proposition 17, we can now prove Theorem 5.

Proof of Theorem 5. Let

$$U^{h,\varepsilon} = (F^\varepsilon)^{-1}\left(\left\{(h,v)\in\mathbb{R}^2 \mid |v| \le \left(\frac{h}{2}\right)^{2/p}\right\}\right),$$
$$U^{h,\varepsilon}_{int} = (F^\varepsilon)^{-1}\left(\left\{(h,v)\in\mathbb{R}^2\setminus\{0\} \mid |v| < \left(\frac{h}{2}\right)^{2/p}\right\}\right).$$

It is clear that $F(U^{h,\varepsilon}_{int})$ is simply-connected. It follows from Theorem 15 that $U^{h,\varepsilon}_{int}$ is symplectomorphic to $\mu^{-1}(\varphi(U^{h,\varepsilon}_{int}))$, where φ are the action coordinates defined in (6). We now define such φ using appropriate sets of curves and a Liouville form λ. Fix $c=(h,v)$, and let r_{min}, r_{max} be the smallest and largest solutions of (10). Set

$$\mathbf{y}_0 = (r_{max}, 0).$$
$$\mathbf{x}_0 = (0, \operatorname{sign}(v)\cdot r_{min}).$$

Let σ_0 be the curve parametrized by $\varphi_t(\mathbf{x}_0,\mathbf{y}_0)$ with $t\in[0,t_0]$, where $\{\varphi_t\}$ is the flow of the vector field $X_{H^\varepsilon_p}$, and t_0 is the smallest $t>0$ such that the norm of the $\mathbf{y}$-component of $\varphi_t(\mathbf{x}_0,\mathbf{y}_0)$ is r_{max}. Let $(\widetilde{\mathbf{x}}_0,\widetilde{\mathbf{y}}_0)=\varphi_{t_0}(\mathbf{x}_0,\mathbf{y}_0)$. Note that there exists $A\in SO(2)$ such that $A\widetilde{\mathbf{x}}_0=\mathbf{x}_0$ and $A\widetilde{\mathbf{y}}_0=\mathbf{y}_0$. Let A^t_1 and A^t_2 be two simple curves in $SO(2)\cong S^1$ connecting the identity with A, and rotating counterclockwise and clockwise, respectively. For $i=1,2$, let σ_i denote the curve parametrized by $(A^t_i\widetilde{\mathbf{x}}_0, A^t_i\widetilde{\mathbf{y}}_0)$, and define γ^c_i to be the composition of the curve σ_0 with σ_i. We observe that by definition $\{\gamma^c_1,\gamma^c_2\}$ generates $H_1\left((F^\varepsilon)^{-1}(c)\right)$.

Next, let $\lambda=\sum_{i=1}^2 x_i dy_i$. We write $(\mathbf{x},\mathbf{y})$ in polar coordinates (p_r,p_θ,r,θ) as in the proof of Proposition 17. If $v=0$, then $\lambda = p_r\,dr$. Hence, in this case

$$\int_{\gamma^c_i}\lambda = \int_{\gamma^c_i} p_r\,dr = \int_{\sigma_0} p_r\,dr. \tag{11}$$

Now suppose that $v\neq 0$. So γ^c_i does not go through the origin $\mathbf{y}=0$. For $i=1,2$,

$$\int_{\gamma^c_i}\lambda = \int_{\gamma^c_i} p_r\,dr + \int_{\gamma^c_i} p_\theta\,d\theta = \int_{\sigma_0} p_r\,dr + v\int_{\gamma^c_i} d\theta. \tag{12}$$

For all v, we can compute the integral

$$\int_{\sigma_0} p_r\,dr = 2\int_{r_{min}}^{r_{max}}\left(h^{2/p}f^\varepsilon(h^{-2/p}r^2)-\frac{v^2}{r^2}\right)^{1/2} dr. \tag{13}$$

Let $g_p^\varepsilon(h,v)$ be the function defined by the expression in (13). We also observe that

$$\begin{aligned} \int_{\gamma_1^c} d\theta &= \begin{cases} 2\pi, & \text{if } v > 0, \\ 0, & \text{if } v < 0. \end{cases} \\ \int_{\gamma_2^c} d\theta &= \begin{cases} 0, & \text{if } v > 0, \\ -2\pi, & \text{if } v < 0. \end{cases} \end{aligned} \tag{14}$$

It follows from (11), (12), (13) and (14) that

$$\varphi^\varepsilon(h,v) = \begin{cases} (g_p^\varepsilon(h,v) + 2\pi v, g_p^\varepsilon(h,v)), & \text{if } v \geq 0, \\ (g_p^\varepsilon(h,v), g_p^\varepsilon(h,v) - 2\pi v), & \text{if } v < 0. \end{cases} \tag{15}$$

It is easy to see that φ^ε extends to a function defined on $U^{h,\varepsilon}$. Finally, to see that the symplectomorphism $U_{int}^{h,\varepsilon} \cong \mu^{-1}(\varphi^\varepsilon(U_{int}^{h,\varepsilon}))$ extends to a symplectomorphism $U^{h,\varepsilon} \cong \mu^{-1}(\varphi^\varepsilon(U^{h,\varepsilon}))$ we use a similar method to the one in [19, Lemma 35]. Namely, we first use a theorem of Eliasson [7] to show that the symplectomorphism extends to the pre-images of the points $(h,v) \neq (0,0)$ such that $|v| = (h/2)^{2/p}$, see [6, 7]. Next, to extend the symplectomorphism to $(0,0)$, we use a Theorem of Gromov-McDuff [16, Theorem 9.4.2] as it was done in the proof of [18, Theorem 3].

Next we observe that by definition $f^\varepsilon(t) \to (1 - t^{p/2})^{2/p}$ as $\varepsilon \to 0$, and hence $h^{2/p} f^\varepsilon(h^{-2/p} r^2) \to (1 - r^p)^{2/p}$ as $(h,\varepsilon) \to (1,0)$. Moreover, r_{min} and r_{max} converge to the two roots of the equation $r^2(1 - r^p)^{2/p} = v^2$, namely

$$r_\pm = \left(\frac{1}{2} \pm \sqrt{\frac{1}{4} - v^p} \right)^{1/p}.$$

It follows from (3), (13) and (15) that $\varphi^\varepsilon(h,v)$ converges to (4) as $(h,\varepsilon) \to (1,0)$. Therefore $\bigcup_{\varepsilon>0} U^{1-\varepsilon,\varepsilon} = X_{\Omega_p}$. Note that X_{Ω_p} is open, unlike $U^{1-\varepsilon,\varepsilon}$

Now let $\{\varepsilon_n\}$ be a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. We can then argue as in the proof of [18, Theorem 3] to conclude that we can find possibly different symplectomorphisms $\Phi_n : U^{1-\varepsilon_n,\varepsilon_n} \to \varphi^{\varepsilon_n}(U^{1-\varepsilon_n,\varepsilon_n})$ for all n such that

$$\Phi_{n+1}|_{U^{1-\varepsilon_n,\varepsilon_n}} = \Phi_n.$$

Finally, define $\Phi : \mathbb{X}_p \to \mathbb{C}^2$ by $\Phi(z) = \Phi_n(z)$ if $z \in U^{1-\varepsilon_n,\varepsilon_n}$. Therefore Φ is a symplectic embedding whose image is X_{Ω_p}, and the proof of the theorem is complete. □

2.2 Convexity/Concavity of the toric image

In this section we prove Proposition 8. We start by establishing the following properties of the function g_p which appear in the toric description (4).

Lemma 18. *The function* $g_p : [0, 1/4^{1/p}] \to \mathbb{R}$ *defined in* (3) *satisfies the following:*

(a) $g_p(0) = A(p)/2$*, and* $g_p(1/4^{1/p}) = 0$*, for every* $p \geq 1$*.*

(b) g_p *is strictly decreasing for all* $p \geq 1$*.*

(c) g_p *is strictly concave if* $1 < p < 2$*, and strictly convex if* $p > 2$*.*

(d) For $p = 2$ *one has* $g_2(v) = \frac{\pi}{2} - \pi v$*.*

(e) $\lim_{v\to 0} g_p'(v) = -\pi$ *and* $\lim_{v\to(1/4)^{1/p}} g_p'(v) = -\sqrt{\frac{2}{p}}\pi$*, for every* $p \geq 1$*.*

(f) If $p \geq 9/2$*, the derivative* g_p' *is injective and its image contains the point* $-2\pi/3$*.*

Proof of Lemma 18. Note first that the boundaries of the integral in (3) guarantee that the integrand is well defined. Moreover, one can check directly that the function $g_p(v)$ is differentiable in the interval $(0, 1/4^{1/p})$, for every $p \geq 1$.

