Geometric and Algebraic Properties of the Group of Hamiltonian Diffeomorphisms

Thesis submitted for the degree "Doctor of Philosophy" by

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This Ph.D. thesis was written under the supervision of Professor Leonid Polterovich.

This work is dedicated to the memory of my father, Adir Ostrover,

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Contents

Introduction.

The main object of this thesis is the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$ associated with a symplectic manifold (M, ω). Symplectic manifolds appear naturally in the Hamiltonian formulation of Classical Mechanics and Geometrical Optics. Symplectic geometry is, in a sense, the geometry underlying the Hamiltonian systems in these settings. The group of Hamiltonian diffeomorphisms plays a fundamental role both in symplectic geometry and in Hamiltonian dynamics. For a symplectic geometer, under some assumptions on the manifold M, this group is the connected component of the identity in the group of all the isometries of the symplectic structure ω. From the point of view of dynamics, the group of Hamiltonian diffeomorphisms serves as the group of all the admissible motions (also called canonical transformations) in some classical dynamic system.

In recent years, the group of Hamiltonian diffeomorphisms has been intensively studied with many new discoveries concerning a wide range of aspects from classical Hamiltonian dynamics through symplectic geometry to algebraic geometry. On the other hand, many features of this group are still waiting to be discovered or completely understood. We refer the reader to [24], [34], [48] and to the references within for symplectic preliminaries, further discussion, and recent developments in the study of the group of Hamiltonian diffeomorphisms.

In this thesis we display several new results regarding geometric and algebraic properties of the group of Hamiltonian diffeomorphisms. The thesis contains four chapters: The first one is an introductive chapter with background from symplectic geometry, and the other three contain the main part of the thesis. The last three chapters of this thesis, although related, are independent of each other. The results in these chapters are published in [43], [44], and [45] respectively. We wish to emphasize that the second chapter is based on a joint work with Roy Wagner. Below is a brief summary of the main results in the thesis.

Geometric Properties of Ham(M, ω)

One of the most remarkable facts regarding the group of Hamiltonian diffeomorphisms is that it carries an intrinsic geometry given by a Finsler bi-invariant metric. This metric was first discovered by Hofer in 1990, in his seminal work [22]. It is important for at least two reasons: Firstly, it yields a geometric intuition for Hamiltonian systems, and secondly it can be used in many ways as a tool in symplectic geometry and dynamics. It is worthwhile to mention that the existence of such a metric is highly unusual for non-compact groups of transformations.

In the years following Hofer's work this new geometry has been intensively studied in the framework of modern symplectic geometry. In fact Hofer's discovery opened a new field, now called Hofer geometry, in which the geometry and the topology of the group of Hamiltonian diffeomorphisms are being studied. A great progress has been made in understanding some of its properties such as geodesics, diameter, and its relation with dynamics. However, many aspects of this group remain unknown. For example, it is not known if there are other Finsler-type bi-invariant metrics on the group Ham(M, ω) which are not equivalent to Hofer's metric, or in other words: whether Hofer's metric is unique.

In the seconed chapter of this thesis we address the question of the uniqueness of Hofer's metric among all the "Finsler-type" bi-invariant metrics on Ham(M, ω). The results in this chapter are based on a joint work with Roy Wagner [44]. In order to describe our contribution toward answering the above question we start with the following preliminaries.

Recall that a Finsler-type metric on the group Ham(M, ω) is obtained by choosing a norm on the Lie algebra $\mathcal{A}$ of Ham(M, ω), extending it to any other tangent space, defining the length of a path in Ham(M, ω) just as in Riemannian geometry, and finally defining the distance between two elements to be the infimum length over all paths connecting them. It is not hard to check that such a distance function is non-negative, symmetric and satisfies the triangle inequality. Moreover, a norm on $\mathcal{A}$ which is invariant under the adjoint action of the group on its Lie Algebra, yields a bi-invariant pseudo-distance function. It is known that for a closed symplectic manifold M, the Lie algebra $\mathcal{A}$ can be identified as the space of all zero-mean normalized smooth functions on M, and the adjoint action under this identification is just the standard action of Hamiltonian diffeomorphisms on functions. The question whether such a distance function is non-degenerate is highly non-trivial.

As mentioned above, a distinguished result by Hofer [22], which was later generalized by Polterovich [46] and finally proven in full generality by Lalonde-McDuff [28], states that the L_∞-norm on $\mathcal{A}$ gives rise to a genuine distance function on $\mathrm{Ham}(M,\omega)$. In the opposite direction, Eliashberg and Polterovich [13] showed that for $1 \leq p < \infty$, the pseudo-distances on $\mathrm{Ham}(M,\omega)$ which correspond to the L_p-norms on $\mathcal{A}$ are all degenerate. Naturally, the following question arises:

Question: What are the invariant norms on $\mathcal{A}$, and which of them give rise to genuine bi-invariant metrics on $\mathrm{Ham}(M,\omega)$?

Our main contributions towards answering this question (appear in [44]) are the following: Let (M,ω) be a closed symplectic manifold.

Theorem I: Any $\mathrm{Ham}(M,\omega)$-invariant norm $\|\cdot\|$ on $\mathcal{A}$ such that $\|\cdot\| \leq C\|\cdot\|_\infty$ for some constant C is invariant under all measure preserving diffeomorphisms on M.

Theorem II: Let $\|\cdot\|$ be a $\mathrm{Ham}(M,\omega)$-invariant norm on $\mathcal{A}$ such that $\|\cdot\| \leq C\|\cdot\|_\infty$, but the two norms are not equivalent. Then the associated pseudo-distance function on $\mathrm{Ham}(M,\omega)$ vanishes identically.

These two theorems will be discussed in the seconde chapter of this thesis together with some further discussion regarding norms which are invariant under measure preserving transformations.

In the third chapter of the thesis we turn to a different direction. We study Hofer's metric in a larger context of Lagrangian submanifolds. Lagrangian submanifolds are submanifolds of half the dimension of the ambient manifold, on which the symplectic form ω vanishes (see Chapter 1 below for a more precise definition). They are among the most important objects in symplectic geometry, and they arise naturally in many physical and geometric situations. For instance, in classical mechanics the systems of partial differential equations of Hamilton-Jacobi type lead to the study of Lagrangian submanifolds. Furthermore, Lagrangian submanifolds are a part of a growing list of mathematically rich special objects that occur naturally in string theory. In fact, according to the "symplectic creed", everything can be thought of as a lagrangian submanifold [59]. In order to make this statement slightly more rigorous we note that symplectic diffeomorphisms admit a simple interpretation as Lagrangian submanifolds. It is not hard to check that the graph of a symplectomorphism in the product symplectic manifold $(M \times M, \omega \oplus -\omega)$ is Lagrangian. This of course leads to a natural link between symplectic diffeomorphisms and Lagrangian manifolds.

The results in the third chapter are based on our work [43]. In this work we compare Hofer's geometries on two spaces associated with a closed symplectic manifold (M, ω). The first space, $\mathrm{Ham}(M, \omega)$, is the group of Hamiltonian diffeomorphisms. The second space $\mathcal{L}$ consists of all Lagrangian submanifolds of $M \times M$ which are exact Lagrangian isotopic to the diagonal. The space $\mathcal{L}$ carries a Finsler-type metric which is analogue to the above mentioned Hofer's metric on $\mathrm{Ham}(M, \omega)$. In chapter three below we show that in the case of a closed symplectic manifold with $\pi_2(M) = 0$, the canonical embedding of $\mathrm{Ham}(M, \omega)$ into $\mathcal{L}$, $f \mapsto graph(f)$ is highly distorted with respect to Hofer's distances in spite of the fact that it preserves lengths of smooth paths. More precisely, denote by d the Hofer metric on $\mathrm{Ham}(M, \omega)$ and by d_{Lag} the analogue Hofer metric on $\mathcal{L}$. Set $\mathbb{1}$ for the identity element of $\mathrm{Ham}(M, \omega)$.

Theorem III: Let (M, ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Then there exist a family $\{\varphi_t\}$, $t \in [0, \infty)$ in $\mathrm{Ham}(M, \omega)$ and a universal constant c such that:

$$d(\mathbb{1}, \varphi_t) \to \infty \quad \text{as} \quad t \to \infty \quad \text{while} \quad d_{Lag}\big(\mathrm{graph}(\mathbb{1}), \mathrm{graph}(\varphi_t)\big) = c.$$

The above family $\{\varphi_t\}$ was constructed explicitly. The following corollaries follow immediately under the conditions of the theorem:

i) The embedding of $\mathrm{Ham}(M, \omega)$ in $\mathcal{L}$ is not isometric, rather, the image of $\mathrm{Ham}(M, \omega)$ in $\mathcal{L}$ is highly distorted. The minimal path between two graphs of Hamiltonian diffeomorphisms in $\mathcal{L}$, might pass through exact Lagrangian submanifolds which are not the graphs of any Hamiltonian diffeomorphisms.

ii) The group $\mathrm{Ham}(M, \omega)$ has an infinite diameter with respect to Hofer's metric d.

iii) The metric d does not coincide with the Viterbo-type metric defined by Schwarz [55].

Moreover, as a by-product of our method we obtained a result regarding the geodesics in $\mathrm{Ham}(M, \omega)$. We showed (under the same conditions on the manifold M) that there exists an element in $\mathrm{Ham}(M, \omega)$ which cannot be joined to the identity by a minimal geodesic.

Theorem IV: Let (M, ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Then there exists an element φ in $(\mathrm{Ham}(M, \omega), d)$ which cannot be joined to the identity by a minimal geodesic.

The first example of this kind was established by Lalonde and McDuff [29] for the case of the two dimensional sphere S^2.

Algebraic Properties of Ham(M, ω)

The second part of this thesis deals with some algebraic properties of the group of Hamiltonian diffeomorphisms. We focus mainly on the existence and the applications of certain *quasi-morphisms*. Recall that a (real valued) quasi-morphism on a group G is a map $r : G \to \mathbb{R}$ that is a bounded distance away from being a homomorphism, i.e., there exists a constant $c = c(r) > 0$ such that

$$|r(gh) - r(g) - r(h)| < c, \quad \text{for all } g, h \in G.$$

A quasi-morphism r is called homogeneous if $r(g^m) = mr(g)$ for all $m \in \mathbb{Z}$. The notion "quasi-morphism" first appeared in the works of Brooks [10] and Gromov [21] on bounded cohomology of groups. Since then, quasi-morphisms have become an important tool in the study of groups. For example, the mere existence of a homogeneous quasi-morphism on a group G which does not vanish on the commutator subgroup G' implies that the commutator subgroup has infinite diameter with respect to the commutator norm (see e.g. [6])

In the case of the group of Hamiltonian diffeomorphisms, a celebrated result by Banyaga [4] states that for a closed symplectic manifold, $\mathrm{Ham}(M, \omega)$ and its universal cover $\widetilde{\mathrm{Ham}}(M, \omega)$ are simple groups. This implies in particular that they do not admit any non-trivial homomorphism to $\mathbb{R}$. On the other hand, the existence of homogeneous quasi-morphisms on the group of Hamiltonian diffeomorphisms and/or its universal cover is known for some classes of closed symplectic manifolds (see e.g. [5], [14], [18], and [19]). In a recent work [15], Entov and Polterovich showed - by using Floer and Quantum homology - that for the class of symplectic manifolds which are monotone and whose quantum homology algebra is semi-simple, $\widetilde{\mathrm{Ham}}(M, \omega)$ admits a homogeneous quasi-morphism to $\mathbb{R}$. In addition to constructing such a quasi-morphism, Entov and Polterovich showed that its value on any diffeomorphism supported in a Hamiltonianly displaceable open subset equals to the Calabi invariant of the diffeomorphism (see Section 4 below for precise definitions). A quasi-morphism with this property is called a *Calabi quasi-morphism*. It turns out that the existence of a Calabi quasi-morphism has several applications regarding Lagrangian intersection and rigidity of intersections in symplectic manifolds. We refer the reader to [8], [16], and to Chapter 4 of this thesis for further information on this subject.

The forth chapter of this thesis is based on our work [45]. There we constructed Calabi quasi-morphisms on the universal cover of the group of Hamiltonian diffeomorphisms for some non-monotone symplectic manifolds. This complements the above mentioned result by Entov and Polterovich which applies in the monotone case.

In order to state our results we first recall some notations. Let $X_\lambda = (S^2 \times S^2,\ \omega_\lambda = \omega \oplus \lambda\omega),\ 1 \leq \lambda \in \mathbb{R}$, where ω is the standard area form on the two-dimensional sphere S^2 with total area 1, and let $Y_\mu = (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2},\ \omega_\mu),\ 0 < \mu < 1$, be the symplectic blow-up of $\mathbb{C}P^2$ at one point, where ω_μ takes the value μ on the exceptional divisor, and 1 on the class of the line $[\mathbb{C}P^1]$. In the monotone case where $\lambda = 1$ and $\mu = 1/3$, Entov and Polterovich [15] proved the existence of a Lipschitz homogeneous Calabi quasi-morphism on the universal covers of $\mathrm{Ham}(X_\lambda)$ and $\mathrm{Ham}(Y_\mu)$. For the definition of the Lipschitz property of a Calabi Quasi-morphism see [15] or Section 4 below. The following theorem (proven in [45]) extend this result to the rational non-monotone case.

Theorem V: Let (M, ω) be either X_λ where $1 < \lambda \in \mathbb{Q}$ or Y_μ where $1/3 \neq \mu \in \mathbb{Q} \cap (0, 1)$. Then there exists a Lipschitz homogeneous Calabi quasi-morphism $\widetilde{r} : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$.

In fact, the examples in the above theorem are special cases of a more general criterion for the existence of a Calabi quasi-morphism (also proven in [45]). This criterion is based on some algebraic properties of the quantum homology algebra associated with the manifold (M, ω). The precise statement is:

Theorem VI: Let (M, ω) be a closed connected rational strongly semi-positive symplectic manifold of dimension $2n$. Suppose that the quantum homology subalgebra $QH_{2n}(M) \subset QH_*(M)$ is a semi-simple algebra over the field Λ_0 and that the minimal Chern number N_M divides n. Then there exists a Lipschitz homogeneous Calabi quasi-morphism $\widetilde{r} : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$.

The precise definitions of the conditions in the theorem and further details will be given in Chapter 4 below.

Moreover, in contrast with Entov-Polterovich's work, we showed that the quasi-morphisms in the theorem above descend to non-trivial homomorphisms on the fundamental group of $\mathrm{Ham}(M,\omega)$. More precisely

Theorem VII: Let M be one of the manifolds listed in Theorem V. Then the restriction of the above mentioned Calabi quasi-morphism to the fundamental group $\pi_1\big(\mathrm{Ham}(M)\big)$ gives rise to a non-trivial homomorphism.

This differs from the situation described by Entov and Polterovich where it was proven that for $M = \mathbb{C}P^n$ endowed with the Fubini-Study form, or for $M = S^2 \times S^2$ equipped with the split symplectic form $\omega \oplus \omega$, the restriction of the Calabi quasi-morphism to the fundamental group $\pi_1\big(\mathrm{Ham}(M)\big)$ vanishes identically.

As a by-product of Theorem V, we also generalize a result regarding rigidity of intersections obtained by Entov and Polterovich in [16]. To describe this result, we recall first the following definitions. For a symplectic manifold M denote by $\{\cdot,\cdot\}$ the standard Poisson brackets on $C^\infty(M)$. A linear subspace $\mathcal{A} \subset C^\infty(M)$ is said to be Poisson-commutative if $\{F,G\} = 0$ for all $F, G \in \mathcal{A}$. We associate to a finite-dimensional Poisson-commutative subspace $\mathcal{A} \subset C^\infty(M)$ its moment map $\Phi_{\mathcal{A}} : M \to \mathcal{A}^*$, defined by $\langle \Phi_{\mathcal{A}}(x), F\rangle = F(x)$. A non-empty subset of the form $\Phi_{\mathcal{A}}^{-1}(p)$, $p \in \mathcal{A}^*$, is called a fiber of $\mathcal{A}$. A fiber $X \subset M$ is said to be displaceable if there exists a Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}(M)$ such that $\varphi(X) \cap X = \emptyset$.

Definition: A closed subset $X \subset M$ is called a stem, if there exists a finite-dimensional Poisson-commutative subspace $\mathcal{A} \subset C^\infty(M)$, such that X is a fiber of $\mathcal{A}$ and each fiber of $\mathcal{A}$, other than X, is displaceable.

In Theorem 2.4 of [16], Entov and Polterovich showed that for a certain class of symplectic manifolds, any two stems have a non-empty intersection. What they used, in fact, was only the existence of a Lipschitz homogeneous Calabi quasi-morphism for manifolds in this class. Using the exact same line of proof, the following theorem follows from Theorem V above.

Theorem IIX: Let M be one of the manifolds listed in Theorem V. Then any two stems in M intersect.

An example of a stem in the case where $M = X_\lambda$ is the product of two equators. More precisely we identify X_λ with $\mathbb{C}P^1 \times \mathbb{C}P^1$ in the obvious way. Denote by $L \subset X_\lambda$ the

Lagrangian torus defined by

$$L = \left\{ ([z_0 : z_1], [w_0 : w_1]) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \mid \quad |z_0| = |z_1|, \ |w_0| = |w_1| \right\}$$

The proof that L is a stem goes along the same line as Corollary 2.5 of [16]. Since the image of a stem under any symplectomorphism of M is again a stem we get:

Corollary: Let X_λ be the first class of manifolds listed in Theorem V. Then for any symplectomorphism ϕ of X_λ we have $L \cap \phi(L) \neq 0$.

Structure of the thesis:

Chapter 1: In the first chapter of the thesis we recall some necessary definitions and notations from symplectic geometry. In particular we introduce the main object of the thesis i.e., the group of Hamiltonian diffeomorphisms.

Chapter 2: In this chapter, after briefly recalling the definition of Hofer's metric, we address the question of the uniqueness of Hofer's metric among the Finsler-type metrics on the group of Hamiltonian diffeomorphisms, and present our main results. The next section contains a fairly detailed outline of the proofs of our main theorems, stressing the main ingredients involved. The following two sections present complete proofs of our main results i.e., Theorem 2.1.3 and Theorem 2.1.2 respectively. Section 2.5 contains proofs of some technical lemmas. The last section in this chapter contains a sketchy treatment of some additional properties of the norm $|||\cdot|||$, together with some references concerning the classification of such norms.

Chapter 3: In this Chapter we study Hofer's metric in a larger context of Lagrangian submanifolds. We consider the natural map which takes a symplectic diffeomorphism to its graph (which is a Lagrangian submanifolds), and discuss it from the viewpoint of Hofer's geometry. The main result stated in the first section of the chapter states that this map has the following surprising feature: it preserves the lengths of curves but strongly distorts the distances. In section 3.2 we prove our main theorem. In section 3.3 we present some result about geodesics in the group of Hamiltonian diffeomorphisms endowed with Hofer's metric. In the last two section of this chapter we prove Proposition 3.2.6 and Proposition 3.2.2, which were stated and used in Section 3.2, respectively.

Chapter 4**:** Here we focus on some algebraic properties of the group of Hamiltonian diffeomorphisms, namely the existence and applications of Calabi Quasi-morphisms. In the first section of this chapter we present some background regarding Calabi Quasi-morphisms, recall previous results concerns the existence of a Calabi Quasi-morphism and some application of it, and state our main results. In Section 4.2 we recall some definitions and notations related to the Calabi quasi-morphism. In Section 4.3 we briefly review the definition of the quantum homology algebra $QH_*(M)$. We then describe the quantum homologies of our main examples and state some of their properties. In Section 4.4 we recall the definition of Floer homology and some relevant notions. Section 4.5 is devoted to the proof of Theorem 4.1.1 and Theorem 4.1.3. In Section 4.6 we discuss the restriction of the Calabi quasi-morphisms on the fundamental group of $\mathrm{Ham}(M)$. In Section 4.7 we prove Theorem 4.1.2 and in the last section we prove the Poincaré duality type lemma which is stated and applied in Section 4.5.

Chapter 1

Preliminaries in Symplectic Geometry

In this chapter we recall some basic definitions and notations from symplectic geometry. We restrict ourselves only to those which serve as a background to the following chapters. For a comprehensive introduction to symplectic geometry see e.g. [34].

1.1 Symplectic Manifolds

A symplectic manifold is a pair (M, ω) consisting of a smooth manifold M together with a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$. The first and most basic example is the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ with the standard linear coordinates $(x_1, y_1, \ldots, x_n, y_n)$, together with the (so-called) standard symplectic structure $\omega_{st} = \sum_{j=1}^{n} dx_j \wedge dy_j$. Note that the dimension of a symplectic manifold must always be even in view of the non-degeneracy condition on ω. Another class of examples are the class of oriented Riemannian surfaces with the symplectic form given by the area form. Our last (fundamental) example of a symplectic manifolds is the cotangent bundle T^*X of an n-dimensional manifold X which carries a canonical symplectic form. This example is especially important since symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds. We refer the reader to [34] for the explanation of the symplectic structures on these spaces.

Given two symplectic manifold (M_1, ω_1) and (M_2, ω_2), a symplectomorphism is a smooth diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^*\omega_2 = \omega_1$. The group of all the symplectomorpisms from M to itself is denoted by $\mathrm{Symp}(M, \omega)$.

1.2 Lagrangian Submanifolds

A particular important class of submanifolds $L \subset M$ of a symplectic manifold (M, ω) are those on which the symplectic form vanishes i.e., $\omega|_{TL} \equiv 0$. Such submanifolds are called *isotropic*. It follows from the non-degeneracy condition of ω that the dimension of an isotropic submanifold is less then or equal to half the dimension of the ambient manifold.

Definition 1.2.1. *A Lagrangian submanifold L of a symplectic manifold M is an isotropic manifold of maximal dimension; in other words, if $\omega|_{TL} \equiv 0$ and $\dim L = \frac{1}{2}\dim M$. An embedding (or immersion) $f : L \to M$, where the dimension of L is half the dimension of M, is called Lagrangian if $f^*\omega \equiv 0$.*

An examples of Lagrangian submanifolds are the torus $T^n = S^1 \times \ldots \times S^1 \subset \mathbb{R}^{2n}$, any curve on an oriented surface, and the zero section in the cotangent bundle. Another class of examples, which will be useful for us in Chapter 3 below, is the following: Let M be a symplectic manifold and let $\psi : M \to M$ be a diffeomorphism. It is not hard to check that ψ is a sympelctomorphism if and only if its graph

$$\mathrm{graph}(\psi) = \{(q, \psi(q)) \ ; \ q \in M\} \subset M \times M,$$

is a Lagrangian submanifold of $(M \times M, \omega \oplus -\omega)$.

We turn now to recall the notion of *exact Lagrangian isotopies* which we will also use in Chapter 3 below. Let (M^{2n}, ω) be a symplectic manifold of dimension $2n$ and let L^n be an n-dimensional closed manifold. Let $\Psi : L \times [0, 1] \to M$ be a family of Lagrangian embeddings. It follows from the definition that $\Psi^*\omega = \alpha_s \wedge dt$, where $\{\alpha_s\}$ is a family of closed 1-forms on L.

Definition 1.2.2. *A Lagrangian isotopy Ψ is called exact if α_s is exact for every s.*

It is known that an exact Lagrangian isotopy can be extended to an ambient Hamiltonian isotopy of M. In chapter 3 below we discuss such an extension property with some additional requirements.

1.3 The Group of Hamiltonian Diffeomorphisms

As mentioned in the introduction, the main object of our study is the infinite-dimensional Lie group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms of M which we now turn to define. Let $\mathcal{A}$ be the space of all time-independent smooth functions on M which are zero-mean normalized with respect to the canonical volume form ω^n i.e.,

$$\mathcal{A} = \Big\{ F \in C^\infty(M) \ ; \ \int_M F\omega^n = 0 \Big\}.$$

A time dependent function $F : M \times [0,1] \to \mathbb{R}$ is called normalized if F_t belongs to $\mathcal{A}$ for all t. For every time-dependent function $F : M \times [0,1] \to \mathbb{R}$, traditionally called Hamiltonian function, we associated a time-dependent Hamiltonian vector field X_{F_t} on M defined by

$$i_{X_{F_t}}\omega = -dF_t, \ \text{where} \ \ F_t(x) = F(t,x).$$

Since the manifold M is closed, the classical Hamiltonian equation

$$\dot{x} = X_{F_t}, \ \ x(0) = x_0 \in M,$$

can be solved over the time interval $[0,1]$ for every initial value $x_0 \in M$. Hence we obtain a 1-parametric family of diffeomorphisms φ_F^t, where $t \in [0,1]$, defined by the condition

$$\varphi_F^t(x_0) = x(t),$$

where $x(t)$ is the solution of the above Hamiltonian equation for the initial value x_0. It is not hard to check (see e.g. [24]) that the maps φ_F^t corresponding to the flow of a Hamiltonian vector field X_{F_t} leave the symplectic form invariant; or in other words, φ_F^t is a family of symplectic diffeomorphisms. In what follow we shall address this family as the Hamiltonian flow generated by the Hamiltonian function F. We denote by $\mathrm{Ham}(M,\omega)$ the set of all the time-1-maps of such Hamiltonian diffeomorphisms i.e.,

$$\mathrm{Ham}(M,\omega) = \big\{ \varphi \mid \varphi = \varphi_F^1 \text{ for some normalized } F : M \times [0,1] \to \mathbb{R} \big\}$$

This set has a group structure with respect to composition (see e.g. [48]), and it is in fact an infinite dimensional Lie group. It is also a normal subgroup of the identity component $\mathrm{Symp}_0(M,\omega)$ of the group of all the symplectomorphisms of M. We refer the readers to [24], [34] and [48] for symplectic preliminaries and further discussions on the group of Hamiltonian diffeomorphisms.

1.4 Hofer's metric on Ham(M, ω)

One of the most remarkable facts concerning the group of Hamiltonian diffeomorphisms is the existence on a bi-invariant metric known as Hofer's metric. We turn now to define it. Let (M, ω) be a closed symplectic manifold. The following important fact is due to Banyaga [4]: For every path $\{\varphi_t\}$, $t \in [0, 1]$ of diffeomorphism with values in $\mathrm{Ham}(M, \omega)$ there exists a normalized time dependent Hamiltonian function $F : M \times [0, 1] \to \mathbb{R}$ which generate $\{\varphi_t\}$ as its Hamiltonian flow. Thus, given a Hamiltonian path $\{\varphi_t\}$ we define its length by

$$\|\{\varphi_t\}\| := \max_{x,t} F(t, x) - \min_{x,t} F(t, x),$$

where the max and the min are taken over all $(x, t) \in M \times [0, 1]$. Next, as in the case of Finsler geometry, we measure the distance from the identity in $\mathrm{Ham}(M, \omega)$ by taking the infimum of the lengths of all the connecting paths i.e., for every $\varphi \in \mathrm{Ham}(M, \omega)$ we define a pseudo-metric d by

$$\widetilde{d}(\mathbb{1}, \varphi) := \inf\{\|\{\varphi_t\}\|\},$$

where the infimum is taken over all the Hamiltonian paths $\{\varphi_t\}$ with $\varphi = \varphi_1$. Next we extend d to a bi-invariant function by setting $d(\varphi, \psi) := \widetilde{d}(\mathbb{1}, \psi\varphi^{-1})$. It is easy to check that d is a bi-invariant pseudo-distance function of $\mathrm{Ham}(M, \omega)$. However is is highly non-trivial to show that

Theorem 1.4.1. *The pseudo-metric d is non-degenerate and therefore defines a genuine bi-invariant metric on the group of Hamiltonian diffeomorphisms.*

This was discovered and proved by Hofer [22] for the case of $\mathbb{R}^{2n}$, then generalized by Polterovich [46] to some larger class of symplectic manifolds, and finally proven in full generality by Lalonde and McDuff in [28]. In what follows we refer to d as Hofer's metric.

Remark 1.4.2. In Chapter 2 below we re-introduce Hofer's metric in a slightly different way, however equivalent. There we consider the group of Hamiltonian diffeomorphisms as a Lie group and define Hofer's metric in terms of Finsler-type metric given by the choice of the supremum norm $\|F\| = \max_M |F|$ on the Lie-algebra of $\mathrm{Ham}(M, \omega)$ which we identified with $\mathcal{A}$. The original metric introduce by Hofer is actually defined by the norm $\|F\| = \max_M F - \min_M F$ on the Lie-algebra of $\mathrm{Ham}(M, \omega)$, but it is equivalent to the one we use in Chapter 2.

Chapter 2

On the Extremality of Hofer's Metric

2.1 Introduction and Results

Let (M,ω) be a closed connected symplectic manifold of dimension $2n$. Let $\mathcal{A}$ be the space of all time-independent smooth functions on M which are zero-mean normalized with respect to the canonical volume form ω^n, and let $\mathrm{Ham}(M,\omega)$ be the group of Hamiltonian diffeomorphisms of M. It is well known (see e.g. [48]) that the Lie algebra of $\mathrm{Ham}(M,\omega)$, that is the space of all Hamiltonian vector fields, can be identified with the space $\mathcal{A}$. Moreover, the adjoint action of $\mathrm{Ham}(M,\omega)$ on its Lie algebra $\mathcal{A}$ is the standard action of diffeomorphisms on functions. The choice of any norm $\|\cdot\|$ on $\mathcal{A}$ gives rise to a pseudo-distance function on $\mathrm{Ham}(M,\omega)$ in the following way: we define the length of a path $\alpha : [0,1] \to \mathrm{Ham}(M,\omega)$ as

$$\mathrm{length}\{\alpha\} = \int_0^1 \|\dot{\alpha}\| dt = \int_0^1 \|F_t\| dt,$$

where $F_t(x) = F(t,x)$ is the Hamiltonian function generating the path α. This is the usual definition of Finsler length. The distance between two Hamiltonian diffeomorphisms is given by

$$\rho(\psi,\varphi) = \inf \mathrm{length}\{\alpha\},$$

where the infimum is taken over all Hamiltonian paths α connecting ψ and φ. It is not hard to check that ρ is non-negative, symmetric and satisfies the triangle inequality. Moreover, a norm on $\mathcal{A}$ which is invariant under the adjoint action yields a bi-invariant pseudo-distance function, i.e. $\rho(\psi,\varphi) = \rho(\theta\,\psi,\theta\,\varphi) = \rho(\psi\,\theta,\varphi\,\theta)$ for every $\psi,\varphi,\theta \in \mathrm{Ham}(M,\omega)$.