(a) This follows immediatly from (2) and (3).

(b) By differentiating (3), one has that for every $p \geq 1$ and $v \in (0, 1/4^{1/p})$

$$g_p'(v) = -2\int_{\left(\frac{1}{2}-\sqrt{\frac{1}{4}-v^p}\right)^{1/p}}^{\left(\frac{1}{2}+\sqrt{\frac{1}{4}-v^p}\right)^{1/p}} \left((1-r^p)^{2/p} - \frac{v^2}{r^2}\right)^{-1/2} \frac{v}{r^2}\, dr. \tag{16}$$

Hence $g_p'(v) < 0$ for all $v \in (0, 1/4^{1/p})$, and therefore g_p is strictly decreasing.

(c) We first change variables in (16) by setting $w = r^p - 1/2$, and obtain

$$\begin{aligned} g_p'(v) &= -\frac{2}{p}\int_{-\sqrt{\frac{1}{4}-v^p}}^{\sqrt{\frac{1}{4}-v^p}} \frac{v}{\left(w+\frac{1}{2}\right)\sqrt{\left(\frac{1}{4}-w^2\right)^{2/p} - v^2}}\, dw \\ &= -\frac{2}{p}\int_{0}^{\sqrt{\frac{1}{4}-v^p}} \frac{v}{\left(\frac{1}{4}-w^2\right)\sqrt{\left(\frac{1}{4}-w^2\right)^{2/p} - v^2}}\, dw. \end{aligned} \tag{17}$$

Next we change variables again by setting

$$x = \frac{v^{p/2} w}{\sqrt{\frac{1}{4} - v^p - w^2}}. \tag{18}$$

It follows from a straight-forward calculation that

$$\frac{dw}{dx} = \frac{v^p\sqrt{\frac{1}{4} - v^p}}{(v^p + x^2)^{3/2}}, \tag{19}$$

$$\frac{1}{4} - w^2 = \frac{v^p}{v^p + x^2}\left(\frac{1}{4} + x^2\right). \tag{20}$$

From (17), (18), (19) and (20) we obtain

$$\begin{aligned} g_p'(v) &= -\frac{2}{p}\int_0^\infty \frac{v\sqrt{\frac{1}{4} - v^p}}{\sqrt{v^p + x^2}\left(\frac{1}{4} + x^2\right)\sqrt{\left(\frac{v^p}{v^p+x^2}\left(\frac{1}{4} + x^2\right)\right)^{2/p} - v^2}}\,dx \\ &= -\frac{2}{p}\int_0^\infty \frac{\sqrt{\frac{1}{4} - v^p}}{\sqrt{v^p + x^2}\left(\frac{1}{4} + x^2\right)\sqrt{\left(\frac{\frac{1}{4}+x^2}{v^p+x^2}\right)^{2/p} - 1}}\,dx \\ &= -\frac{2}{p}\int_0^\infty \frac{1}{\frac{1}{4} + x^2}\cdot\sqrt{\frac{\frac{\frac{1}{4}+x^2}{v^p+x^2} - 1}{\left(\frac{\frac{1}{4}+x^2}{v^p+x^2}\right)^{2/p} - 1}}\,dx. \end{aligned} \tag{21}$$

Note that for $u > 1$, the function $u \mapsto \frac{u-1}{u^{2/p}-1}$ is strictly decreasing or increasing if $0 < p < 2$ or $p > 2$, respectively. Moreover, for fixed x the function $v \mapsto \frac{\frac{1}{4}+x^2}{v^p+x^2}$ is strictly decreasing. Therefore, g_p' is strictly decreasing or increasing if $0 < p < 2$ or $p > 2$, respectively, which proves the claim.

(d) If we let $p = 2$ in (21) then,

$$g_2'(v) = -\int_0^\infty \frac{1}{\frac{1}{4} + x^2}\,dx = -\pi.$$

From (a), it follows that $g_2(0) = \pi/2$, and hence $g_2(v) = \frac{\pi}{2} - \pi v$, as required.

(e) It follows from (21) that

$$\lim_{v\to 0} g_p'(v) = -\frac{1}{p}\int_0^\infty \frac{dx}{x\left(\frac{1}{4} + x^2\right)\sqrt{\left(\frac{1}{4x^2} + 1\right)^{2/p} - 1}}.$$

Setting $t = (\frac{1}{4x^2} + 1)^{1/p}$, we obtain

$$\lim_{v\to 0} g_p'(v) = -2\int_1^\infty \frac{dt}{t\sqrt{t^2 - 1}} = -2\arctan\sqrt{t^2 - 1}\,\Big|_1^\infty = -\pi.$$

Next, observe that

$$\lim_{v\to(1/4)^{1/p}} \frac{\left(\frac{1}{4}+x^2\right)^{2/p}-(v^p+x^2)^{2/p}}{\frac{1}{4}-v^p} = \frac{2}{p}\left(\frac{1}{4}+x^2\right)^{2/p-1}. \tag{22}$$

Relations (21) and (22) yield

$$\begin{aligned}\lim_{v\to(1/4)^{1/p}} g_p'(v) &= -\frac{2}{p}\int_0^\infty \left(\frac{1}{4}+x^2\right)^{\frac{1}{p}-\frac{3}{2}} \sqrt{\lim_{v\to(1/4)^{1/p}} \frac{\frac{1}{4}-v^p}{\left(\frac{1}{4}+x^2\right)^{2/p}-(v^p+x^2)^{2/p}}}\,dx \\ &= -\frac{2}{p}\cdot\sqrt{\frac{p}{2}}\int_0^\infty \left(\frac{1}{4}+x^2\right)^{-1} dx = -\sqrt{\frac{2}{p}}\pi.\end{aligned}$$

(f) We proved in (c) that g_p' is strictly increasing if $p > 2$. In particular, g_p' is injective if $p > 2$. It follows from (e) that the image of g_p' is the interval $\left[-\sqrt{\frac{2}{p}}\pi, -\pi\right]$, which contains the point $-2\pi/3$ if $p \geq 9/2$.

□

Proof of Proposition 8. Let $p \geq 1$, and let $(x(v), y(v))$ be the parametrization of the curve given by (4). It follows from Proposition 18 that $g_p'(v) \geq -\sqrt{2}\pi$ for $v \geq 0$. Thus, $x'(v) > 0$ for all v, and hence this curve is the graph of a decreasing function φ. Moreover, by definition one has

$$y''(v)x'(v) - x''(v)y'(v) = 2\pi g_p''(|v|).$$

Thus, it follows from Lemma 18 (c),(d) that X_{Ω_p} is a concave toric domain if $1 \leq p \leq 2$, and a convex toric domain if $p \geq 2$. □

3 The ECH capacities of toric domains

In [9], Hutchings defined a sequence of symplectic capacities for 4-dimensional symplectic manifolds using embedded contact homology (ECH). In particular, for a Liouville domain $X \subset \mathbb{R}^4$, he defined a sequence of numbers $(c_k(X))_{k\in\mathbb{N}} \subset \mathbb{R}\cup\{\infty\}$ satisfying:

- $0 = c_0(X) \leq c_1(X) \leq c_2(X) \leq \cdots \leq \infty$,
- $c_k(a\cdot X) = a^2 \cdot c_k(X)$, for all $k \in \mathbb{N}$ and $a > 0$,
- $X_1 \hookrightarrow X_2 \Rightarrow c_k(X_1) \leq c_k(X_2)$, for all $k \in \mathbb{N}$.

- $(c_k(B(a)))_{k\in\mathbb{N}} = (0, a, a, 2a, 2a, 2a, 3a, 3a, 3a, 3a, \dots)$.

The ECH capacities turn out to give sharp obstructions for many symplectic embedding problems (see e.g., [14]). Moreover, for convex and concave toric domains, they can be computed combinatorially as explained in [4, 9, 10]. We will now review some relevant properties of the ECH capacities, and in particular describe the first two capacities of symmetric concave/convex toric domains.

Let X_Ω be a concave toric domain. The weight expansion $w(\Omega)$, associated with X_Ω, is a multiset which was defined inductively in [4] as follows. Let $T(c) \subset \mathbb{R}^2$ be the triangle whose vertices are $(0,0)$, $(c,0)$ and $(0,c)$. For a set $\Omega \subset \mathbb{R}^2_{\geq 0}$ which is bounded by the coordinate axes and the graph of a decreasing concave function $\varphi : [0,a] \to \mathbb{R}_{\geq 0}$ with $\varphi(a) = 0$, we define

$$\tau(\Omega) := \sup\{c \,|\, T(c) \subseteq \Omega\}. \tag{23}$$

We write $\Omega \setminus T(\tau(\Omega)) = \widetilde{\Omega}_1 \sqcup \widetilde{\Omega}_2$, where $\widetilde{\Omega}_1$ does not intersect the y-axis and $\widetilde{\Omega}_2$ does not intersect the x-axis. Note that these sets can be empty, otherwise their closures have a unique obtuse corner. Consider the closures of $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ translated so that each obtuse corner is mapped to the origin. Let Ω_1 and Ω_2 be the images of these translations under multiplication by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

respectively. Note that X_{Ω_1} and X_{Ω_2} are again concave toric domains if non-empty. The weight expansion is defined inductively by

$$w(\Omega) = \{\tau(\Omega)\} \sqcup w(\Omega_1) \sqcup w(\Omega_2),$$

where this union is considered with repetition, and $w(\Omega_j) = \emptyset$ if $\Omega_j = \emptyset$. Another way of seeing it is the following. The process above defines a directed tree of domains starting from Ω. We denote the elements of this tree by $\Omega_{i_1\dots i_p}$, where the indices $i_1, \dots, i_p \in \{1, 2\}$, and the domains derived from $\Omega_{i_1\dots i_p}$ are $\Omega_{i_1\dots i_p 1}$ and $\Omega_{i_1\dots i_p 2}$. It is possible that some $\Omega_{i_1\dots i_p}$ are empty. Then,

$$w(\Omega) = \{\tau(\Omega_{i_1\dots i_p}) \mid p \in \mathbb{N};\ i_1, \dots, i_p \in \{1, 2\}\}.$$

With a slight abuse of notation, we now write $w(\Omega) = (w_1, w_2, w_3, \dots)$, where

$$w_1 \geq w_2 \geq w_3 \geq \cdots \tag{24}$$

Note that $w_1 = \tau(\Omega)$ and that $w_2 = \max(\tau(\Omega_1), \tau(\Omega_2))$. It was shown in [4] that for a concave toric domain X_Ω one has

$$c_k(X_\Omega) = c_k\left(\bigsqcup_{j=1}^{\infty} B(w_j)\right) = \max_{i_1+\cdots+i_k=k} \sum_{j=1}^{k} c_{i_j}(B(w_j)). \tag{25}$$

Next, we say that a toric domain X_Ω is *symmetric* if it is invariant under the reflection about the line $y = x$. The following lemma is a computation of the first two ECH capacities that will be relevant for all of the domains in this paper.

Lemma 19. *Let X_Ω be a symmetric toric domain where $\Omega \subset \mathbb{R}^2_{\geq 0}$ is bounded by the coordinate axes and a C^1-curve γ parametrized by $(x(v), y(v))$, which connects the points $(a, 0)$ and $(0, a)$. Let $b \in (0, a)$ such that $(b, b) = (x(v), y(v))$ for some v.*

(a) If X_Ω is a convex toric domain, then

$$\begin{aligned} c_1(X_\Omega) &= a, \\ c_2(X_\Omega) &= 2b. \end{aligned}$$

(b) If X_Ω is a concave toric domain such that $-2 \leq y'(v)/x'(v) \leq -1/2$, then

$$\begin{aligned} c_1(X_\Omega) &= 2b, \\ c_2(X_\Omega) &= a. \end{aligned}$$

(c) If X_Ω is a concave toric domain such that $y'(v)/x'(v) > -1/2$ for some v, and let v_0 such that $y'(v_0)/x'(v_0) = -1/2$. Then

$$\begin{aligned} c_1(X_\Omega) &= 2b, \\ c_2(X_\Omega) &= 2y(v_0) + x(v_0). \end{aligned}$$

Proof of Lemma 19. (a) If X_Ω is a convex toric domain bounded by the coordinate axes and a C^1 curve γ connecting $(x_0, 0)$ to $(0, y_0)$, then it follows from [10, Proposition 5.6] that

$$c_1(X_\Omega) = \min(x_0, y_0), \quad c_2(X_\Omega) = \min(2x_0, 2y_0, x_1 + y_1) = x_1 + y_1,$$

where (x_1, y_1) is a point on the curve at which the slope of the tangent line is -1. Since we assume that γ is symmetric about the line $y = x$, we conclude that

$$c_1(X_\Omega) = a, \quad c_2(X_\Omega) = 2b.$$

We turn now to prove (b) and (c). Suppose that X_Ω is a concave toric domain and let $(w_1, w_2, \dots)$ be its weight expansion. It follows from the (24), (25) and the computation of $c_k(B(c))$ that

$$c_1(X_\Omega) = w_1, \quad c_2(X_\Omega) = w_1 + w_2. \tag{26}$$

Since γ is a C^1 curve, w_1 is the unique real number such that the line $x + y = w_1$ is tangent to γ. Since this curve is symmetric about the line $y = x$, it follows that

$c_1(X_\Omega) = w_1 = 2b$. Let $x_{1/2}$ be the x-intercept of the line of slope $-1/2$ whose intersection with the first quadrant is as large as possible but still contained in Ω. Simple linear algebra shows that $w_2 = x_{1/2} - w_1$. So if $-2 \le y'(v)/x'(v) \le -1/2$ for all v, then $x_{1/2} = a$, and from (26) we obtain

$$c_2(X_\Omega) = w_1 + (a - w_1) = a.$$

On the other hand, suppose that $y'(v)/x'(v) > -1/2$ for some v (which implies that $x_{1/2} < a$), and let v_0 such that $y'(v_0)/x'(v_0) = -1/2$. In this case the upper right side of the triangle $T'(w_2)+(w_1,0)$ is tangent to γ. So the point $(w_2+w_1,0)$ belongs to the line of slope $-1/2$ going through $(x(v_0), y(v_0))$. Therefore, in this case

$$c_2(X_\Omega) = w_1 + w_2 = 2y(v_0) + x(v_0). \tag{27}$$

This completes the proof of Lemma 19. □

The first two ECH capacities of the toric domain X_{Ω_p} introduced in Theorem 5 are now a straight forward consequence of Lemma 18 and Lemma 19.

Proposition 20. *The first two ECH capacities of X_{Ω_p} are given as follows.*

(a) If $p \in [1,2]$, then

$$\begin{aligned} c_1(X_{\Omega_p}) &= 2\pi(1/4)^{1/p}, \\ c_2(X_{\Omega_p}) &= A(p). \end{aligned} \tag{28}$$

(b) If $p \ge 2$, then

$$\begin{aligned} c_1(X_{\Omega_p}) &= A(p) \\ c_2(X_{\Omega_p}) &= \begin{cases} 2\pi(1/4)^{1/p}, & \text{if } p \le 9/2, \\ 2\pi \left(g_p'\right)^{-1}(-2\pi/3) + 3g_p\left(\left(g_p'\right)^{-1}(-2\pi/3)\right), & \text{if } p > 9/2. \end{cases} \end{aligned} \tag{29}$$

3.1 Ball packings and symplectic embeddings

In this section we provide a criterion for the embedding of a symmetric concave domain into a ball in $\mathbb{R}^4$ (see Proposition 23 below). We start with recalling the following result proved by Cristofaro-Gardiner in [5].

Theorem 21. *Let X_Ω be a concave toric domain with weight expansion $(w_1, w_2, \dots)$, and let $X_{\Omega'}$ be a convex toric domain. Then $X_\Omega \hookrightarrow X_{\Omega'}$ if and only if*

$$\bigsqcup_{i=1}^{N} B(w_i) \hookrightarrow X_{\Omega'}, \quad \forall N.$$

Next we recall a criterion for the existence of a ball packing

$$\coprod_{i=1}^{N} B(a_i) \hookrightarrow B(c), \tag{30}$$

based on the so called "Cremona transformations". From now on we assume that $N \geq 3$. A vector $(c; a_1, a_2, \ldots, a_N) \in \mathbb{R}^{N+1}$ is called *ordered* if $c \geq a_1 \geq a_2 \geq \cdots$. An ordered vector is called *reduced* if $c \geq a_1 + a_2 + a_3$ and $a_i \geq 0$ for all i. The Cremona transform is the linear transformation

$$(c; a_1, a_2, \ldots, a_N) \mapsto (2c - a_1 - a_2 - a_3; c - a_2 - a_3, c - a_1 - a_3, c - a_1 - a_2, a_4, \ldots, a_N).$$

A *Cremona move* takes a vector $(c; a_1, a_2, \ldots, a_N)$ to the vector obtained by ordering its image under the Cremona transform. The following theorem is a combination of results in [11, 12, 15], as explained for example in [3].