From now on we will deal only with such norms and we will refer to ρ as the pseudo-distance generated by the norm $\|\cdot\|$.

It is highly non-trivial to check whether such a distance function is non-degenerate, that is $\rho(\mathbb{1},\psi) > 0$ for $\psi \neq \mathbb{1}$. In fact, for compact symplectic manifolds, a bi-invariant pseudo-metric ρ on $\mathrm{Ham}(M,\omega)$ is either a genuine metric or identically zero. This is an immediate corollary of a well known theorem by Banyaga [4], which states that $\mathrm{Ham}(M,\omega)$ is a simple group, combined with the fact that the null-set

$$\mathrm{null}(\rho) = \{\psi \in \mathrm{Ham}(M,\omega) \mid \rho(\mathbb{1},\psi) = 0\}$$

is a normal subgroup of $\mathrm{Ham}(M,\omega)$.

A distinguished result by Hofer [22] states that the L_∞ norm $\|\cdot\|_\infty$ on $\mathcal{A}$ gives rise to a genuine distance function on $\mathrm{Ham}(M,\omega)$. This was discovered and proved by Hofer for the case of $\mathbb{R}^{2n}$, then generalized by Polterovich [46] to some larger class of symplectic manifolds, and finally proven in full generality by Lalonde and McDuff in [28]. The above mentioned distance function is known as Hofer's metric and has been intensively studied since its discovery (see e.g. [24], [34], [48]). We also refer the reader to Oh's paper [42] for another approach to the non-degeneracy of Hofer's metric, and to Chekanov's paper [12] for a proof that an analogue of Hofer's metric is (up to scaling) the only non-degenerate Hamiltonian-invariant Finsler metric (see [12] for the precise definition) on the space of Lagrangian submanifolds Hamiltonian isotopic to a given closed Lagrangian. In the opposite direction, Eliashberg and Polterovich showed in [13] that for $1 \leq p < \infty$, the pseudo-distances on $\mathrm{Ham}(M,\omega)$ which correspond to the L_p norms on $\mathcal{A}$ vanish identically. Thus, the following question arises from [13] and [48]:

Question: What are the invariant norms on $\mathcal{A}$, and which of them give rise to genuine bi-invariant metrics on $\mathrm{Ham}(M,\omega)$?

Our main contributions towards answering this question are

Theorem 2.1.1. *Let $\|\cdot\|$ be a $Ham(M,\omega)$-invariant norm on $\mathcal{A}$ such that $\|\cdot\| \leq C\|\cdot\|_\infty$ for some constant C. Then $\|\cdot\|$ is invariant under all measure preserving diffeomorphisms of M.*

Theorem 2.1.2. *Let $\|\cdot\|$ be a $Ham(M,\omega)$-invariant norm on $\mathcal{A}$ such that $\|\cdot\| \leq C\|\cdot\|_\infty$ for some constant C, but the two norms are not equivalent. Then the associated pseudo-distance function ρ on $Ham(M,\omega)$ vanishes identically.*

Here, two norms are said to be *equivalent*, if each bounds the other up to a multiplicative constant.

The next result is a strengthened formulation of Theorem 2.1.1 and a key ingredient in the proof of Theorem 2.1.2. As the discussion below explains, it also bears on the question of classifying $\mathrm{Ham}(M,\omega)$-invariant norms.

Theorem 2.1.3. *Let $\|\cdot\|$ be a Ham(M,ω)-invariant norm on $\mathcal{A}$ such that $\|\cdot\| \le C\|\cdot\|_\infty$ for some constant C. Then $\|\cdot\|$ can be extended to a semi-norm $|||\cdot||| \le C\|\cdot\|_\infty$ on $L_\infty(M)$, which is invariant under all measure preserving bijections on M.*

The formulation of the theorem states only what is necessary for the proofs of Theorems 2.1.1 and 2.1.2. In fact we know more about $|||\cdot|||$. First, $|||\cdot|||$ is a norm, rather than just a semi-norm (namely, $|||\cdot|||$ does not vanish on non-zero functions). Second, $|||\cdot|||$, when restricted to zero-mean functions, coincides with the completion of $||\cdot||$. Third, this completion can be viewed as a dense subspace of the space of zero-mean functions in $L_1(M)$, equipped with a norm invariant under measure preserving bijections. The argument for the first claim is briefly sketched in Remark 2.5.1, and that for the second and third claims is outlined in the final section. The final section also refers to literature concerning the classification of such norms, and indicates their possible pathologies.

2.2 A short outline of the Proofs

As explained in the introduction, the degeneracy of the pseudo-distance function ρ (Theorem 2.1.2) is proved in [13] for L_p norms, $1 \le p < \infty$. The only property of L_p actually used in that proof is, roughly speaking, that uniformly bounded functions with small support have small norm. More precisely, in Section 2.4 we reproduce an argument from [13] to show that the proof of Theorem 2.1.2 can be reduced to the following

Claim 2.2.1. *If* $\sup\{\|F_n\|_\infty\} < \infty$ *and* $\mathrm{Vol}\big(\mathrm{Support}(F_n)\big) \to 0$*, then* $\|F_n\| \to 0$.

Therefore, our main task is to prove this property for any norm which satisfies the requirements of Theorem 2.1.2. As will be explained below, Theorem 2.1.3 allows us to carry out the proof of this claim in a more amenable setting.

A natural approach to Claim 2.2.1 would be to consider characteristic functions with small-measure support first, then make the standard move to step functions, and conclude with any smooth bounded function with small-measure support. The obvious obstacle is that characteristic functions are not smooth, and are therefore outside our space. Here one may choose to approximate them by smooth functions and work from there. We chose, however, to extend our setting so as to include genuine characteristic functions. This is where Theorem 2.1.3 comes in. We will interrupt the discussion on the proof of Claim 2.2.1 to discuss the proof of Theorem 2.1.3.

Recall that our aim in Theorem 2.1.3 is to extend the given norm $\|\cdot\|$ to $L_\infty(M)$. For this purpose, we first extend our norm to all smooth functions, with average not necessarily zero (since this adds just one dimension to our original space of functions, any two extensions are equivalent). Next, we take advantage of the fact that $C^\infty(M)$ is dense in $L_\infty(M)$ with respect to the topology of convergence in measure. We define

$$|||F||| = \inf\{\liminf_{n\to\infty} ||F_n||\},$$

where the infimum is taken over all sequences $\{F_n\}$ of uniformly bounded smooth functions which converge in measure to F.

Such constructions occur occasionally in functional analysis, for instance in the extension of the Riemann integral from continuous to semi-continuous functions (using pointwise convergence from above/below), and in the extension of operator norms from finite-rank operators on a Banach space to approximable operators (using uniform convergence on compacta). However, we are not aware of any similar construction which relies on convergence in measure.

We study $|||\cdot|||$ in Section 2.3. First we confirm that $|||\cdot|||$ is a semi-norm on $L_\infty(M)$ which is dominated from above by $\|\cdot\|_\infty$. We then go on to prove the non-trivial properties of $|||\cdot|||$: it coincides with $\|\cdot\|$ on smooth functions, and is invariant under measure preserving bijections. Formally:

Claim 2.2.2. *For every $F \in \mathcal{A}$ we have $\|F\| = |||F|||$.*

Claim 2.2.3. *For every $F \in L_\infty(M)$ and every measure preserving bijection φ on M we have*

$$|||F \circ \varphi||| = |||F|||$$

In order to prove this second property, recall that our original norm $\|\cdot\|$ is already invariant under Hamiltonian diffeomorphisms. To extend the invariance we invoke Katok's "Basic Lemma" from [25], which allows to approximate in measure any measure preserving bijection by a Hamiltonian diffeomorphism. More precisely, fix an arbitrary Riemannian metric d on M. We claim

Lemma 2.2.4. *For every measure preserving bijection φ of M (not necessarily continuous) and every $\varepsilon > 0$, there exists a Hamiltonian diffeomorphism g on M which ε-approximates φ in measure, namely*

$$\mathrm{Vol}\big(\{x \in M\,;\, d(\varphi(x), g(x)) > \varepsilon\}\big) < \varepsilon$$

This result is of course independent of the specific Riemannian structure chosen. We postpone the proof of the lemma to the last section of this paper. The proof of Claim 2.2.3 follows easily from Lemma 2.2.4 and the definition of $|||\cdot|||$. Claim 2.2.2 and Claim 2.2.3 conclude the proof of Theorem 2.1.3.

With a measure-preserving-bijection-invariant extension of $||\cdot||$ at our disposal, let's return to the proof of Claim 2.2.1. Note that Claim 2.2.2 implies that it is sufficient to prove Claim 2.2.1 for the norm $|||\cdot|||$. The rest of this section is devoted to this issue.

Our argument depends on the fact, inspired by an argument from [47], that an operator, which performs piecewise averaging on functions, is bounded. More precisely, relying on the fact that $|||\cdot|||$ is invariant under measure preserving bijections, we prove that

Lemma 2.2.5 (Piecewise-Averaging property). *For every continuous F and every measurable partition $\{S_i\}$ of M, we have*

$$|||\sum_i \langle F\rangle_{S_i} \mathbb{1}_{S_i}||| \le |||F|||,$$

where $\langle F\rangle_{S_i} = \frac{1}{\mathrm{Vol}(S_i)}\int_{S_i} F\omega^n$ denotes the average of F over S_i.

The proof of the lemma is postponed to the last section. Let us now explain how this property serves to prove Claim 2.2.1. Fix $\varepsilon > 0$. The hypothesis of Theorem 2.1.2 provides us with smooth functions F such that $\|F\|_\infty = 1$ while $\|F\| = |||F||| \le \varepsilon$. Partition M into A and $A^c = M \setminus A$, where A is a small enough neighborhood of the maximum of F, such that $||1\!\!1_A - \langle F\rangle_A 1\!\!1_A + \langle F\rangle_{A^c} 1\!\!1_{A^c}||_\infty < \varepsilon$. Next, it follows from Lemma 2.2.5, the fact that $|||\cdot|||$ is dominated from above by $\|\cdot\|_\infty$, and the triangle inequality that

$$|||1\!\!1_A||| \le ||1\!\!1_A - \langle F\rangle_A 1\!\!1_A + \langle F\rangle_{A^c} 1\!\!1_{A^c}||_\infty + |||\langle F\rangle_A 1\!\!1_A + \langle F\rangle_{A^c} 1\!\!1_{A^c}||| \le \varepsilon + |||F||| \le 2\varepsilon$$

Since $|||\cdot|||$ is invariant under measure preserving bijections, this applies to every set B with the same measure as A. Thus, we establish Claim 2.2.1 for sequences of characteristic functions on sets with measure tending to zero. It is now a simple approximation argument, which establishes Claim 2.2.1 as stated for smooth functions. The details are given in Section 2.4.

2.3 Extending the norm and Proof of Theorem 2.1.3

In this section we construct the semi-norm $|||\cdot|||$, and prove its properties as stated in Theorem 2.1.3. The first step towards the construction of $|||\cdot|||$ is an extension of the given norm to $C^\infty(M)$. Let C be a constant such that $\|\cdot\| \le C\|\cdot\|_\infty$. Endow the space $C^\infty(M)$ of all smooth function on M with the norm $\|\cdot\|'$ defined by

$$\|F\|' = \inf\big\{\|F_1\| + C\|F_2\|_\infty \ ; \ F = F_1 + F_2, \quad F_1 \in \mathcal{A}, \quad F_2 \in C^\infty(M)\big\}.$$

The above definition is just the analytic presentation of the norm corresponding to the convex hull of the unit ball of $(\mathcal{A}, \|\cdot\|)$ with the unit ball of $(C^\infty(M), \|\cdot\|_\infty)$, the latter homothetically shrunk so as to fit inside the former when restricted to $\mathcal{A}$. The homogeneity of the new norm is clear. To see that the new norm satisfies the triangle inequality, let $F = F_1 + F_2$ and $G = G_1 + G_2$ such that $\|F_1\| + C\|F_2\|_\infty \le \|F\|' + \varepsilon$, and $\|G_1\| + C\|G_2\|_\infty \le \|G\|' + \varepsilon$. Then

$$\begin{aligned}\|F+G\|' &\le \|F_1 + G_1\| + C\|F_2 + G_2\|_\infty \\ &\le (\|F_1\| + C\|F_2\|_\infty) + (\|G_1\| + C\|G_2\|_\infty) \\ &\le \|F\|' + \|G\|' + 2\varepsilon.\end{aligned}$$

The new norm is obviously $\mathrm{Ham}(M,\omega)$-invariant. To see that $\|F\|' \le C\|F\|_\infty$, just substitute $F_1 = 0$ and $F_2 = F$ in the definition. To see that $\|\cdot\|' = \|\cdot\|$ on $\mathcal{A}$, let $F = F_1 + F_2$ where $F_1 \in \mathcal{A}$ and $F_2 \in C^\infty(M)$. Choosing $F_1 = F$ and $F_2 = 0$ proves that

$\|F\|' \leq \|F\|$. For the opposite direction note that since $F, F_1 \in \mathcal{A}$, and since $F_2 = F - F_1$, the function F_2 must also be in $\mathcal{A}$. Therefore

$$\|F\|' = \inf_{F=F_1+F_2} \left\{\|F_1\| + C\|F_2\|_\infty\right\} \geq \inf_{F=F_1+F_2} \left\{\|F_1\| + \|F_2\|\right\} \geq \inf_{F=F_1+F_2} \|F_1 + F_2\| = \|F\|$$

Now we are ready to extend our norm to the entire $L_\infty(M)$. Using the same convex-hull trick won't do (it would fail invariance under measure preserving bijections). Instead, we take advantage of the classical fact that any measurable function can be approximated in measure arbitrarily well by smooth functions (see e.g. [53]). We define a new functional by taking the least $\|\cdot\|'$ norm among all such approximations. Formally, we endow the space $L_\infty(M)$ with

$$|||F||| = \inf \left\{\liminf_{n\to\infty} \|F_n\|'\right\},$$

where the infimum is taken over all sequences of uniformly bounded smooth functions $\{F_n\}$ which converge in measure to F.

It is clear that the new functional is homogeneous. To see that it obeys the triangle inequality, take $\{F_n\}$ and $\{G_n\}$ which satisfy $\liminf \|F_n\|' \leq |||F||| + \varepsilon$ and $\liminf \|G_n\|' \leq |||G||| + \varepsilon$. Then

$$|||F + G||| \leq \liminf \|F_n + G_n\|' \leq \liminf(\|F_n\|' + \|G_n\|') \leq |||F||| + |||G||| + 2\varepsilon.$$

To see that the new functional is still bounded by $C\|\cdot\|_\infty$, note that any essentially bounded function F can be approximated in measure by smooth F_n's with at most the same essential supremum. Indeed, take any approximation in measure F_n of F, and replace it with $\mathrm{sign}(F_n) \cdot (f_n \circ |F_n|)$, where f_n is a good enough smooth approximation from below of the function $f(s) : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(s) = \min\{s, \|F\|_\infty\}$. Taking such F_n's we get

$$|||F||| \leq \liminf \|F_n\|' \leq C \liminf \|F_n\|_\infty \leq C\|F\|_\infty.$$

In order to complete the proof of Theorem 2.1.3 we need the following two claims.

Claim 2.3.1. *For every $F \in \mathcal{A}$ we have $\|F\| = \|F\|' = |||F|||$*

Claim 2.3.2. *For every $F \in L_\infty(M)$ and every measure preserving bijection φ on M we have*

$$|||F \circ \varphi||| = |||F|||$$

In order to prove the first claim a certain technical lemma is needed. To state the lemma, fix from now on an arbitrary Riemannian structure on M, and denote by d the corresponding distance function. Our results are, of course, independent of the specific Riemannian structure chosen.

Lemma 2.3.3 (Covering Evenly by Many Packings). *For every $\delta > 0$ and $\varepsilon > 0$ there exists a covering of M by connected open subsets $\{U_i^j\}$, where $j = 1, \dots, J$ and $i = 1, \dots, L_j$, such that*

(i) for every fixed j, each pair of sets $\{U_i^j\}$ have a positive distance from each other.

(ii) the diameter of U_i^j with respect to d is at most δ for all i and j.

(iii) for every $x \in M$, the number of j's for which $x \notin \cup_i U_i^j$ is at most εJ.

The proof of the lemma is postponed to the last section of this paper.

Proof of Claim 2.3.1: The restricted equality $\|\cdot\| = \|\cdot\|'$ has been proved along with the definition of $\|\cdot\|'$ above. Let's prove the restricted equality $|||\cdot||| = \|\cdot\|'$. By choosing $F_n = F$ for all n in the definition of $|||\cdot|||$, we get $|||\cdot||| \le \|\cdot\|'$. In order to show that $\|\cdot\|' \le |||\cdot|||$, let $F \in \mathcal{A}$ and let $\{F_n\}$ be a sequence of uniformly bounded smooth functions, which converges in measure to F. We need to show that

$$\liminf_{n\to\infty} \|F_n\|' \ge \|F\|'.$$

For this purpose we will construct a sequence $\{\widetilde{F}_n\}$ which converges uniformly to F, such that $\|F_n\|' \ge \|\widetilde{F}_n\|'$. Since $\|\cdot\|' \le C\|\cdot\|_\infty$, uniform convergence implies convergence in $\|\cdot\|'$, and we can conclude

$$\liminf_{n\to\infty} \|F_n\|' \ge \liminf_{n\to\infty} \|\widetilde{F}_n\|' = \|F\|'.$$

Let us construct the sequence $\{\widetilde{F}_n\}$. Fix $\varepsilon > 0$, and let $\delta > 0$ such that every open neighborhood of diameter 2δ in M can be viewed as a neighborhood in $\mathbb{R}^{2n}$ such that the original d and the Euclidian distance are equivalent up to a factor 2. Take a covering $\{U_i^j\}$ of M as in Lemma 2.3.3 with the given ε and δ. Take $\eta < \delta/6$ such that the 3η-extensions of any two sets U_i^j with the same j still have a positive distance between them, and such that

$$d(x, y) \le 2\eta \;\Rightarrow\; |F(x) - F(y)| \le \varepsilon. \tag{2.1}$$

Set V_i^j to be the 3η-extension of U_i^j with respect to the distance d on M. The radius η was chosen such that V_i^j has diameter at most 2δ, and can therefore be viewed as a neighborhood in $\mathbb{R}^{2n}$ where d and the Euclidean distance are equivalent up to a factor 2. In particular, any closed Euclidean ball of radius η centered inside U_i^j is contained in V_i^j. Denote by $B_\eta(x)$ the Euclidean ball of radius η around x. Requirement (2.1) guarantees that

$$\left|\langle F\rangle_{B_\eta(x)} - F(x)\right| \leq \varepsilon. \tag{2.2}$$

Next fix n such that

$$\mathrm{Vol}(\{x : |F_n(x) - F(x)| > \varepsilon\}) \ < \ \frac{\varepsilon \cdot |B_\eta|}{\max\{||F_n||_\infty, ||F||_\infty\}},$$

where $|B_\eta|$ is the measure of a Euclidean ball of radius η. This is possible since $\{F_n\}$ converges to F in measure, and since the F_n's are uniformly bounded. This choice of n implies that

$$\left|\langle F_n\rangle_{B_\eta(x)} - \langle F\rangle_{B_\eta(x)}\right| \leq 3\varepsilon. \tag{2.3}$$

By the definition of the integral, and the uniform continuity of F_n, there exist points $\{x^k\}_{k=1}^K \subseteq B_\eta(0)$, where K may be depend on n, such that for every $x \in U_i^j$

$$\left|\frac{1}{K}\sum_{k=1}^{K} F_n\left(x + x^k\right) - \langle F_n\rangle_{B_\eta(x)}\right| \leq \varepsilon.$$

Note that we have arranged that V_i^j contains the closure of the η-extension of U_i^j. Thus, using a standard cut-off argument, we consider Hamiltonian diffeomorphisms $g_{i,j}^1, \ldots, g_{i,j}^K$, all supported inside V_i^j, defined by $g_{i,j}^k(x) = x + x^k$ inside U_i^j and $g_{i,j}^k(x) = x$ outside a small neighborhood of U_i^j. We therefore get for all $x \in U_i^j$

$$\left|\frac{1}{K}\sum_{k=1}^{K} F_n\left(g_{i,j}^k(x)\right) - \langle F_n\rangle_{B_\eta(x)}\right| \leq \varepsilon. \tag{2.4}$$

Note that for fixed j and k, the Hamiltonian diffeomorphisms $\{g_{i,j}^k\}$ have disjoint supports, and can therefore be bundled together to form a single diffeomorphism. We set

$$\widetilde{F_n}(x) = \frac{1}{J}\sum_{j=1}^{J}\Big(\frac{1}{K}\sum_{k=1}^{K} F_n\big(\prod_i g_{i,j}^k(x)\big)\Big).$$

From the triangle inequality and the fact that the norm $\|\cdot\|'$ is invariant under Hamiltonian diffeomorphisms we conclude that $\|\widetilde{F_n}\|' \leq \|F_n\|'$. Hence, we need only to show that $\|\widetilde{F_n} - F\|_\infty \to 0$ as $\varepsilon \to 0$. Indeed,

$$\widetilde{F_n}(x) = \frac{1}{J}\Big(\sum_{j\in\mathcal{J}(x)}\Big(\frac{1}{K}\sum_k F_n\big(\prod_i g^k_{i,j}(x)\big)\Big) + \sum_{j\in\mathcal{J}^c(x)}\Big(\frac{1}{K}\sum_k F_n\big(\prod_i g^k_{i,j}(x)\big)\Big)\Big),$$

where $\mathcal{J}(x) = \{j \,|\, x \in \cup_i U_i^j\}$, $\mathcal{J}^c(x) = \{j \,|\, x \notin \cup_i U_i^j\}$. Recall that the third item of Lemma 2.3.3 limited the cardinality of $\mathcal{J}^c(x)$ to at most εJ for all x. Together with (2.4) this implies that

$$\Big|\widetilde{F_n}(x) - \frac{1}{J}\sum_{j=1}^{J}\big(\langle F_n\rangle_{B_\eta(x)}\big)\Big| \leq \varepsilon\frac{|\mathcal{J}(x)|}{J} + \frac{|\mathcal{J}^c(x)|}{J}\cdot 2\max\|F_n\|_\infty \leq \varepsilon + 2\varepsilon\cdot\max\|F_n\|_\infty$$

Together with (2.2) and (2.3) we conclude that

$$\begin{aligned}
\big|\widetilde{F_n}(x) - F(x)\big| \;&\leq\; \Big|\widetilde{F_n}(x) - \frac{1}{J}\sum_{j=1}^{J}\big(\langle F_n\rangle_{B_\eta(x)}\big)\Big| + \Big|\frac{1}{J}\sum_{j=1}^{J}\big(\langle F_n\rangle_{B_\eta(x)}\big) - \frac{1}{J}\sum_{j=1}^{J}\big(\langle F\rangle_{B_\eta(x)}\big)\Big| \\
&+\; \Big|\frac{1}{J}\sum_{j=1}^{J}\big(\langle F\rangle_{B_\eta(x)}\big) - \frac{1}{J}\sum_{j=1}^{J}\big(F(x)\big)\Big| + \Big|\frac{1}{J}\sum_{j=1}^{J}\big(F(x)\big) - F(x)\Big| \\
&\leq\; \varepsilon + 2\varepsilon\cdot\max\|F_n\|_\infty + 3\varepsilon + \varepsilon \leq 5\varepsilon + 2\varepsilon\cdot\max\|F_n\|_\infty
\end{aligned}$$

Since the F_n's are uniformly bounded, $\widetilde{F_n}$ indeed converges uniformly to F as ε goes to zero.

As explained in Section 2.2, the proof of Claim 2.3.2 is based on a powerful result by Katok [25] which is used for the proof of Lemma 2.2.4.

Proof of Claim 2.3.2: Take $F \in L_\infty(M)$ and φ a measure-preserving bijection on M. Consider a sequence $\{F_n\}$ of uniformly bounded smooth functions which converges in measure to F. Let ε_n such that F_n is an ε_n-approximation in measure of F. Choose positive numbers δ_n so that $d(x,y) \leq \delta_n \Rightarrow |F_n(x) - F_n(y)| \leq \varepsilon_n$. By repeatedly using Lemma 2.2.4 we get a family of Hamiltonian diffeomorphisms $\{g_n\}$ such that

$$\mathrm{Vol}\big(\{x \,|\, d\left(g_n(x), \varphi(x)\right) > \delta_n\}\big) \leq \varepsilon_n.$$

Obviously

$$\big|F_n\big(g_n(x)\big) - F\big(\varphi(x)\big)\big| \leq \big|F_n\big(g_n(x)\big) - F_n\big(\varphi(x)\big)\big| + \big|F_n\big(\varphi(x)\big) - F\big(\varphi(x)\big)\big|.$$

Our choice of ε_n, δ_n and g_n guarantees that the above sum is smaller than $2\varepsilon_n$ outside a $2\varepsilon_n$-measure exceptional set, and therefore that $\{F_n \circ g_n\}$ converges in measure to $F \circ \varphi$. This and the invariance of $\|\cdot\|'$ imply that

$$|||F \circ \varphi||| \leq \liminf_n \|F_n \circ g_n\|' = \liminf_n \|F_n\|'.$$

Since this is true for any sequence $\{F_n\}$ of uniformly bounded smooth functions which converges in measure to F, we conclude that $|||F \circ \varphi||| \leq |||F|||$. Moreover, by applying the same argument to $F \circ \varphi$ and φ^{-1} we obtain that $|||F||| \leq |||F \circ \varphi|||$, and the proof is complete.

2.4 Proof of Theorem 2.1.2

Let ρ be an intrinsic bi-invariant pseudo-distance function on $\mathrm{Ham}(M, \omega)$ induced by some invariant norm on $\mathcal{A}$. In order to determine whether ρ is degenerate or not we will use a criterion by Eliashberg and Polterovich [13]. This criterion is based on the following notion of "displacement energy" introduced by Hofer [22].

Definition 2.4.1. *For every open subset $A \subset M$ define its displacement energy with respect to the pseudo-distance ρ as*

$$e(A) = \inf \left\{ \rho(\mathbb{1}, \psi) \mid \psi \in \mathrm{Ham}(M, \omega),\ \psi(A) \cap A = \emptyset \right\},$$

and set $e(A) = \infty$ if the above set is empty.

Theorem 2.4.2 (Eliashberg-Polterovich). *If ρ is a genuine metric on Ham(M, ω) then the displacement energy of every non-empty open set is strictly positive.*

This theorem allows to reduce the proof of Theorem 2.1.2 to showing that the displacement energy of some small ball vanishes. An argument borrowed from [13], to be presented immediately below, further reduces the problem to

Claim 2.4.3. *For every $C' > 0$ and every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, C')$ such that for every function F with $\|F\|_\infty \leq C'$ and $\mathrm{Vol}\big(\mathrm{Support}(F)\big) < \delta$ we have $\|F\| < \varepsilon$.*

Indeed, choose an embedded open ball $B \subset M$ such that its boundary ∂B is an embedded sphere, and such that there exists some Hamiltonian isotopy $\{g_t\}$, $t \in [0, 1]$, generated by a Hamiltonian function $G(t,x)$ with $g_1(B) \cap B = \emptyset$. Denote by Σ_t the sphere $g_t(\partial B)$. Consider the function $K(t,x)$ obtained from G by smoothly cutting-off outside a neighborhood U_t of Σ_t. Note that the time-one-map of $K(t,x)$ also displaces B, i.e. $k_1(B) \cap B = \emptyset$. This is true since for every $t \in [0,1]$ we have $k_t(\partial B) = g_t(\partial B)$. Using Claim 2.4.3, we note that by decreasing the sizes of the neighborhoods U_t we can make $\|K_t\|$ arbitrarily small. Hence the displacement energy of the ball B vanishes.

We are thus left with proving Claim 2.4.3. As explained in Section 2.2, instead of proving it for $\|\cdot\|$, we shall prove it for the extension $|||\cdot|||$ announced in Theorem 2.1.3.

Proof of Claim 2.4.3

Let $\mathbb{1}_V$ stand for the characteristic function of the set V. We first prove that

$$|||\mathbb{1}_V||| \to 0 \text{ as } \mathrm{Vol}(V) \to 0. \tag{2.5}$$

Since $\|\cdot\|$ is not equivalent to $\|\cdot\|_\infty$, and since $|||\cdot|||$ is an extension of $\|\cdot\|$, for every $\varepsilon > 0$ there exists some function $F \in \mathcal{A}$ with $\|F\| = |||F||| \le \varepsilon$, while $\|F\|_\infty = 1$. Assume that the maximum of F is obtained at some point $x_0 \in M$ and set U to be a small-radius open set around x_0. Continuity allows us to choose U in such a way that $|F(x)| > 1 - \varepsilon$ for every $x \in U$. Using the triangle inequality we obtain:

$$|||\langle F\rangle_U \cdot \mathbb{1}_U||| \le |||\langle F\rangle_U \cdot \mathbb{1}_U + \langle F\rangle_{U^c} \cdot \mathbb{1}_{U^c}||| + |||\langle F\rangle_{U^c} \cdot \mathbb{1}_{U^c}|||,$$

where $U^c = M \setminus U$. The left summand is estimated via Lemma 2.2.5. To estimate the right summand, recall that $\mathrm{Vol}(U)\langle F\rangle_U + \mathrm{Vol}(U^c)\langle F\rangle_{U^c} = \langle F\rangle_M = 0$. Together with Lemma 2.2.5 we therefore get:

$$|||\langle F\rangle_U \cdot \mathbb{1}_U||| \le |||F||| + |||\frac{\langle F\rangle_U \cdot \mathrm{Vol}(U)}{\mathrm{Vol}(U^c)} \cdot \mathbb{1}_{U^c}|||.$$

Now, since $|||\cdot||| \le C\|\cdot\|_\infty$, and since $\|\frac{\langle F\rangle_U \cdot \mathrm{Vol}(U)}{\mathrm{Vol}(U^c)} \cdot \mathbb{1}_{U^c}\|_\infty$ goes to zero with $\mathrm{Vol}(U)$, for U with small enough measure we get

$$|||\langle F\rangle_U \cdot \mathbb{1}_U||| \le |||F||| + \varepsilon \le 2\varepsilon.$$

Due to the fact that $|\langle F\rangle_U| > 1 - \varepsilon$, taking $\varepsilon < 1/2$ we get $|||\mathbb{1}_U||| < 4\varepsilon$. Since $|||\cdot|||$ is invariant under measure preserving bijections, this applies to every set V with the same measure as U .