Theorem 22. *Let $(c; a_1, a_2, \ldots, a_N)$ be an ordered vector, and $N \geq 3$. Then there exists a symplectic embedding*

$$\coprod_{i=1}^{N} B(a_i) \hookrightarrow B(c)$$

if and only if the vector $(c; a_1, a_2, \ldots, a_N)$ is taken to a reduced vector under a finite number of Cremona moves, and

$$\sum_{i=1}^{N} a_i^2 \leq c^2.$$

Now suppose that X_Ω is a symmetric concave toric domain. Its weight sequence is of the form $(w_1, w_2, w_2, w_3, w_3, \ldots)$. As an application of the two theorems above we prove the following result, which will be used later and may be also of independent interest.

Proposition 23. *Let X_Ω be a symmetric concave toric domain with weight sequence $(w_1, w_2, w_2, w_3, w_3, \ldots)$, and let $c = c_2(X_\Omega)$. Suppose that $vol(X_\Omega) \leq vol(B(c))$, and*

$$\tau(\Omega_1) \geq \tau(\Omega_{11}) + \tau(\Omega_{111}), \tag{31}$$

where $\tau(\cdot)$ is defined in (23). Then there exists a symplectic embedding

$$X_\Omega \hookrightarrow B(c).$$

Proof of Proposition 23. We assume that X_Ω is not a ball, otherwise the result is trivial. Thus, $w_2 \neq 0$ and it follows from (27) that $c_2(X_\Omega) = w_1 + w_2$. By Theorems 21 and 22, it suffices to show that for every $N \geq 2$, the vector

$$(w_1 + w_2; w_1, w_2, w_2, w_3, w_3, w_4, w_4, \ldots, w_N, w_N) \tag{32}$$

can be turned into a reduced vector after a finite number of Cremona moves. We remark that we are always assuming that a vector of this form is ordered. After one Cremona move, (32) becomes

$$(w_1; w_1 - w_2, w_3, w_3, w_4, w_4 \ldots, w_N, w_N, 0, 0). \tag{33}$$

We need to check that $w_1 - w_2 \geq w_3$. Note that $w_2 = \tau(\Omega_1) = \tau(\Omega_2)$. Since Ω is symmetric, it follows that the height of Ω_1 is $w_1/2$. Hence $w_2 \leq w_1/2$, and consequently

$$w_1 - w_2 \geq \frac{w_1}{2} \geq w_2 \geq w_3.$$

If $w_3 = 0$, then (33) is reduced, so we assume that $w_3 \neq 0$. We claim that for $k \geq 2$

$$w_2 < w_k + w_{k+1} \Rightarrow w_2 + \cdots + w_k \leq \frac{w_1}{2}. \tag{34}$$

Note that (34) holds for $k = 2$ since $w_2 \leq w_1/2$. Now suppose that

$$w_2 < w_k + w_{k+1} \tag{35}$$

for some $k > 2$. We can write $w_k = \tau(\Omega_{1i_1\ldots i_p})$ and $w_{k+1} = \tau(\Omega_{1j_1\ldots j_l})$, where $p, l \neq 0$. First suppose that $i_q \neq j_q$ for some q, and let r be the smallest such q. Then,

$$\tau(\Omega_{1i_1\ldots i_p}) + \tau(\Omega_{1j_1\ldots j_l}) \leq \tau(\Omega_{1i_1\ldots i_r1}) + \tau(\Omega_{1i_1\ldots i_r2}). \tag{36}$$

We now observe (see Figure 2(a)) that

$$\tau(\Omega_{1i_1\ldots i_r1}) + \tau(\Omega_{1i_1\ldots i_r2}) \leq \tau(\Omega_{1i_1\ldots i_r}). \tag{37}$$

This is because $\tau(\Omega_{1i_1\ldots i_r1})$ and $\tau(\Omega_{1i_1\ldots i_r2})$ are the y-coordinate and x-coordinate of the intersections between the line $x+y = \tau(\Omega_{1i_1\ldots i_r})$ and the lines of slope $-1/2$ and -2 which intersect $\partial\Omega_{1i_1\ldots i_r} \cap \mathbb{R}^2_{>0}$ and not $\mathbb{R}^2_{>0} \setminus \Omega_{1i_1\ldots i_r}$, respectively. The latter condition mean that these lines are tangent to $\partial\Omega_{1i_1\ldots i_r} \cap \mathbb{R}^2_{>0}$ if the function defining this domain is C^1. It follows from (36) and (37) that

$$w_k + w_{k+1} = \tau(\Omega_{1i_1\ldots i_p}) + \tau(\Omega_{1j_1\ldots j_l}) \leq \tau(\Omega_{1i_1\ldots i_r}) \leq \tau(\Omega_1) = w_2,$$

which contradicts (35). Now suppose that $i_q = j_q$ for all $1 \leq q \leq \min(p, l)$. Note

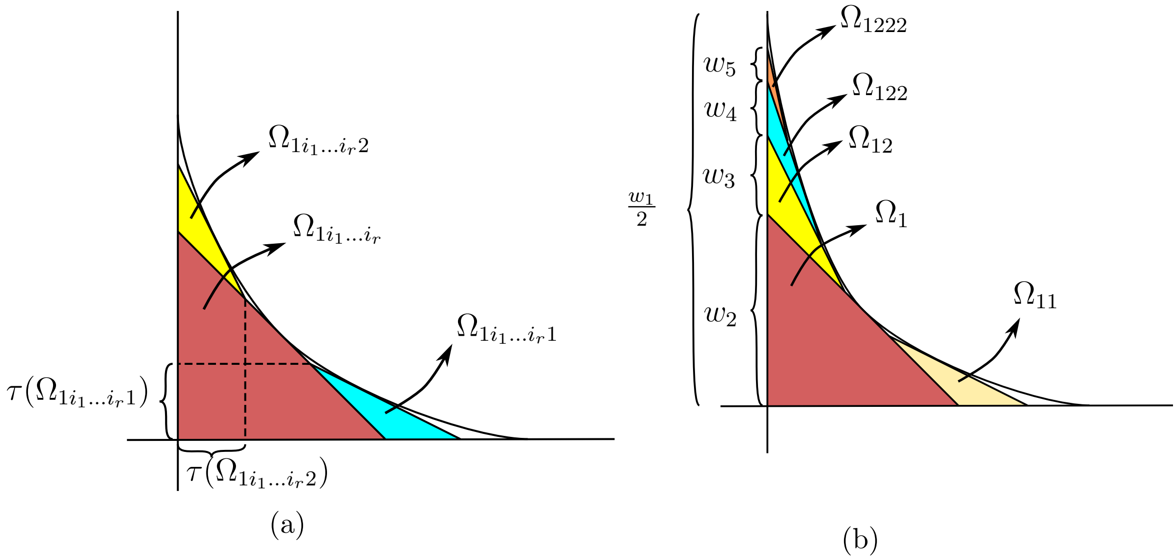

Figure 2

that $p < l$ since $w_k \geq w_{k+1}$. Therefore,

$$\tau(\Omega_{1i_1\ldots i_p}) + \tau(\Omega_{1j_1\ldots j_l}) \leq \tau(\Omega_{1i_1\ldots i_p}) + \tau(\Omega_{1i_1\ldots i_pi_{p+1}}). \tag{38}$$

Let us first assume that $i_{p+1} = 1$. If $i_q = 2$ for some $1 \leq q \leq p$, then

$$\tau(\Omega_{1i_1\ldots i_p}) + \tau(\Omega_{1i_1\ldots i_pi_{p+1}}) \leq \tau(\Omega_{1i_1\ldots i_{q-1}2}) \leq \tau(\Omega_1) = w_2. \tag{39}$$

Combining (38) and (39), we again reach a contradiction with (35). So the only way to satisfy (35) in this case is if $i_q = 1$ for all q. Now let us assume that $i_{p+1} = 2$. Similarly, if $i_q = 1$ for some $1 \leq q \leq p$, then

$$\tau(\Omega_{1i_1\ldots i_p}) + \tau(\Omega_{1i_1\ldots i_pi_{p+1}}) \leq \tau(\Omega_{1i_1\ldots i_{q-1}1}) \leq \tau(\Omega_1) = w_2, \tag{40}$$

and again we reach a contradiction. Thus, to satisfy (35) it is necessary that $i_q = 2$ for all q. We note that if $l < p$, then the only way to satisfy (35) is for one of the following two conditions to hold:

$$w_k + w_{k+1} = \tau(\Omega_{11\ldots1}) + \tau(\Omega_{11\ldots11\ldots j_l}) \tag{41}$$

$$w_k + w_{k+1} = \tau(\Omega_{12\ldots2}) + \tau(\Omega_{12\ldots22\ldots j_l}). \tag{42}$$