Now we can complete the proof of Claim 2.4.3. Let $F \in C^\infty(M)$ be supported in some compact set $U \subset M$ with measure ε. Consider a finite partition of U into measurable sets $\{S_i\}_{i=1}^N$ with radius so small that uniform continuity affirms $\max(F|_{S_i}) - \min(F|_{S_i}) \leq \varepsilon$ for every $1 \leq i \leq N$. We have

$$|||F||| = |||\sum_{i=1}^N F \cdot \mathbb{1}_{S_i}||| \leq |||\sum_{i=1}^N \big(F - F(\eta_i)\big) \cdot \mathbb{1}_{S_i}||| + |||\sum_{i=1}^N F(\eta_i) \cdot \mathbb{1}_{S_i}|||,$$

where η_i is an arbitrary point in S_i. Without loss of generality we assume that $F(\eta_1) \leq F(\eta_2) \leq \ldots \leq F(\eta_N)$. Using the fact that $|||\cdot||| \leq C\|\cdot\|_\infty$ and the choice of the S_i's we get

$$|||F||| \leq C\|\sum_{i=1}^N \big(F - F(\eta_i)\big) \cdot \mathbb{1}_{S_i}\|_\infty + |||\sum_{i=1}^N F(\eta_i) \cdot \mathbb{1}_{S_i}||| \leq C\varepsilon + |||\sum_{i=1}^N F(\eta_i) \cdot \mathbb{1}_{S_i}|||$$

Next, in order to bound the last term on the right, we use Abel's summation trick

$$|||\sum_{i=1}^N F(\eta_i) \cdot \mathbb{1}_{S_i}||| = |||\sum_{i=1}^N \big(F(\eta_i) - F(\eta_{i-1})\big) \cdot \mathbb{1}_{\bigcup_{k=i}^N S_k}|||,$$

where $F(\eta_0)$ is defined to be zero. Substituting this in the above inequality we conclude

$$|||F||| \leq C\varepsilon + \Big(\sum_{i=1}^N F(\eta_i) - F(\eta_{i-1})\Big) \cdot \max_i |||\mathbb{1}_{\bigcup_{k=i}^N S_k}||| \leq C\varepsilon + 2\|F\|_\infty \cdot \max_i |||\mathbb{1}_{\bigcup_{k=i}^N S_k}|||.$$

Applying this estimate to a sequence of functions as in the statement of the claim, recalling that $\varepsilon = \mathrm{Vol}(\bigcup_{k=1}^N S_k)$ is the volume of the support, and relying on (2.5), the proof of the claim is complete.

2.5 Technical Lemmas

Here we prove Lemma 2.2.4, Lemma 2.2.5 and Lemma 2.3.3. Recall that M is a closed connected symplectic manifold and d is some Riemannian metric on M.

Proof of Lemma 2.2.4: Fix $\varepsilon > 0$. Let $\{A_i\}_{i=1}^N$ be a family of compact measurable disjoint subsets of M such that the following two conditions hold:

1. The diameter of each set A_i is at most ε,

2. $\mathrm{Vol}(\cup_{i=1}^N A_i) \geq \mathrm{Vol}(M) - \varepsilon$.

Next, let $\widetilde{B}_i = \varphi^{-1}(A_i)$, and let B_i be compact subsets of $\widetilde{B}_i$, such that $\sum_{i=1}^N \mathrm{Vol}(\widetilde{B}_i \backslash B_i) < \varepsilon$. From Katok's Basic Lemma [25] we get a Hamiltonian diffeomorphism g satisfying

$$\sum_{i=1}^N \mathrm{Vol}(g(B_i) \setminus A_i) \leq \varepsilon.$$

We claim that g is a good approximation in measure of φ. To see this, let $C_i = \{x \in B_i \,|\, g(x) \in A_i\}$ and denote by $C = \cup_{i=1}^N C_i$. Note that

$$\mathrm{Vol}(C) = \sum_{i=1}^N \mathrm{Vol}(C_i) \geq \sum_{i=1}^N \mathrm{Vol}(B_i) - \varepsilon \geq \sum_{i=1}^N \mathrm{Vol}(A_i) - 2\varepsilon \geq \mathrm{vol}(M) - 3\varepsilon.$$

Moreover, for every $x \in C$ the points $g(x)$ and $\varphi(x)$ belong to the same A_i. Since the diameter of the sets A_i is at most ε, we conclude that g is a 3ε-approximation in measure of φ.

Proof of Lemma 2.2.5: In order to keep notation simple, let's assume we have only two parts, S_1 and S_2. Fix $\varepsilon > 0$. Partition S_1 and S_2 into disjoint measurable sets $\{U_j^1\}_{j=1}^{J_1}$ and $\{U_j^2\}_{j=1}^{J_2}$, respectively, such that the following two conditions hold:

1. All U_j^1's have the same measure, and all U_j^2's have the same measure,
2. Inside all U_j^i's the function F does not oscillate by more than ε.

Choose $\varphi_{j,k}^i$ to be measure preserving bijections, not necessarily continuous, which map U_j^i onto U_k^i. For every permutation π of the set $\{1, \dots, J_i\}$, define $\varphi_\pi^i(x) = \varphi_{j,\pi(j)}^i(x)$ if $x \in U_j^i$, and $\varphi_\pi^i(x) = x$ if $x \notin S_i$. Finally, define the measure preserving bijections $\varphi_{\pi,\sigma} = \varphi_\pi^1 \circ \varphi_\sigma^2$. Since $|||\cdot|||$ is invariant under measure preserving bijections, the triangle inequality yields

$$|||\frac{1}{J_1!J_2!} \sum_{\pi,\sigma} F \circ \varphi_{\pi,\sigma}||| \leq |||F|||.$$

Now, our choice of U_j^i is such that for every $x \in U_j^i$ we have $|\langle F \rangle_{U_j^i} - F(x)| \leq \varepsilon$. Together with the equal measures of the U_j^i's, this means that for $x \in U_k^i$ we get

$$|\frac{1}{J_i} \sum_{j=1}^{J_i} F \circ \varphi_{k,j}^i(x) - \langle F \rangle_{S_i}| \leq \varepsilon.$$

We thus infer the inequality

$$\|\frac{1}{J_1!J_2!}\sum_{\pi,\sigma}(F\circ\varphi_{\pi,\sigma}) - (\langle F\rangle_{S_1}\mathbb{1}_{S_1} + \langle F\rangle_{S_2}\mathbb{1}_{S_2})\|_\infty \le \varepsilon.$$

Since $|||\cdot||| \le C\|\cdot\|_\infty$, taking ε to zero concludes the proof.

Remark 2.5.1. If F is not continuous, then the choice of equal measure U^i_j's where F has small oscillations may not be possible. The argument, however, can be easily adapted to include the non-continuous case as well. Lemma 2.2.5 is also the key behind the proof that $|||\cdot|||$ is indeed a norm, namely that it vanishes only on the zero function. Indeed, if $|||F|||$ were zero for a non-zero F, a piecewise averaging of F would generate a non-zero step function with vanishing norm. Then, further piecewise averagings may be used to produce a sequence of zero norm step functions, which converge uniformly to a non-zero smooth function. This would mean that the original norm, which coincides with $|||\cdot|||$ on smooth functions, was already only a seminorm. We omit the details, because our main results still hold even if $|||\cdot|||$ were only a seminorm.

Proof of Lemma 2.3.3: According to Whitney's embedding theorem there exists a smooth embedding $\Psi : M \to \mathbb{R}^N$ for some (large enough) N. Next, for $\alpha,\beta \in \mathbb{R}$ set $\alpha\mathbb{K}+\beta = \{x \in \mathbb{R}^N \,|\, \exists\ i \text{ such that } x_i - \beta \in \alpha\mathbb{Z}\}$. Roughly speaking, $\alpha\mathbb{K}+\beta$ stands for the homothetic image of the "standard" grid in $\mathbb{R}^N$ translated in the direction of the vector $(1,\ldots,1)$. Fix $J \in \mathbb{N}$. For every $1 \le j \le J$, let G_j be the $\frac{\alpha}{4J}$-extension of the grid $\alpha\mathbb{K}+\frac{\alpha j}{J}$. Set $\{V^j_i\}_{i=1}^{L_j}$ to be the connected components of $\Psi(M)\cap(G_j)^c$. Note that a single "cell" of $(G_j)^c$ may be split into several connected components when intersected with $\Psi(M)$. However, by choosing the embedding coordinate-functions Ψ_i to be Morse functions, we can guarantee that the number of connected components is indeed finite. It may well be the case that for a given j some V^j_i's are zero-distance apart, but since our coordinates are Morse functions, arbitrarily small translations of G_j suffice to guarantee positive distance separation between all V^j_i's.

Now set $U^j_i = \Psi^{-1}(V^j_i)$. The first property in the statement follows from the positive distance between the V^j_i's. Compactness guarantees that a small enough α implies the second property. The last property follows from the fact that, regardless of J, the intersection of any $N+1$ different extended grids G_j is empty. Taking J such that $\frac{N+1}{J} < \varepsilon$ we are done.

2.6 Further Information Concerning our Norms

Let $\|\cdot\|$ be a $\mathrm{Ham}(M,\omega)$-invariant norm on $\mathcal{A}$ such that $\|\cdot\| \le C\|\cdot\|_\infty$ for some constant C. Let $|||\cdot|||$ be the extension of $\|\cdot\|$ to $L_\infty(M)$ constructed in the proof of Theorem 2.1.3. The main objective of this section is to place the normed spaces $(\mathcal{A}, \|\cdot\|)$ and $(L_\infty(M), |||\cdot|||)$ in the context of Banach (i.e. topologically complete) spaces of functions, so that existing knowledge from this field be made applicable to our context. For this purpose, we need to be able to view our spaces as subspaces of Banach spaces of functions. It can be easily seen that if the original norm $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$, so is $|||\cdot|||$, and we are in a Banach space setting. In the non-equivalent case we claim the following. Here $\|\cdot\|'$ is the extensions of $\|\cdot\|$ to $C^\infty(M)$ constructed in the proof of Theorem 2.1.3.

Proposition 2.6.1. *Let $\|\cdot\|$ be a $Ham(M,\omega)$-invariant norm on $\mathcal{A}$ which is dominated from above by $\|\cdot\|_\infty$, but not equivalent to it. The space $(L_\infty(M), |||\cdot|||)$ then coincides with a dense subspace of the completion of $(C^\infty(M), \|\cdot\|')$. Moreover, this completion can be viewed as a dense subspace of the space $L_1(M)$ of integrable measurable functions on M, equipped with a norm which is invariant under measure preserving bijections.*

Sketch of the proof: To establish the relation between $|||\cdot|||$ and the completion of $\|\cdot\|'$, we need to show that if $\{F_n\}$ is a sequence of uniformly bounded smooth functions tending in measure to F, then $\{F_n\}$ is a Cauchy sequence in $\|\cdot\|'$ (which is equivalent to showing that it is a Cauchy sequence in $|||\cdot|||$). Indeed, let F_n and F_m both ε-approximate F in measure for some arbitrary small ε. We can then write $F_n - F_m = G_{n,m} + H_{n,m}$, where $G_{n,m}$ and $H_{n,m}$ are smooth and uniformly bounded, $\|G_{n,m}\|_\infty \le 2\varepsilon$, and the measure of the support of $H_{n,m}$ is at most 2ε. Claim 2.4.3 now proves that $|||F_n - F_m||| \to 0$ as $n, m \to \infty$.

We turn now to the second part of the proposition. First we claim that there exists some constant C such that $|||F||| \ge C\|F\|_{L_1}$ for any essentially bounded measurable function F. Indeed, set M_F to be the median of F, namely the unique number for which both $\mathrm{Vol}(\{x \in M \,|\, F(x) \ge M_F\})$ and $\mathrm{Vol}(\{x \in M \,|\, F(x) \le M_F\})$ are at least half. Without loss of generality we may assume that $M_F \ge 0$. Let $\{x \in M \,|\, F(x) > M_F\} \subseteq A \subseteq \{x \in M \,|\, F(x) \ge M_F\}$, such that $\mathrm{Vol}(A) = \frac{1}{2}$. Finally, let $B = \{x \in M \,|\, F(x) \ge 0\}$. We will argue under the assumption that F is zero-mean; the extension to general F involves adding just one dimension to our space of functions, and therefore follows immediately.

By Lemma 2.2.5 we obtain that

$$|||F||| \geq |||\langle F\rangle_A \mathbb{1}_A + \langle F\rangle_{A^c}\mathbb{1}_{A^c}||| = \langle F\rangle_A |||\mathbb{1}_A - \mathbb{1}_{A^c}|||.$$

We clearly have

$$\langle F\rangle_A = 2\int_A F(x)\,\omega^n \geq 2\int_{B-A} F(x)\,\omega^n,$$

and therefore

$$\langle F\rangle_A \geq \int_B F(x)\,\omega^n = \frac{1}{2}||F||_{L_1}.$$

Together with the above estimate for $|||F|||$, this yields $|||F||| \geq \frac{1}{2}||F||_{L_1} \cdot |||\mathbb{1}_A - \mathbb{1}_{A^c}|||$. The invariance property of the norm $|||\cdot|||$ implies that the value of $|||\mathbb{1}_A - \mathbb{1}_{A^c}|||$ depends only on the fixed $\mathrm{Vol}(A) = \frac{1}{2}$. Thus, the inequality is proved.

Now, every Cauchy sequence of smooth functions in $\|\cdot\|'$ (not necessarily uniformly bounded) is also a Cauchy sequence in L_1, and is therefore convergent in measure. In order to regard this limit in measure as an element of the completion of $\|\cdot\|'$, we need to show that if two Cauchy sequences in $\|\cdot\|'$, $\{F_n\}$ and $\{G_n\}$, converge in measure to the same F, then both sequences have the same limit in the completion, namely $\|F_n - G_n\|' \to 0$. For this purpose set $H_n = F_n - G_n$. By definition, H_n is a Cauchy sequence in $\|\cdot\|'$ converging in L_1 and in measure to the zero function. We need to prove that $\|H_n\|' \to 0$. Again, we will carry out the proof in $|||\cdot|||$. By taking small uniform perturbations we may also assume that $\mathrm{Vol}(A_n) \to 0$, where $A_n = \mathrm{Support}(H_n)$. Next, by applying a slight variant of Lemma 2.2.5 we get

$$|||H_n - H_m||| \geq |||(H_n - H_m)\mathbb{1}_{A_m} + \langle H_n - H_m\rangle_{A_m^c}\mathbb{1}_{A_m^c}|||.$$

Since $\langle H_n - H_m\rangle_{A_m^c} \leq \|H_n - H_m\|_{L_1} \cdot \mathrm{Vol}(A_m^c)$, this term goes to zero. We therefore conclude that $|||(H_n - H_m)\mathbb{1}_{A_m}||| = |||H_m - H_n \cdot \mathbb{1}_{A_m}|||$ converges to zero as n, m increase. Due to Claim 2.4.3, for every fixed n the term $|||H_n \cdot \mathbb{1}_{A_m}|||$ tends to zero with m. We therefore conclude, as announced, that $|||H_m||| \to 0$.