If (41) holds, then

$$w_2 < \tau(\Omega_{11}) + \tau(\Omega_{111}),$$

which contradicts (31). Finally, if (42) hold, then for all $j \leq k$, w_j is of the form $\tau(\Omega_{12\ldots2})$, otherwise we would get a contradiction to (35). So

$$w_j = \tau(\Omega_{1\underbrace{2\ldots2}_{j-2}}).$$

Therefore,

$$w_2 + \cdots + w_k \leq \frac{w_1}{2},$$

see Figure 2(b). This completes the proof of (34). Next, let

$$m = \max\{n \mid w_2 < w_n + w_{n+1}\}.$$

Since $w_3 \neq 0$, it follows that $m \geq 2$. We claim that for $2 \leq k \leq m$, after $k-2$ Cremona moves (33) turns into a vector of the form

$$(w_1 + kw_2 - 2(w_2 + \cdots + w_k); w_1 + (k-1)w_2 - 2(w_2 + \cdots + w_k), w_{k+1}, w_{k+1}, \ldots, 0). \quad (43)$$

We will prove this claim by induction on k. First observe that for $k = 2$, (43) is simply (33). Now suppose that (43) holds for some $k < m$. Applying one Cremona move, we obtain

$$\begin{aligned}&(w_1 + (k+1)w_2 - 2(w_2 + \cdots + w_k + w_{k+1}); w_1 + kw_2 - 2(w_2 + \cdots + w_k + w_{k+1}),\\ &w_{k+2}, w_{k+2}, w_2 - w_{k+1}, w_2 - w_{k+1}, \ldots, 0).\end{aligned} \quad (44)$$

We need to check that (44) is ordered. Note that since $k < m$, it follows that $w_{k+2} > w_2 - w_{k+1}$. Moreover, from (34) we conclude that

$$w_1 + kw_2 - 2(w_2 + \cdots + w_k + w_{k+1}) - w_{k+2} \geq 0.$$

So (44) is ordered. Moreover all of its components are non-negative. In particular after $(m-2)$ Cremona moves we obtain a non-negative vector

$$(w_1 + mw_2 - 2(w_2 + \cdots + w_m); w_1 + (m-1)w_2 - 2(w_2 + \cdots + w_m), w_{m+1}, w_{m+1}, \ldots, 0). \quad (45)$$

If $2w_{m+1} \leq w_2$, then (45) is reduced. If not, we apply one more Cremona move to obtain

$$\begin{aligned}&(w_1 + (m+1)w_2 - 2(w_2 + \cdots + w_{m+1}); w_1 + mw_2 - 2(w_2 + \cdots + w_{m+1}),\\ &w_2 - w_{m+1}, w_2 - w_{m+1}, \ldots, 0).\end{aligned} \quad (46)$$

Note that (46) is ordered since $w_2 - w_{m+1} \geq w_{m+2}$. Since $w_2 < 2w_{m+1}$, it follows that (46) is reduced. Hence, we conclude that (32) can be turned into a reduced vector after a finite number of Cremona moves. Therefore by Theorems 21 and 22,

$$X_\Omega \hookrightarrow B(w_1 + w_2),$$

and the proof of the proposition is complete. □

4 Proof of the main results

In this section we prove Theorems 1, 10, and 12, and also Proposition 13.

4.1 Lagrangian ℓ_p-sum : rigidity

Here we prove Theorem 10 (a,b,c,d), and the corresponding parts of Theorem 1.

Proof of Theorem 10 (a,b,c,d). Recall first that the first two ECH capacities of the Euclidean ball satisfy $c_1(B(r)) = c_2(B(r)) = r$. Throughout the proof we shall frequently use the fact that the interior of $\mathbb{X}_p$ is symplectomorphic to the toric domain X_{Ω_p}, as proved in Theorem 5 above. In particular, combining Theorem 5 and Proposition 20 shows that the first two ECH capacities of $\mathbb{X}_p$ are given by (28) and (29). We now split the proof into four parts.

(a) Suppose that $p \in [1, 2]$ and that $B(r) \hookrightarrow \mathbb{X}_p$. Then, by Theorem 5, Proposition 20, and the properties of the ECH capacities,

$$r \leq c_1(\mathbb{X}_p) = 2\pi(1/4)^{1/p}, \tag{47}$$

and hence $r_S(\mathbb{X}_p) \leq 2\pi(1/4)^{1/p}$. On the other hand, let $(\mathbf{x}, \mathbf{y}) \in B(r)$, where $r = 2\pi(1/4)^{1/p}$. It follows from (47) and the Hölder inequality that

$$\|\mathbf{x}\|^p + \|\mathbf{y}\|^p \leq \left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right)^{\frac{p}{2}} \cdot 2^{1-\frac{p}{2}} < 2^{\frac{p}{2}} \cdot \left(\tfrac{1}{4}\right)^{\frac{1}{2}} \cdot 2 \cdot 2^{-\frac{p}{2}} = 1.$$

Thus, one has that $B(r) \subset \mathbb{X}_p$ for $r = 2\pi(1/4)^{1/p}$. We conclude that $r_S(\mathbb{X}_p) = 2\pi(1/4)^{1/p}$, and moreover that the symplectic embedding problem $B \overset{?}{\hookrightarrow} \mathbb{X}_p$ is both rigid and torically rigid for $1 \leq p \leq 2$.

(b) Suppose now that $p \in [2, \infty)$ and that $B(r) \hookrightarrow \mathbb{X}_p$. Using again Theorem 5, Proposition 20, and the properties of the ECH capacities, one has in this case

$$r \leq c_1(\mathbb{X}_p) = A(p),$$

where $A(p)$ is given by (2). Hence $r_S(\mathbb{X}_p) \leq A(p)$. Since by Proposition 8 X_{Ω_p} is a concave toric domain, it follows from Remark 7 that $T \subset \Omega_p$, where T is the triangle bounded by the coordinate axes and the line $x + y = A(p)$. Thus we conclude that $B(A(p)) \subset X_{\Omega_p}$. Therefore, $r_S(\mathbb{X}_p) = A(p)$, and the symplectic embedding problem $B \overset{?}{\hookrightarrow} \mathbb{X}_p$ is torically rigid.

(c) Suppose that $p \in [1, 2]$ and that $\mathbb{X}_p \hookrightarrow B(r)$. Looking now at the second ECH capacitiy (see (28)), one has

$$A(p) = c_2(\mathbb{X}_p) \leq r.$$

Hence, $R_S(\mathbb{X}_p) \geq A(p)$. On the other hand, let (t,t) be the intersection of the curve (4) and the line $y = x$. It follows from Proposition 8 that X_{Ω_p} is now a convex toric domain, and thus Remark 7 shows that $\Omega_\alpha \subset T$, where T is the triangle bounded by the coordinate axes and the line $x + y = 2t$. In particular, $X_T = B(2t)$. From (2),(3), and (4) we conclude that

$$t = g_p(0) = 2\int_0^1 (1 - r^p)^{1/p}\, dr = \frac{A(p)}{2}. \tag{48}$$

So $X_{\Omega_p} \subset B(A(p))$. Therefore, $R_S(\mathbb{X}_p) = A(p)$, and the symplectic embedding problem $\mathbb{X}_p \overset{?}{\hookrightarrow} B(r)$ is torically rigid.

(d) Finally, suppose that $p \in [2, 9/2]$ and that $\mathbb{X}_p \hookrightarrow B(r)$. As in the previous case, from the second ECH capacity we conclude that

$$2\pi(1/4)^{1/p} = c_2(\mathbb{X}_p) \leq r, \tag{49}$$

which implies that $R_S(\mathbb{X}_p) \geq 2\pi(1/4)^{1/p}$. On the other hand, if $(\mathbf{x}, \mathbf{y}) \in \mathbb{X}_p$, then by Hölder's inequality

$$\pi\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right) \leq \pi\left((\|\mathbf{x}\|^p + \|\mathbf{y}\|^p)^{2/\alpha} \cdot 2^{1-2/p}\right) < 2\pi\left(\tfrac{1}{4}\right)^{1/p},$$

and so $\mathbb{X}_p \subseteq B\left(2\pi\left(\frac{1}{4}\right)^{1/p}\right)$. Therefore, $R_S(\mathbb{X}_p) = 2\pi\left(\frac{1}{4}\right)^{1/p}$, and the symplectic embedding problem $\mathbb{X}_p \overset{?}{\hookrightarrow} B(r)$ is both rigid and torically rigid.