Since we already know that $|||\cdot|||$ is invariant under measure preserving bijections, the proof that the completion is also invariant is straightforward. Note that since we assume $\|\cdot\|$ is dominated by $\|\cdot\|_\infty$, but not equivalent to it, Banach's Open Map Theorem implies that the completion of $\|\cdot\|$ must in fact exceed the space of essentially bounded measurable functions. The proof is now complete.

The literature contains much information concerning a special subclass of the class of Banach norms on spaces of functions, which are invariant under measure preserving bijections. This is the subclass of the so called *Rearrangement Invariant Function Spaces*. The main (but not only!) requirement is that the norm be monotone with respect to the natural partial order on non-negative functions. Since an explicit formulation will drag us into a long list of definitions which are not relevant for this paper, we will make do here with a reference. The book [9] introduces Rearrangement Invariant Function Spaces in Chapter 2, Definition 1.4 (which relies on Definitions 1.1 and 1.3 from Chapter 1). The main classification results are announced in Chapter 2, Theorem 5.15 and in Chapter 3, Theorem 2.12. Another thorough analysis from a somewhat different point of view is available in [31] (Definitions 1.b.17 and 2.a.1, and the results of the second section).

We cannot rule out the possibility that all normed spaces $(\mathcal{A}, \|\cdot\|)$, which are invariant under Hamiltonian diffeomorphisms, can be viewed as subspaces of Rearrangement Invariant Function Spaces. The following example, while not relating directly to the issue under discussion, serves to indicate the kind of pathologies one might expect from norms outside this class. Take the space $\mathcal{A} \bigoplus \mathcal{B}$, where $\mathcal{B}$ is the space of functions on M which attain only finitely many values. It is straightforward to see that the sum is indeed an algebraically direct sum. For an element $a+b \in \mathcal{A} \bigoplus \mathcal{B}$ consider the functional $\|a+b\| = \|a\|_1 + \|b\|_\infty$. It is easy to check that $\|\cdot\|$ is a norm invariant under measure preserving diffeomorphisms, but not under measure preserving bijections (we restrict our attention, of course, to measure preserving bijections which keep the 'rearranged' function inside our space). It is also not hard to see that this norm is not bounded by $\|\cdot\|_\infty$.

Chapter 3

A Comparison of Hofer's Metrics

3.1 Introduction and Main Results

In this chapter we compare Hofer's geometries on two remarkable spaces associated with a closed symplectic manifold (M, ω). The first space $\mathrm{Ham}(M, \omega)$ is the group of Hamiltonian diffeomorphisms. The second consists of all Lagrangian submanifolds of $(M \times M, -\omega \oplus \omega)$ which are exact Lagrangian isotopic to the diagonal $\triangle \subset M \times M$. Let us denote this second space by $\mathcal{L}$. The canonical embedding

$$j : \mathrm{Ham}(M, \omega) \to \mathcal{L}, \quad f \mapsto graph(f)$$

preserves Hofer's length of smooth paths. Thus, it naturally follows to ask whether j is an isometric embedding with respect to Hofer's distance. Here, we provide a negative answer to this question for the case of a closed symplectic manifold with $\pi_2(M) = 0$. In fact, our main result shows that the image of $\mathrm{Ham}(M, \omega)$ inside $\mathcal{L}$ is "strongly distorted" (see Theorem 3.1.1 below).

Let us proceed with precise formulations. We recall the following definition from the Introduction. Given a path $\alpha = \{f_t\}$, $t \in [0, 1]$ of Hamiltonian diffeomorphisms of (M, ω), define its Hofer's length (see [22]) as

$$\mathrm{length}(\alpha) = \int_0^1 \left\{ \max_{x \in M} F(x,t) - \min_{x \in M} F(x,t) \right\} dt$$

where $F(x, t)$ is the Hamiltonian function generating $\{f_t\}$. For two Hamiltonian diffeomorphisms ϕ and ψ, define the Hofer distance $d(\phi, \psi) = \inf \mathrm{length}(\alpha)$ where the infimum is taken over all smooth paths α connecting ϕ and ψ. For further discussion see e.g. [28], [34], and [46].

Hofer's metric can be defined in a more general context of Lagrangian submanifolds (see [12]). Let (P, σ) be a closed symplectic manifold, and let $\triangle \subset P$ be a closed Lagrangian submanifold. Consider a smooth family $\alpha = \{L_t\}$, $t \in [0,1]$ of Lagrangian submanifolds, such that each L_t is diffeomorphic to $\triangle$. Recall that α is called an *exact path* connecting L_0 and L_1, if there exists a smooth map $\Psi : \triangle \times [0,1] \to P$ such that for every t, $\Psi(\triangle \times \{t\}) = L_t$, and in addition $\Psi^*\sigma = dH_t \wedge dt$ for some smooth function $H : \triangle \times [0,1] \to \mathbb{R}$. The Hofer length of an exact path is defined by

$$\operatorname{length}(\alpha) = \int_0^1 \left\{ \max_{x \in \triangle} H(x,t) - \min_{x \in \triangle} H(x,t) \right\} dt.$$

It is easy to check that the above notion of length is well-defined. Denote by $\mathcal{L}(P, \triangle)$ the space of all Lagrangian submanifolds of P which can be connected to $\triangle$ by an exact path. For two Lagrangian submanifolds L_1 and L_2 in $\mathcal{L}(P, \triangle)$, define the Hofer distance d_{Lag} on $\mathcal{L}(P, \triangle)$ as follows: $d_{Lag}(L_1, L_2) = \inf \operatorname{length}(\alpha)$, where the infimum is taken over all exact paths on $\mathcal{L}(P, \triangle)$ that connect L_1 and L_2.

In what follows we choose $P = M \times M$, $\sigma = -\omega \oplus \omega$ and take $\triangle$ to be the diagonal of $M \times M$. We abbreviate $\mathcal{L} = \mathcal{L}(P, \triangle)$ as in the beginning of the paper. Based on a result by Banyaga [4], it can be shown that every smooth path on $\mathcal{L}(P, \triangle)$ is necessarily exact. Our main result is the following:

Theorem 3.1.1. *Let (M, ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Then there exist a family $\{\varphi_t\}$, $t \in [0, \infty)$ in $Ham(M, \omega)$ and a constant c such that:*

1. $d(\mathbb{1}, \varphi_t) \to \infty$ *as* $t \to \infty$.
2. $d_{Lag}\left(\operatorname{graph}(\mathbb{1}), \operatorname{graph}(\varphi_t)\right) = c$.

In fact, we construct the above family $\{\varphi_t\}$ explicitly:

Example 3.1.2. Consider an open set $B \subset M$. Suppose that there exists a Hamiltonian diffeomorphism h such that $h(B) \cap \text{Closure } (B) = \emptyset$. By perturbing h slightly, we may assume that all the fixed points of h are non-degenerate. Let $F(x,t)$, where $x \in M$,

$t \in [0,1]$ be a Hamiltonian function such that $F(x,t) = c_0 < 0$ for all $x \in M \setminus B,\ t \in [0,1]$. Assume that $F(t,x)$ is normalized such that for every t, $\int_M F(t,\cdot)\omega^n = 0$. We define the family $\{\varphi_t\},\ t \in [0,\infty)$ by $\varphi_t = hf_t$, where $\{f_t\}$ is the Hamiltonian flow generated by $F(t,x)$. As we'll see below, the family $\{\varphi_t\}$ satisfies the requirements of Theorem 3.1.1.

Theorem 3.1.1 has some corollaries:

1. The embedding of $\mathrm{Ham}(M,\omega)$ in $\mathcal{L}$ is not isometric, rather, the image of $\mathrm{Ham}(M,\omega)$ in $\mathcal{L}$ is highly distorted. The minimal path between two graphs of Hamiltonian diffeomorphisms in $\mathcal{L}$, might pass through exact Lagrangian submanifolds which are not the graphs of any Hamiltonian diffeomorphisms. Compare with the situation described in [37], where it was proven that in the case of a compact manifold, the space of Hamiltonian deformations of the zero section in the cotangent bundle is locally flat in the Hofer metric.

2. The group of Hamiltonian diffeomorphisms of a closed symplectic manifold with $\pi_2(M) = 0$ has an infinite diameter with respect to Hofer's metric.

3. Hofer's metric d on $\mathrm{Ham}(M,\omega)$ *does not* coincide with the Viterbo-type metric on $\mathrm{Ham}(M,\omega)$ defined by Schwarz in [55].

As a by-product of our method we obtain the following result (see Section 3.3 below):

Theorem 3.1.3. *Let (M,ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Then there exists an element φ in $(\mathrm{Ham}(M,\omega), d)$ which cannot be joined to the identity by a minimal geodesic.*

The first example of this kind of result was established by Lalonde and McDuff [29] for the case of S^2.

3.2 Proof of The Main Theorem

In this section we prove Theorem 3.1.1. Throughout this section let (M,ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Let $\{\varphi_t\},\ t \in [0,\infty)$ the family of Hamiltonian diffeomorphisms defined in Example 3.1.2. We begin with the following lemma which states that Hamiltonian diffeomorphisms act as isometries on the space $(\mathcal{L}, d_{Lag})$. The proof of the lemma follows immediately from the definitions.

Lemma 3.2.1. *Let $\Gamma : \triangle \times [0,1] \to M \times M$ be an exact Lagrangian isotopy in $\mathcal{L}$ and let $\Phi : M \times M \to M \times M$ be a Hamiltonian diffeomorphism. Then*

$$\text{length}\{\Gamma\} = \text{length}\{\Phi \circ \Gamma\}.$$

In particular, $d_{Lag}(L_1, L_2) = d_{Lag}(\Phi(L_1), \Phi(L_2))$ for every $L_1, L_2 \in \mathcal{L}$.

Next, consider the following exact isotopy of the Lagrangian embeddings $\Psi : \triangle \times [0,\infty) \to M \times M$, $\Psi(x,t) = (x, \varphi_t(x))$. We denote by $L_t = \Psi(\triangle \times \{t\})$ the graph of $\varphi_t = hf_t$ in $M \times M$. The following proposition will be proved in Section 3.5 below.

Proposition 3.2.2. *For every $t \in [0,\infty)$ there exists a Hamiltonian isotopy $\{\Phi_s\}$, $s \in [0,t]$, of $M \times M$, such that $\Phi_s(L_0) = L_s$ and such that for every s, $\Phi_s(\triangle) = \triangle$.*

Hence, it follows from Proposition 3.2.2 and Lemma 3.2.1, that the family $\{\varphi_t\}$, $t \in [0,\infty)$ satisfies the second conclusion of Theorem 3.1.1 with constant $c = d_{Lag}(\triangle, L_0)$.

Let us now verify the first statement of Theorem 3.1.1. For this purpose we will use a theorem by Schwarz [55] stated below. First, recall the definitions of the action functional and the action spectrum. Consider a closed symplectic manifold (M, ω) with $\pi_2(M) = 0$. Let $\{f_t\}$ be a Hamiltonian path generated by a Hamiltonian function $F : [0,1] \times M \to \mathbb{R}$. We denote by $\text{Fix}^\circ(f_1)$ the set of fixed points, x, of the time-1-map f_1 whose orbits $\gamma = \{f_t(x)\}$, $t \in [0,1]$ are contractible. For $x \in \text{Fix}^\circ(f_1)$, take any 2-disc $\Sigma \subset M$ with $\partial\Sigma = \gamma$, and define the symplectic action functional by

$$\mathcal{A}(F, x) = \int_\Sigma \omega - \int_0^1 F(t, f_t(x))dt.$$

The assumption $\pi_2(M) = 0$ ensures that the integral $\int_\Sigma \omega$ does not depend on the choice of Σ.

Remark 3.2.3. In the case of a closed symplectic manifold with $\pi_2(M) = 0$, a result by Schwarz [55], implies that for a Hamiltonian path $\{f_t\}$ with $f_1 \neq \mathbb{1}$ there exist two fixed points x, $y \in \text{Fix}^\circ(f_1)$ with $\mathcal{A}(F, x) \neq \mathcal{A}(F, y)$. Moreover, the action functional does not depend on the choice of the Hamiltonian path generating f_1. Therefore, we can speak about the action of a fixed point of a Hamiltonian diffeomorphism, regardless of the Hamiltonian function used to define it.

Definition 3.2.4. *For each f in* $\mathrm{Ham}(M,\omega)$ *we define the action spectrum*

$$\Sigma_f = \{\mathcal{A}(f,x) \mid x \in \mathrm{Fix}^\circ(f)\} \subset \mathbb{R}.$$

The action spectrum Σ_f is a compact subset of $\mathbb{R}$ (see e.g. [55], [24]).

Theorem 3.2.5. [55] . *Let (M,ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Then, for every f in* $\mathrm{Ham}(M,\omega)$

$$d(\mathbb{1}, f) \geq \min \Sigma_f.$$

Next, consider the family $\{\varphi_t\} = \{hf_t\}$, $t \in [0,\infty)$. Note that $\mathrm{Fix}^\circ(h) = \mathrm{Fix}^\circ(\varphi_t)$ for every t. The following proposition shows that the action spectrum of φ_t is a linear translation of the action spectrum of h. Its proof is carried out in Section 3.4.

Proposition 3.2.6. *For every $t \in [0,\infty)$, and for every fixed point $z \in \mathrm{Fix}^\circ(\varphi_t) = \mathrm{Fix}^\circ(h)$,*

$$\mathcal{A}(\varphi_t, z) = \mathcal{A}(h,z) - tc_0$$

where c_0 is the negative (constant) value that F attains on $M \setminus B$ (see Example 3.1.2).

We are now in a position to complete the proof of Theorem 3.1.1. Indeed, the action spectrum is a compact subset of $\mathbb{R}$, hence its minimum is finite. By proposition 3.2.6 the minimum of Σ_{φ_t} tends to infinity as $t \to \infty$. Thus,

$$d(\mathbb{1}, \varphi_t) \to \infty \ as \ t \to \infty$$

as follows from Theorem 3.2.5. This completes the proof of Theorem 3.1.1. □

3.3 Geodesics in Ham(M,ω) and Proof of Theorem 3.1.3.

In this section we describe our result about geodesics in the group of Hamiltonian diffeomorphisms endowed with the Hofer metric d. We refer the reader to [7], [28], [29], and [48] for further details on this subject.

Let $\gamma = \{\phi_t\}$, $t \in [0,1]$ be a smooth regular path in $\mathrm{Ham}(M,\omega)$, i.e. $\frac{d}{dt}\phi_t \neq 0$ for every $t \in [0,1]$. The path γ is called a *minimal geodesic* if it minimizes the distance between its end-points:

$$\mathrm{length}(\gamma) = d(\phi_0, \phi_1).$$

The graph of a Hamiltonian path $\gamma = \{\phi_t\}$ is the family of embedded images of M in $M \times M$ defined by the map $\Gamma : M \times [0,1] \to M \times M,\ (x,t) \mapsto (x, \phi_t(x))$. Next, consider the family $\{\varphi_t\},\ t \in [0,\infty)$ that was constructed in Example 3.1.2. We will show that there exists no minimal geodesic joining the identity and φ_{t_0}, for some t_0.

Proof of Theorem 3.1.3: Assume (by contradiction) that for every t, there exists a minimal geodesic in $\mathrm{Ham}(M,\omega)$ joining the identity with φ_t. Fix $t_0 \in [0,\infty)$. There exists a Hamiltonian path $\alpha = \{g_s\},\ s \in [0,1]$ in $\mathrm{Ham}(M,\omega)$ such that

$$d_{t_0} := d(\mathbb{1}, \varphi_{t_0}) = \mathrm{length}(\alpha).$$

Expressed in Lagrangian submanifolds terms, $\Psi = \{\mathrm{graph}(g_s)\},\ s \in [0,1]$ is an exact path in $M \times M$ joining the diagonal with $\mathrm{graph}(\varphi_{t_0})$. By Proposition 3.2.2, there exists a Hamiltonian isotopy Φ such that for every t, $\Phi_t(\mathrm{graph}(\varphi_{t_0})) = \mathrm{graph}(\varphi_t)$, and $\Phi_t(\triangle) = \triangle$. We choose t_1 to be sufficiently close to t_0 so as to ensure that $\{\Phi_{t_1}(\mathrm{graph}(g_s))\},\ s \in [0,1]$ is the graph of some Hamiltonian path γ in $\mathrm{Ham}(M,\omega)$. Indeed, this can be done since it follows from the proof of Proposition 3.2.2, that the Hamiltonian diffeomorphism Φ is C^1-close to the identity in a small neighborhood of $\mathrm{graph}(\varphi_{t_0})$. Moreover, using a compactness argument, we can choose a finite number of points $S = \{s_1 < \ldots < s_n\}$ in $[0,1]$, and repeat the construction of Φ in a small neighborhood of $\mathrm{graph}(g_{s_i})$ for $i = 1,\ldots,n$. Then, by smoothly patching together those Hamiltonian flows, we conclude that for every $s \in [0,1]$, $\Phi_{t_1}(\mathrm{graph}(g_s))$ is the graph of some Hamiltonian diffeomorphism. Next, we claim the following

$$d_{t_1} \le \mathrm{length}(\gamma) = \mathrm{length}\{\mathrm{graph}(\gamma)\} = \mathrm{length}\{\mathrm{graph}(\alpha)\} = \mathrm{length}\{\alpha\} = d_{t_0}.$$

Indeed, a straightforward computation yields that the embedding $f \mapsto graph(f)$ preserves Hofer's length, and from Lemma 3.2.1, $\mathrm{length}\{\mathrm{graph}(\alpha)\} = \mathrm{length}\{\mathrm{graph}(\gamma)\}$. We have shown that for every t_0 there exists $\varepsilon > 0$ such that if $|t - t_0| \le \varepsilon$ then $d_t \le d_{t_0}$. Since d_t is a continuous function, we conclude that d_t is a constant function. On the other hand, by Theorem 3.1.1, $d_t = d(\mathbb{1}, \varphi_t) \to \infty$ as $t \to \infty$. Hence there is a contradiction.

3.4 Proof of Proposition 3.2.6.

We investigate the expression $\mathcal{A}(\varphi_t, z)$ for some fixed t. Since the action functional does not depend on the choice of the Hamiltonian path generating the time-1-map (see Remark 3.2.3), we consider the following path generating φ_t.

$$\gamma(s) = \begin{cases} f_{2st} & , \quad s \in [0,\frac{1}{2}] \\ h_{2s-1}f_t & , \quad s \in (\ \frac{1}{2},1]. \end{cases}$$

Note that since $h(B) \cap B = \emptyset$ and f_t is supported in B, then for $z \in \mathrm{F}ix^\circ(\varphi_t) = \mathrm{F}ix^\circ(h)$ the path $\{\gamma_s(z)\}, s \in [0,1]$ coincides with the path $\{h_s(z)\},\ s \in [0,1]$. Denote by α the loop $\{\gamma_s(z)\},\ s \in [0,1]$ and let Σ be any 2-disc with $\partial\Sigma = \alpha$. The details of the calculation of $\mathcal{A}(\varphi_t, z)$ are as follows:

$$\mathcal{A}(\varphi_t, z) = \int_\Sigma \omega - \int_0^1 tF(s,z)ds - \int_0^1 H(s, h_s(z))ds,$$

where F and H are the Hamiltonian functions generating $\{h_t\}$ and $\{f_t\}$ respectively. Recall that by definition, F is equal to a constant c_0 in $M \setminus B$. This implies that

$$\mathcal{A}(\varphi_t, z) = \int_\Sigma \omega - \int_0^1 H(s, h_s(z))ds - tc_0.$$

The right hand side is exactly $\mathcal{A}(h,z) - tc_0$. Hence, the proof is complete.

3.5 Extending the Hamiltonian Isotopy

In this section we prove Proposition 3.2.2. Let us first recall some relevant notations. Let $\{\varphi_t\},\ t \in [0,\infty)$ the family of Hamiltonian diffeomorphisms defined in Example 3.1.2. Consider the following exact isotopy of Lagrangian embeddings $\Psi : \triangle \times [0,\infty) \to M \times M,\ \Psi(x,t) = (x, \varphi_t(x))$. We denote by $L_t = \Psi(\triangle \times \{t\})$ the graph of $\varphi_t = hf_t$ in $M \times M$, and by $\triangle$ the diagonal in $M \times M$. It follows from the construction of the family $\{\varphi_t\}$, that for every t, $\mathrm{Fix}(\varphi_t) = \mathrm{Fix}(h)$. Hence, L_t intersects the diagonal at the same set of points for every t. Moreover, we assumed that all the fixed points of h are non-degenerate, therefore for every t, L_t transversely intersect the diagonal. In order to prove Proposition 3.2.2, we first need the following lemma.

Lemma 3.5.1. *Let $x, y \in Fix^\circ(\varphi_t) = Fix^\circ(h)$, i.e., intersection points of the family $\{L_t\}$ and the diagonal in $M \times M$. Take a smooth curve $\alpha : [0,1] \to M$ with $\alpha(0) = x$ and $\alpha(1) = y$ and let $\Sigma : [0,1] \times [0,1] \to M,\ \Sigma(t,s) = \varphi_t(\alpha(s))$ be a 2-disc with $\partial\Sigma([0,t]\times[0,1]) = \varphi_t\alpha - \varphi_0\alpha = \varphi_t\alpha - h\alpha$. Then the symplectic area of $\Sigma_t = \Sigma([0,t]\times[0,1])$ vanishes for all t.*

Proof: By a direct computation of the symplectic area of Σ_t, we obtain that

$$\int_{\Sigma_t} \omega = \int_{[0,t]\times[0,1]} \Sigma_t^* \omega = -\int_0^t dt \int_0^1 d\widehat{F}_t(\frac{\partial}{\partial s}\varphi_t \alpha(s))\, ds = \int_0^t \widehat{F}_t(\varphi_t(x))dt - \int_0^t \widehat{F}_t(\varphi_t(y))dt,$$

where $\widehat{F}$ is the Hamiltonian function generating the flow $\{\varphi_t\}$. A straightforward computation shows that $\widehat{F}(t,x) = F(t, h^{-1}x)$, where F is the Hamiltonian function generating the flow $\{f_t\}$. Recall that by definition, $F(x,t)$ is equal to a constant c_0 outside the ball B. Moreover, since x, $y \in Fix^\circ(h)$ and $h(B) \cap B = \emptyset$, then $x, y \notin$ B. Therefore, $\widehat{F}_t(\varphi_t(x)) = \widehat{F}_t(\varphi_t(y)) = c_0$ for every t. Thus, we conclude that for every t, the symplectic area of Σ_t vanishes as required.

Proof of Proposition 3.2.2: We shall proceed along the following lines. By the Lagrangian tubular neighborhood theorem (see [58]), there exists a symplectic identification between a small tubular neighborhood U_s of L_s in $M \times M$ and a tubular neighborhood V_s of the zero section in the cotangent bundle T^*L_s. Moreover, it follows from a standard compactness argument that there exists $\delta_s = \delta(s, U_s) > 0$ such that $L_{s'} \subset U_s$ for every s' with $|s' - s| \leq \delta_s$. Next, denote $I_s = (s - \delta_s, s + \delta_s) \cap [0,t]$, and consider an open cover of the interval $[0,t]$ by the family $\{I_s\}$, that is $[0,t] = \bigcup_{s\in[0,t]} I_s$. By compactness we can choose a finite number of points $S = \{s_1 < \ldots < s_n\}$ such that $[0,t] = \bigcup_{i=1}^n I_{s_i}$. Without loss of generality we may assume that $I_{s_j} \cap I_{s_{j+2}} = \emptyset$. Now, for every $s \in$ S, we will construct a Hamiltonian function $\widetilde{H}_s : U_s \to \mathbb{R}$ such that the corresponding Hamiltonian flow sends L_s to $L_{s'}$ for $s' \in I_s$, and leave the diagonal invariant. Next, by smoothly patching together those Hamiltonian flows on the intersections $U_{s_i} \cap U_{s_{i+1}}$, we will achieve the required Hamiltonian isotopy Φ.

We fix $s_0 \in$ S. Let (p,q) be canonical local coordinates on $T^*L_{s_0}$ (where q is the coordinate on L_{s_0} and p is the coordinate on the fiber). Moreover, we fix a Riemannian metric on L_{s_0}, and denote by $\|\cdot\|_{s_0}$ the induced fiber norm on $T^*L_{s_0}$. Consider the aforementioned tubular neighborhood U_{s_0} of L_{s_0} in $M \times M$. For every $x \in L_{s_0} \cap \triangle$ denote by $\sigma_{s_0}(x)$ the component of the intersection of U_{s_0} and $\triangle$ containing the point x. Note that we may choose U_{s_0} small enough such that the sets $\{\sigma_{s_0}(x)\}$, $x \in L_{s_0} \cap \triangle$, are mutually disjoint. In what follows we shall denote the image of $\sigma_{s_0}(x)$ under the above identification between U_{s_0} and V_{s_0}, by $\sigma_{s_0}(x)$ as well.

We first claim that there exists a Hamiltonian symplectomorphism $\widetilde{\varphi} : V_{s_0} \to V_{s_0}$ which for every intersection point $x \in L_{s_0} \cap \triangle$ sends $\sigma_{s_0}(x)$ to the fiber over x and which leaves L_{s_0} invariant. Indeed, since L_{s_0} transversely intersects the diagonal, and since $\sigma_{s_0}(x)$ is a Lagrangian submanifold, $\sigma_{s_0}(x)$ is the graph of a closed 1-form of p-variable i.e, $\sigma_{s_0}(x) = \{(p, \alpha(p))\}$ where $\alpha(p)$ is locally defined near the intersection point x, and $\alpha(0) = 0$. Define a family of local diffeomorphisms by $\varphi_t(p,q) = (p, q - t\alpha(p))$. Since the 1-form $\alpha(p)$ is closed, $\{\varphi_t\}$ is a Hamiltonian flow. Denote by $K(p,q)$ the Hamiltonian function generating $\{\varphi_t\}$. A simple computation shows that $K(p,q) = -\int \alpha(p)dp$. Hence $K(p,q)$ is independent on the q-variable i.e, $K(p,q) = K(p)$. Furthermore, we may assume that $K(0) = 0$. Next, we cut off the Hamiltonian function $K(p)$ outside a neighborhood of the intersection point x. Let $\beta(r)$ be a smooth cut-off function that vanishes for $r \geq 2\varepsilon$ and equal to 1 when $r \leq \varepsilon$, for sufficiently small ε. Define

$$\widetilde{K}(p,q) = \beta(\|p\|) \cdot \beta(\|q\|) \cdot K(p).$$

A straightforward computation shows that, $\frac{\partial \widetilde{K}}{\partial q}(0, \cdot) = \frac{\partial \widetilde{K}}{\partial p}(0, \cdot) = 0$. Hence the time-1-map of the Hamiltonian flow corresponding to $\widetilde{K}(p,q)$ is the required symplectomorphism. Therefore, we now can assume that $\sigma_{s_0}(x)$ coincide with the fiber over the point x.

Next, since Ψ is an exact Lagrangian isotopy, we have that for every $s \in I_{s_0}$, L_s is a graph of an exact 1-form dG_s in the symplectic tubular neighborhood V_{s_0} of L_{s_0}. Hence, in the above local coordinates (p,q) on $T^*L_{s_0}$, L_s takes the form $L_s = (dG_s(q), q)$. Moreover, note that $dG_s(0) = 0$.

Define

$$\widetilde{H}_{s_0}(p,q) = \beta(\|p\|) \cdot G_s(q).$$

Consider the Hamiltonian vector field corresponding to $\widetilde{H}_{s_0}$,

$$\widetilde{\xi} = \begin{cases} \dot{p} = -\frac{\partial \widetilde{H}}{\partial q} = -\beta(\|p\|) \cdot \frac{\partial G_s(q)}{\partial q} \\ \dot{q} = \frac{\partial \widetilde{H}}{\partial p} = \frac{\partial}{\partial p}\{\beta(\|p\|)\} \cdot G_s(q) \end{cases}$$

It follows that for every $s \in I_{s_0}$ such that $L_s \subset \{(p,q) \mid \|p\| < \varepsilon\}$, the Hamiltonian flow is given by

$$(p,q) \to \left(p + \frac{\partial G_s(q)}{\partial q}, q\right)$$

Hence, locally, the Hamiltonian flow sends L_{s_0} to L_s as required. It remains to prove that $\widetilde{\xi}$ vanishes on the diagonal. First, since $dG_s(0) = 0$, it follows that $\dot{p} = 0$. Next, consider x and y, two intersection points of the family $\{L_s\}$ and the diagonal. It follows from Lemma 3.2.2 that the symplectic area between L_{s_0} and L_s in V_{s_0} vanishes for every $s \in I_{s_0}$. Hence, by the same argument as in Lemma 3.5.1, for every such s we have

$$0 = \int_{\Sigma_s} \omega = \int_{[0,s]\times[0,1]} \Sigma_s^* \omega = \int_0^s \Big(G_s(x) - G_s(y) \Big) ds$$

Thus, we get that $G_s(x) - G_s(y) = 0$. Note that by changing the functions $\{G_s\}$ by a summand depending only on s, we can assume that for every s, G_s vanishes on $L_s \cap \triangle$. Hence, we obtain that $\widetilde{\xi}_{|\triangle} = 0$. Therefore, we have that the diagonal is invariant under the Hamiltonian flow. Finally, by smoothly patching together all the Hamiltonian flows corresponding to the Hamiltonian functions $\widetilde{H}_{s_i}$, for $i = 1, \dots, n$, we conclude that there exists a Hamiltonian isotopy Φ such that $\Phi_s(L_0) = L_s$ and $\Phi_s(\triangle) = \triangle$. This completes the proof of the proposition.