This completes the proof of the first four parts of Theorem 10, and of the corresponding parts of Theorem 1. $\square$

4.2 Lagrangian ℓ_p-sum : flexibility for $p > 9/2$

In this section, we finish the proofs of Theorems 1 and 10. More precisely, we will compute the outer radius $R_S(\mathbb{X}_p)$ for $p > 9/2$, and show that the corresponding symplectic embedding problem is non-rigid. Assume $p > 9/2$. It follows from Theorem 5 and Proposition 20 that

$$c_2(\mathbb{X}_p) = c_2(X_{\Omega_p}) = 2\pi\left(g_p'\right)^{-1}(-2\pi/3) + 3g_p\left(\left(g_p'\right)^{-1}(-2\pi/3)\right). \tag{50}$$

Recall from the proof of Proposition 20 that this number is the x-intercept of a tangent line to the curve (4). Moreover, the x-intercept of this curve is $2\pi(1/4)^{1/p}$ by Lemma 18 (a). This implies that

$$X_{\Omega_p} \subset B(c) \iff c \geq 2\pi(1/4)^{1/p}.$$

The same calculation as in the proof of Theorem 10 (d) above shows that

$$\mathbb{X}_p \subset B(c) \iff c \geq 2\pi(1/4)^{1/p}.$$

Since X_{Ω_p} is a concave toric domain, it follows that

$$c_2(\mathbb{X}_p) = c_2(X_{\Omega_p}) < 2\pi(1/4)^{1/p}.$$

Therefore the following proposition implies that $\mathbb{X}_p \overset{?}{\hookrightarrow} B$ is non-rigid, and moreover that $R_S(\mathbb{X}_p) = c_2(\mathbb{X}_p)$, as claimed in Theorems 1 and 10.

Proposition 24. *For $p > 9/2$, there is a symplectic embedding $X_{\Omega_p} \hookrightarrow B(c_2(X_{\Omega_p}))$.*

Proof of Proposition 24. Let $p > 9/2$. To establish the required embedding, we verify the conditions of Proposition 23 above. We first claim that

$$\mathrm{Vol}(X_{\Omega_p}) = \mathrm{Vol}(\mathbb{X}_p) \leq \mathrm{Vol}(B(c_2(\mathbb{X}_p))) = \mathrm{Vol}(B(c_2(X_{\Omega_p}))).$$

Indeed, since for $p < \tilde{p}$ one has $\mathbb{X}_p \subset \mathbb{X}_{\tilde{p}}$, it follows from (29) that for all $p > 9/2$

$$\begin{aligned} \mathrm{Vol}(\mathbb{X}_p) \leq \mathrm{Vol}(\mathbb{X}_\infty) = \pi^2 &\leq \tfrac{1}{2}\left(2\pi\left(\tfrac{1}{4}\right)^{2/9}\right)^2 = \tfrac{1}{2}c_2(\mathbb{X}_{9/2})^2 \\ &\leq \tfrac{1}{2}c_2(\mathbb{X}_p)^2 = \mathrm{Vol}(B(c_2(\mathbb{X}_p))). \end{aligned}$$

We now need to verify (31). We claim that $\tau((\Omega_p)_1)$ and $\tau((\Omega_p)_{11})+\tau((\Omega_p)_{111})$ are increasing functions of p. For $p > 9/2$, we now compute the x-intercept of the lines of slope -1, $-1/2$, $-1/3$ and $-1/4$, which intersect the curve (4), but not $\mathbb{R}^2_{\geq 0}\backslash\Omega_p$. For $p \geq 25/2$, these lines are all tangent to (4). Let $x_{-1}(p), x_{-1/2}(p), x_{-1/3}(p), x_{-1/4}(p)$ be the x-intercepts of these lines, see Figure 3(a). We observe that

$$\begin{aligned} \tau((\Omega_p)_1) &= x_{-1/2}(p) - x_{-1}(p), \\ \tau((\Omega_p)_{11}) &= x_{-1/3}(p) - x_{-1/2}(p), \\ \tau((\Omega_p)_{111}) &= x_{-1/4}(p) - x_{-1/3}(p). \end{aligned}$$

Now let $w_2(p) = \tau((\Omega_p)_1)$ and $d(p) = \tau((\Omega_p)_{11}) + \tau((\Omega_p)_{111})$. So,

$$\begin{aligned} w_2(p) &= x_{-1/2}(p) - x_{-1}(p), \\ d(p) &= x_{-1/4}(p) - x_{-1/2}(p). \end{aligned} \tag{51}$$

If we denote by $v_{-1}(p)$, $v_{-1/2}(p)$,$v_{-1/3}(p)$ and $v_{-1/4}(p)$, the parameter values of v where the lines defined above interesct the curve (4), respectively, then

$$\begin{aligned} v_{-1}(p) &= 0, \\ v_{-1/2}(p) &= \left(g_p'\right)^{-1}(-2\pi/3), \\ v_{-1/4}(p) &= \begin{cases} \left(\frac{1}{4}\right)^{1/p} = \left(g_p'\right)^{-1}\left(-\sqrt{\frac{2}{p}}\pi\right), & \text{if } \frac{9}{2} < p < \frac{25}{2}, \\ \left(g_p'\right)^{-1}(-2\pi/5), & \text{if } p \geq \frac{25}{2}. \end{cases} \end{aligned} \tag{52}$$

Since g_p is a concave function,

$$v_{-1}(p) < v_{-1/2}(p) \leq v_{-1/3}(p) \leq v_{-1/4}(p). \tag{53}$$

Moreover

$$\begin{aligned} x_{-1}(p) &= 2g_p(0) = A(p), \\ x_{-1/2}(p) &= 2\pi v_{-1/2}(p) + 3g_p(v_{-1/2}(p)), \\ x_{-1/4}(p) &= 2\pi v_{-1/4}(p) + 5g_p(v_{-1/4}(p)). \end{aligned} \tag{54}$$

In order to prove that the functions $w_2(p)$ and $d(p)$ are increasing, note first that they are continuous in $(9/2, \infty]$ and differentiable in $(9/2, 25/2) \cup (25/2, \infty)$. So it suffices to show that

$$w_2'(p), d'(p) > 0, \quad \text{for all } p \in (9/2, 25/2) \cup (25/2, \infty). \tag{55}$$

It follows from (17) that for $v < (1/4)^{1/p}$, the function $p \mapsto g_p'(v)$ is increasing. Since the function $(p, v) \mapsto g_p(v)$ is C^∞ in the interior of its domain, we obtain

$$\frac{\partial}{\partial v}\frac{\partial}{\partial p} g_p(v) = \frac{\partial}{\partial p} g_p'(v) > 0,$$

Hence for a fixed p, the function $v \mapsto \frac{\partial}{\partial p} g_p(v)$ is increasing. Further, differentiating (3) for $v > 0$ we obtain

$$\frac{\partial}{\partial p} g_p(v) = \int_{\left(\frac{1}{2}-\sqrt{\frac{1}{4}-v^p}\right)^{1/p}}^{\left(\frac{1}{2}+\sqrt{\frac{1}{4}-v^p}\right)^{1/p}} \frac{(1-r^p)^{2/p}}{\left((1-r^p)^{2/p} - \frac{v^2}{r^2}\right)^{1/2}} \left(\ln \frac{1}{(1-r^p)^{\frac{2}{p^2}} r^{\frac{2r^p}{p(1-r^p)}}} \right) dr \quad > 0, \tag{56}$$

since $r \in (0, 1)$. So

$$\begin{aligned} w_2'(p) &= x_{-1/2}'(p) - x_{-1}'(p) = 3\tfrac{\partial}{\partial p} g_p(v_{-1/2}(p)) - 2\tfrac{\partial}{\partial p} g_p(0) \\ &> 2\left(\tfrac{\partial}{\partial p} g_p(v_{-1/2}(p)) - \tfrac{\partial}{\partial p} g_p(0)\right) > 0. \end{aligned}$$

Let $p \in (9/2, 15/2)$. It follows from (52) that $v_{-1/4}'(p) > 0$ and that $g_p'(v_{-1/4}(p)) > -2\pi/3$. Using (51), (53), (54) and (56) we obtain

$$\begin{aligned} d'(p) &= x_{-1/4}'(p) - x_{-1/2}'(p) \\ &= 2\pi v_{-1/4}'(p) + 5g_p'(v_{-1/4}(p))v_{-1/4}'(p) + 5\frac{\partial}{\partial p} g_p(v_{-1/4}(p)) - 3\frac{\partial}{\partial p} g_p(v_{-1/2}(p)) \\ &> 5\frac{\partial}{\partial p} g_p(v_{-1/4}(p)) - 3\frac{\partial}{\partial p} g_p(v_{-1/2}(p)) \\ &> 3\left(\frac{\partial}{\partial p} g_p(v_{-1/4}(p)) - \frac{\partial}{\partial p} g_p(v_{-1/2}(p))\right) \geq 0. \end{aligned}$$

Let $p \in (15/2/\infty)$. Then, by (51), (52), (53), (54) and (56)

$$\begin{aligned}
d'(p) &= x'_{-1/4}(p) - x'_{-1/2}(p)\\
&= 2\pi v'_{-1/4}(p) + 5g'_p(v_{-1/4}(p))v'_{-1/4}(p) + 5\frac{\partial}{\partial p}g_p(v_{-1/4}(p)) - 3\frac{\partial}{\partial p}g_p(v_{-1/2}(p))\\
&= 5\frac{\partial}{\partial p}g_p(v_{-1/4}(p)) - 3\frac{\partial}{\partial p}g_p(v_{-1/2}(p))\\
&> 3\left(\frac{\partial}{\partial p}g_p(v_{-1/4}(p)) - \frac{\partial}{\partial p}g_p(v_{-1/2}(p))\right) \geq 0.
\end{aligned}$$

We conclude that $w_2(p)$ and $d(p)$ are increasing. A simple calculation using (5) shows that

$$d(\infty) = 10\sin\frac{\pi}{5} - 6\sin\frac{\pi}{3} = 5\sqrt{\frac{5-\sqrt{5}}{2}} - 3\sqrt{3} < 0.69.$$

Moreover it follows from (2) that

$$w_2(9/2) = 2\pi\left(\frac{1}{4}\right)^{2/9} - A\left(\frac{9}{2}\right) = 2\pi\left(\frac{1}{4}\right)^{2/9} - \frac{4\cdot\Gamma\left(\frac{11}{9}\right)^2}{\Gamma\left(\frac{13}{9}\right)} > 0.85.$$

So for any $p \in [9/2, \infty]$,

$$d(p) \leq d(\infty) < w_2(9/2) \leq w_2(p).$$

Hence, (31) is satisfied, and Proposition 23 implies that

$$X_{\Omega_p} \hookrightarrow B(c_2(X_{\Omega_p})).$$

□

4.3 The symplectic ℓ^p-sum of two discs

Here we prove Theorem 12 and Proposition 13.

Proof of Theorem 12. It is clear from the definition that $\mathbb{B}_p(\mathbb{C}^2)$ is a concave toric domain for $0 < p < 2$, and a convex toric domain for $p > 2$. Thus, it follows from Lemma 19 that

$$c_1(\mathbb{B}_p(\mathbb{C}^2)) = \min(1, 2^{1-2/p}), \text{ if } p > 0, \tag{57}$$

$$c_2(\mathbb{B}_p(\mathbb{C}^2)) = \begin{cases} 2^{1-2/p}, & \text{if } p \geq 2\\ \left(1 + 2^{\frac{p}{p-2}}\right)^{1-2/p}, & \text{if } 1 \leq p < 2.\end{cases} \tag{58}$$

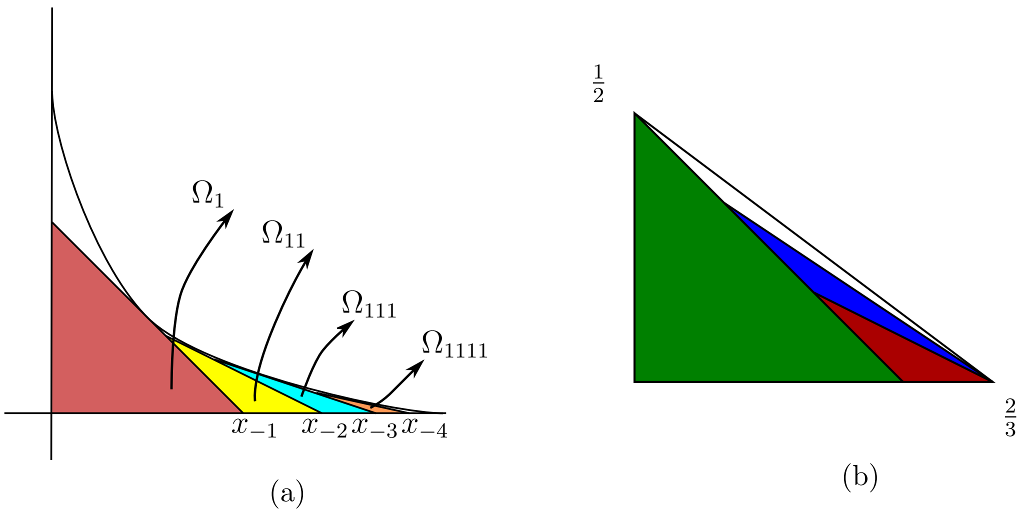

Figure 3

A straight forward calculation shows that

$$B(c_1(\mathbb{B}_p(\mathbb{C}^2))) \subset \mathbb{B}_p(\mathbb{C}^2), \text{ for all } p > 0, \tag{59}$$

and

$$\mathbb{B}_p(\mathbb{C}^2) \subset B(c_2(\mathbb{B}_p(\mathbb{C}^2))), \text{ for } p \geq 2. \tag{60}$$

The combination of (57), (58), (59), (60) implies parts (a) and (b) of the theorem. In order to prove (c), we apply Proposition 23. Let $p \in [1, 2)$ and let $\tilde{\Omega}_p = \mu(\mathbb{B}_p(\mathbb{C}^2))$, where μ is the moment map. Note that $\tilde{\Omega}_p \subset \mathbb{R}^2_{\geq 0}$ is the region bounded by the coordinate axes and the curve

$$x^{p/2} + y^{p/2} = 1 \quad (\text{and } \max\{\|x\|, \|y\|\} = 1 \text{ for } p = \infty).$$

As in the proof of Proposition 24 above, we can define $x_{-1}(p)$, $x_{-1/2}(p)$, $x_{-1/3}(p)$ and $x_{-1/4}(p)$ for $\tilde{\Omega}_p$, and a simple computation shows that for $n \geq 1$

$$x_{-1/n}(p) = \left(\frac{n^{\frac{p}{2-p}}}{1 + n^{\frac{p}{2-p}}} \right)^{\frac{2-p}{p}}. \tag{61}$$

Thus, as in the proof of Proposition 24, one has

$$\begin{aligned} c_2(p) &= x_{-1/2}(p), \\ w_2(p) &= \tau((\tilde{\Omega}_p)_1) = x_{-1/2}(p) - x_{-1}(p), \\ d(p) &= \tau((\tilde{\Omega}_p)_{11}) + \tau((\tilde{\Omega}_p)_{111}) = x_{-1/4}(p) - x_{-1/2}(p). \end{aligned} \tag{62}$$

Moreover, a direct computation gives

$$\operatorname{vol}(B_p(1,1)) = \int_0^1 (1-t^{p/2})^{2/p}\,dt = \frac{1}{p}B\left(\frac{2}{p},\frac{2}{p}\right), \tag{63}$$

where here $B(\alpha,\beta)$ stands for the Euler beta function. It follows from (61), (62) and (63) that the assumptions for Lemma 23 are:

$$\frac{1}{p}B\left(\frac{2}{p},\frac{2}{p}\right) \le \frac{1}{2}\left(\left(\frac{2^{\frac{p}{2-p}}}{1+2^{\frac{p}{2-p}}}\right)^{\frac{2}{p}-1}\right)^2 \tag{64}$$

$$\left(\frac{4^{\frac{p}{2-p}}}{1+4^{\frac{p}{2-p}}}\right)^{\frac{2-p}{p}} - \left(\frac{2^{\frac{p}{2-p}}}{1+2^{\frac{p}{2-p}}}\right)^{\frac{2-p}{p}} \le \left(\frac{2^{\frac{p}{2-p}}}{1+2^{\frac{p}{2-p}}}\right)^{\frac{2-p}{p}} - \left(\frac{1}{2}\right)^{\frac{2-p}{p}}. \tag{65}$$

We will first prove (64). We claim that the function $x \mapsto xB(x,x)$ is convex in $(1,2)$. In fact, for $x \in (1,2)$,

$$\begin{aligned}\frac{d^2}{dx^2}(xB(x,x)) &= \int_0^1 \frac{d^2}{dx^2}\left(x(t(1-t))^{x-1}\right)\,dt \\ &= \int_0^1 (t(1-t))^{x-1}\left(\ln(t(1-t)) + (\ln(t(1-t)))^2 x\right)\,dt.\end{aligned} \tag{66}$$

The maximum of the function $t \mapsto -1/\ln(t(1-t))$ is $1/\ln(4)$, which is smaller than 1, so the integrand in (66) is positive. Therefore $x \mapsto xB(x,x)$ is convex in $(1,2)$. Hence, for all $x \in (1,2)$,

$$xB(x,x) \le B(1,1) + (2B(2,2) - B(1,1))(x-1) = \frac{5}{3} - \frac{2x}{3}. \tag{67}$$

Now it is a simple calculus problem to check that for $x \in (1,2)$,

$$\frac{5}{3} - \frac{2x}{3} \le \frac{4}{(1+2^{\frac{1}{x-1}})^{2(x-1)}} = \left(\frac{2^{\frac{1}{x-1}}}{1+2^{\frac{1}{x-1}}}\right)^{2(x-1)}. \tag{68}$$

Combining (67) and (68) for $x = 2/p$, we obtain (64).

We now prove (65) for $1 \le p < 2$. It is clear that

$$2^{\frac{3p}{2-p}} - 3\cdot 2^{\frac{2p}{2-p}} + 3\cdot 2^{\frac{p}{2-p}} - 1 = \left(2^{\frac{p}{2-p}} - 1\right)^3 > 0.$$

Consequently

$$\frac{2^{\frac{p}{2-p}}}{1+2^{\frac{p}{2-p}}} > \frac{1+3\cdot 2^{\frac{2p}{2-p}}}{4\left(1+2^{\frac{2p}{2-p}}\right)} = \frac{1}{2}\left(\frac{1}{2} + \frac{4^{\frac{p}{2-p}}}{1+4^{\frac{p}{2-p}}}\right).$$

Since the function $x \mapsto x^{\frac{2-p}{p}}$ is concave, it follows that

$$\left(\frac{2^{\frac{p}{2-p}}}{1+2^{\frac{p}{2-p}}}\right)^{\frac{2-p}{p}} > \left(\frac{1}{2}\left(\frac{1}{2}+\frac{4^{\frac{p}{2-p}}}{1+4^{\frac{p}{2-p}}}\right)\right)^{\frac{2-p}{p}} \geq \frac{1}{2}\left(\left(\frac{1}{2}\right)^{\frac{2-p}{p}}+\left(\frac{4^{\frac{p}{2-p}}}{1+4^{\frac{p}{2-p}}}\right)^{\frac{2-p}{p}}\right)$$

and so

$$\left(\frac{1}{2}\right)^{\frac{2-p}{p}}+\left(\frac{4^{\frac{p}{2-p}}}{1+4^{\frac{p}{2-p}}}\right)^{\frac{2-p}{p}} \leq 2\left(\frac{2^{\frac{p}{2-p}}}{1+2^{\frac{p}{2-p}}}\right)^{\frac{2-p}{p}}.$$

Therefore (65) holds. □

Proof of Proposition 13. It is enough to prove that $\mathbb{B}_1(\mathbb{C}^2) \hookrightarrow E(1/2, 2/3)$. The domain $\mathbb{B}_1(\mathbb{C}^2)$ is a symmetric concave toric domain, and a direct computation shows that the first few numbers of its weight sequence are

$$\begin{array}{llllll} w_1 = \frac{1}{2}, & w_2 = \frac{1}{6}, & w_3 = \frac{1}{6}, & w_4 = \frac{1}{12}, & w_5 = \frac{1}{12}, & w_6 = \frac{1}{20}, \\ w_7 = \frac{1}{20}, & w_8 = \frac{1}{30}, & w_9 = \frac{1}{30}, & w_{10} = \frac{1}{30}, & w_{11} = \frac{1}{30}. & \end{array}$$

Note that one can easily fit the ball $B(1/2)$ and two balls $B(1/6)$ into $E(1/2, 2/3)$, see Figure 3(b). The remaining domain is equivalent to the ball $B(1/6)$ under an $SL(2, \mathbb{Z})$ transformation. So for $N \geq 4$,

$$\bigsqcup_{i=1}^{N} B(w_i) \hookrightarrow E(1/2, 2/3) \iff \bigsqcup_{i=4}^{N} B(w_i) \hookrightarrow B(1/6). \tag{69}$$

We now consider the ordered vector

$$(1/6; 1/12, 1/12, 1/20, 1/20, 1/30, 1/30, 1/30, 1/30, \ldots, w_N).$$

After appling one Cremona move (and re-ordering), we obtain the vector

$$(7/60; 1/20, 1/30, 1/30, 1/30, 1/30, 1/30, 1/30, 1/30, \ldots, w_N, 0),$$

which is reduced. Thus, Theorem 22 implies that the embeddings in (69) exist, and therefore from Theorem 21 we conclude that $\mathbb{B}_1(\mathbb{C}^2) \hookrightarrow E(1/2, 2/3)$, as required. □

References

[1] V. I. Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag, New York-Heidelberg, 1978. Translated from the Russian by K. Vogtmann and A. Weinstein, Graduate Texts in Mathematics, 60.

[2] S. Artstein-Avidan and Y. Ostrover, *Bounds for Minkowski billiard trajectories in convex bodies*, Int. Math. Res. Not. IMRN, (2014), pp. 165–193.

[3] O. Buse and M. Pinsonnault, *Packing numbers of rational ruled four-manifolds*, J. Symplectic Geom., 11 (2013), pp. 269–316.

[4] K. Choi, D. Cristofaro-Gardiner, D. Frenkel, M. Hutchings, and V. G. B. Ramos, *Symplectic embeddings into four-dimensional concave toric domains*, J. Topol., 7 (2014), pp. 1054–1076.

[5] D. Cristofaro-Gardiner, *Symplectic embeddings from concave toric domains into convex ones*, J. Differential Geom., 112 (2019), pp. 199–232.

[6] J.-P. Dufour and P. Molino, *Compactification d'actions de $\mathbf{R}^n$ et variables action-angle avec singularités*, in Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), vol. 20 of Math. Sci. Res. Inst. Publ., Springer, New York, 1991, pp. 151–167.

[7] L. H. Eliasson, *Hamiltonian systems with poisson commuting integrals*, PhD thesis, University of Stockholm, (1984).

[8] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math., 82 (1985), pp. 307–347.

[9] M. Hutchings, *Quantitative embedded contact homology*, J. Differential Geom., 88 (2011), pp. 231–266.

[10] ——, *Beyond ECH capacities*, Geom. Topol., 20 (2016), pp. 1085–1126.

[11] B.-H. Li and T.-J. Li, *Symplectic genus, minimal genus and diffeomorphisms*, Asian J. Math., 6 (2002), pp. 123–144.

[12] D. McDuff, *Blowups and symplectic embeddings in dimension 4*, Topology, 30 (1991), pp. 409–421.

[13] ——, *Symplectic embeddings of 4-dimensional ellipsoids*, J. Topol., 2 (2009), pp. 1–22.

[14] ——, *The Hofer conjecture on embedding symplectic ellipsoids*, J. Differential Geom., 88 (2011), pp. 519–532.

[15] D. McDuff and L. Polterovich, *Symplectic packings and algebraic geometry*, Invent. Math., 115 (1994), pp. 405–434. With an appendix by Yael Karshon.

[16] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, vol. 52 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2004.

[17] D. McDuff and F. Schlenk, *The embedding capacity of 4-dimensional symplectic ellipsoids*, Ann. of Math. (2), 175 (2012), pp. 1191–1282.

[18] V. G. B. Ramos, *Symplectic embeddings and the Lagrangian bidisk*, Duke Math. J., 166 (2017), pp. 1703–1738.

[19] V. G. B. Ramos and D. Sepe, *On the rigidity of lagrangian products*, Journal of Symplectic Geometry, (to appear).

[20] F. Schlenk, *Symplectic embedding problems, old and new*, Bull. Amer. Math. Soc. (N.S.), 55 (2018), pp. 139–182.