Chapter 4

Calabi Quasi-morphisms

4.1 Introduction and Results

Let (M, ω) be a closed connected symplectic manifold of dimension $2n$. Let $\mathrm{Ham}(M, \omega)$ denote the group of Hamiltonian diffeomorphisms of (M, ω) and let $\widetilde{\mathrm{Ham}}(M, \omega)$ be its universal cover. A celebrated result by Banyaga [4] states that for a closed symplectic manifold, $\mathrm{Ham}(M, \omega)$ and $\widetilde{\mathrm{Ham}}(M, \omega)$ are simple groups and therefore they do not admit any non-trivial homomorphism to $\mathbb{R}$. However, in some cases, these groups admit non-trivial homogeneous quasi-morphisms to $\mathbb{R}$. Recall that a (real-valued) quasi-morphism of a group G is a map $r \colon G \to \mathbb{R}$ satisfying the homomorphism equation up to a bounded error, i.e., there exists a constant $C \geq 0$ such that

$$|r(g_1 g_2) - r(g_1) - r(g_2)| \leq C, \quad \text{for every } g_1, g_2 \in G.$$

A quasi-morphism r is called homogeneous if $r(g^n) = nr(g)$ for all $g \in G$ and $n \in \mathbb{Z}$. The existence of homogeneous quasi-morphisms on the group of Hamiltonian diffeomorphisms and/or its universal cover is known for some classes of closed symplectic manifolds (see e.g. [5], [14], [18], and [19]). In a recent work [15], Entov and Polterovich showed - by using Floer and Quantum homology - that for the class of symplectic manifolds which are monotone and whose quantum homology algebra is semi-simple, $\widetilde{\mathrm{Ham}}(M, \omega)$ admits a homogeneous quasi-morphism to $\mathbb{R}$. In addition to constructing such a quasi-morphism, Entov and Polterovich showed that its value on any diffeomorphism supported in a Hamiltonianly displaceable open subset equals to the Calabi invariant of the diffeomorphism (see Section 4.2 below for precise definitions). A quasi-morphism with this property is called a Calabi quasi-morphism.

The notion “quasi-morphism" first appeared in the works of Brooks [10] and Gromov [21] on bounded cohomology of groups. Since then, quasi-morphisms have become an important tool in the study of groups. For example, the mere existence of a homogeneous quasi-morphism on a group G which does not vanish on the commutator subgroup G' implies that the commutator subgroup has infinite diameter with respect to the commutator norm (see e.g. [6]). Two well known examples of quasi-morphisms are the Maslov quasi-morphism on the universal cover of the group of linear symplectomorphisms of $\mathbb{R}^{2n}$, and the rotation quasi-morphism defined on the universal cover of the group of orientation-preserving homeomorphisms of S^1. We refer the readers to [6] and [26] and the references cited therein for further details on this subject. Recently, Biran, Entov and Polterovich [8], and Entov and Polterovich [16] established several other applications of the existence of a Calabi quasi-morphism regarding rigidity of intersections in symplectic manifolds. An example of this type is given in Theorem 4.1.5 and Corollary 4.1.6 below.

In view of the work by Entov and Polterovich [15], it is natural to ask which classes of symplectic manifolds admit a Calabi quasi-morphism. In a very recent work, Py [51] constructed a homogeneous Calabi quasi-morphism for closed oriented surfaces with genus greater than 1. In this note we concentrate on the case of non-monotone symplectic manifolds. We will provide some examples of non-monotone rational ruled surfaces admitting a Calabi quasi-morphism. More precisely, let

$$X_\lambda = (S^2 \times S^2,\ \omega_\lambda = \omega \oplus \lambda\,\omega), \quad 1 \leq \lambda \in \mathbb{R},$$

where ω is the standard area form on the two-sphere S^2 with area 1, and let

$$Y_\mu = (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2},\ \omega_\mu), \quad 0 < \mu < 1,$$

be the symplectic blow-up of $\mathbb{C}P^2$ at one point (see e.g. [33], [47]), where ω_μ takes the value μ on the exceptional divisor, and 1 on the class of the line $[\mathbb{C}P^1]$. The manifold Y_μ is the region

$$\left\{(z_1, z_2) \in \mathbb{C}^2 \mid \mu \leq |z_1|^2 + |z_2|^2 \leq 1\right\}$$

with the bounding spheres collapsed along the Hopf flow. It is known that any symplectic form on these manifolds is, up to a scaling by a constant, diffeomorphic to one of the above symplectic forms (see [30]).

In the monotone case where $\lambda = 1$ and $\mu = \frac{1}{3}$, Entov and Polterovich [15] proved the existence of a homogeneous Calabi quasi-morphism on the universal covers of $\mathrm{Ham}(X_\lambda)$ and $\mathrm{Ham}(Y_\mu)$. Moreover, they shows that these quasi-morphisms are Lipschitz with respect to Hofer's metric. For the precise definition of the Lipschitz property of a quasi-morphism, see Section 4.2 below. Here we prove the following:

Theorem 4.1.1. *Let (M, ω) be one of the following symplectic manifolds:*

(i) $X_\lambda = (S^2 \times S^2, \omega_\lambda)$, *where* $1 < \lambda \in \mathbb{Q}$.

(ii) $Y_\mu = (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\mu)$, *where* $\frac{1}{3} \neq \mu \in \mathbb{Q} \cap (0, 1)$.

Then there exists a homogeneous Calabi quasi-morphism $\widetilde{r} \colon \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$, *which is Lipschitz with respect to Hofer's metric.*

It can be shown in the monotone case that any homogeneous quasi-morphism on the universal cover of $\mathrm{Ham}(X_1)$ descends to a quasi-morphism on $\mathrm{Ham}(X_1)$ itself [15]. This is due to the finiteness of the fundamental group $\pi_1\big(\mathrm{Ham}(X_1)\big)$, which was proved by Gromov in [20]. He also pointed out that the homotopy type of the group of symplectomorphisms of $S^2 \times S^2$ changes when the spheres have different areas. McDuff [32], and Abreu and McDuff [1], showed that the fundamental groups, $\pi_1\big(\mathrm{Ham}(X_\lambda)\big)$ and $\pi_1\big(\mathrm{Ham}(Y_\mu)\big)$, contain elements of infinite order for every $0 < \mu < 1$ and for every $\lambda > 1$. Thus, the above argument will no longer hold in these cases. Furthermore we claim:

Theorem 4.1.2. *Let M be one of the manifolds listed in Theorem 4.1.1. Then the restriction of the above mentioned Calabi quasi-morphism* $\widetilde{r} \colon \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$ *to the fundamental group* $\pi_1\big(\mathrm{Ham}(M, \omega)\big) \subset \widetilde{\mathrm{Ham}}(M, \omega)$ *gives rise to a non-trivial homomorphism.*

This differs from the situation described in [15] where it was proven that for $M = \mathbb{C}P^n$ endowed with the Fubini-Study form, or for $M = S^2 \times S^2$ equipped with the split symplectic form $\omega \oplus \omega$, the restriction of the Calabi quasi-morphism to the fundamental group $\pi_1\big(\mathrm{Ham}(M)\big)$ vanishes identically.

For technical reasons, we shall assume in what follows that M is a rational strongly semi-positive symplectic manifold. Recall that a symplectic manifold M is rational if the set $\{\omega(A) \mid A \in \pi_2(M)\}$ is a discrete subset of $\mathbb{R}$ and strongly semi-positive if for every

$A \in \pi_2(M)$ one has

$$2 - n \leq c_1(A) < 0 \Longrightarrow \omega(A) \leq 0,$$

where $c_1 \in H^2(M, \mathbb{Z})$ denotes the first Chern class of M. The assumption that M is strongly semi-positive is a standard technical assumption (see e.g. [50], [56]) which guarantees, roughly speaking, the good-behavior of some moduli spaces of pseudo-holomorphic curves. Note that every symplectic manifold of dimension 4 or less, in particular the manifolds listed in Theorem 4.1.1, is strongly semi-positive. The rationality assumption is also a technical assumption. It plays a role, for example, in Lemma 4.5.1 below, where for non-rational symplectic manifolds the action spectrum is a non-discrete subset of $\mathbb{R}$ and our method of proof fails.

In fact, the examples in Theorem 4.1.1 are special cases of a more general criterion for the existence of a Calabi quasi-morphism. In [15], such a criterion was given for closed monotone symplectic manifolds. This criterion is based on some algebraic properties of the quantum homology algebra of (M, ω). More precisely, recall that as a module the quantum homology of M is defined as $QH_*(M) = H_*(M) \otimes \Lambda$, where Λ is the standard Novikov ring

$$\Lambda = \Big\{ \sum_{A \in \Gamma} \lambda_A q^A \mid \lambda_A \in \mathbb{Q},\ \#\{A \in \Gamma \mid \lambda_A \neq 0,\ \omega(A) > c\} < \infty,\ \forall\, c \in \mathbb{R} \Big\}.$$

Here $\Gamma = \pi_2(M) \ / \ (\ker c_1 \cap \ker \omega)$, where c_1 is the first Chern class. A grading on Λ is given by $\deg(q^A) = 2c_1(A)$. We shall denote by Λ_k all the elements in Λ with degree k. We refer the readers to [35] and to Subsection 4.3.1 below for a more detailed exposition and for the precise definition of the quantum product on $QH_*(M)$. In the monotone case i.e., where there exists $\kappa > 0$ such that $\omega = \kappa \cdot c_1$, the Novikov ring Λ can be identified with the field of Laurent series $\sum \alpha_j x^j$, with coefficients in $\mathbb{Q}$, and all α_j vanish for j greater than some large enough j_0. In this case we say that the quantum homology $QH_*(M)$ is *semi-simple* if it splits with respect to multiplication into a direct sum of fields, all of which are finite dimension linear spaces over Λ. It was shown in [15] that for monotone symplectic manifolds with semi-simple quantum homology algebra there exists a Lipschitz homogenous Calabi quasi-morphism on the universal cover of the group of Hamiltonian diffeomorphisms.

In the non-monotone case the above definition of semi-simplicity will no longer hold since Λ is no longer a field. However, it turns out that a similar criterion to the above still exists in this case. More precisely, we focus upon the sub-algebra $QH_{2n}(M) \subset QH_*(M)$ over the sub-ring $\Lambda_0 \subset \Lambda$. This sub-algebra is the degree component of the identity in $QH_*(M)$. Using the fact that in the non-monotone case the sub-ring Λ_0 can be identified with the field of Laurent series, we say as before that $QH_{2n}(M)$ is semi-simple over Λ_0 if it splits into a direct sum of fields with respect to multiplication. Denote by N_M the minimal Chern number of M defined as the positive generator of the image $c_1(\pi_2(M)) \subseteq \mathbb{Z}$ of the first Chern class c_1. The following criterion is a generalization of Theorem 1.5 from [15] to the rational strongly semi-positive case.

Theorem 4.1.3. *Let (M, ω) be a closed connected rational strongly semi-positive symplectic manifold of dimension $2n$. Suppose that the quantum homology subalgebra $QH_{2n}(M) \subset QH_*(M)$ is a semi-simple algebra over the field Λ_0 and that N_M divides n. Then there exists a Lipschitz homogeneous Calabi quasi-morphism $\widetilde{r}\colon \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$.*

For the manifolds X_λ and Y_μ listed in Theorem 4.1.1 the minimal Chern number N_M is 2 and 1 respectively. Thus, one of our main tasks is to prove that for these manifolds the top-dimension quantum homology subalgebra $QH_4(M)$ is semi-simple over the field Λ_0.

As a by-product of Theorem 4.1.1, we generalize a result regarding rigidity of intersections obtained by Entov and Polterovich in [16]. To describe the result, we recall first the following definitions. For a symplectic manifold M denote by $\{\cdot, \cdot\}$ the standard Poisson brackets on $C^\infty(M)$. A linear subspace $\mathcal{A} \subset C^\infty(M)$ is said to be Poisson-commutative if $\{F, G\} = 0$ for all $F, G \in \mathcal{A}$. We associate to a finite-dimensional Poisson-commutative subspace $\mathcal{A} \subset C^\infty(M)$ its moment map $\Phi_{\mathcal{A}}\colon M \to \mathcal{A}^*$, defined by $\langle \Phi_{\mathcal{A}}(x), F\rangle = F(x)$. A non-empty subset of the form $\Phi_{\mathcal{A}}^{-1}(p)$, $p \in \mathcal{A}^*$, is called a fiber of $\mathcal{A}$. A fiber $X \subset M$ is said to be displaceable if there exists a Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}(M)$ such that $\varphi(X) \cap X = \emptyset$. The following definition was introduced in [16]:

Definition 4.1.4. *A closed subset $X \subset M$ is called a stem, if there exists a finite-dimensional Poisson-commutative subspace $\mathcal{A} \subset C^\infty(M)$, such that X is a fiber of $\mathcal{A}$ and each fiber of $\mathcal{A}$, other than X, is displaceable.*

In Theorem 2.4 of [16], Entov and Polterovich showed that for a certain class of symplectic manifolds, any two stems have a non-empty intersection. What they used, in fact, was only the existence of a Lipschitz homogeneous Calabi quasi-morphism for manifolds in this class. Using the exact same line of proof, the following theorem follows from Theorem 4.1.1 above.

Theorem 4.1.5. *Let M be one of the manifolds listed in Theorem 4.1.1. Then any two stems in M intersect.*

An example of a stem in the case where $M = X_\lambda$ is the product of two equators. More precisely, we identify X_λ with $\mathbb{C}P^1 \times \mathbb{C}P^1$ in the obvious way. Denote by $L \subset X_\lambda$ the Lagrangian torus defined by

$$L = \left\{ ([z_0 : z_1], [w_0 : w_1]) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \mid \quad |z_0| = |z_1|,\ |w_0| = |w_1| \right\}$$

The proof that L is a stem goes along the same line as Corollary 2.5 of [16]. Since the image of a stem under any symplectomorphism of M is again a stem we get:

Corollary 4.1.6. *Let X_λ be one of the manifolds in the first class of manifolds listed in Theorem 4.1.1 above. Then for any symplectomorphism φ of X_λ we have $L \cap \varphi(L) \neq 0$.*

4.2 Preliminaries on Calabi quasi-morphism

In this section we recall the definition of a Calabi quasi-morphism introduced in [15]. We start with the definition of the classical Calabi invariant (see [4] and [11]). Let (M, ω) be a closed connected symplectic manifold. Given a Hamiltonian function $H\colon S^1 \times M \to \mathbb{R}$, set $H_t := H(t, \cdot)$ and denote by φ the time-1-map of the Hamiltonian flow $\{\varphi_H^t\}$. The group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$ consists of all such time-1-maps. Let $\widetilde{\mathrm{Ham}}(M, \omega)$ be the universal cover of $\mathrm{Ham}(M, \omega)$. For a non-empty open subset U of M, we denote by $\widetilde{\mathrm{Ham}}_U(M, \omega)$ the subgroup of $\widetilde{\mathrm{Ham}}(M, \omega)$, consisting of all elements that can be represented by a path $\{\varphi_H^t\}_{t \in [0,1]}$ starting at the identity and generated by a Hamiltonian function H_t supported in U for all t. For $\varphi \in \widetilde{\mathrm{Ham}}_U(M, \omega)$ we define $\mathrm{Cal}_U\colon \widetilde{\mathrm{Ham}}_U(M, \omega) \to \mathbb{R}$ by

$$\varphi \mapsto \int_0^1 dt \int_M H_t \, \omega^n.$$

This map is well defined, i.e., it is independent of the specific choice of the Hamiltonian function generating φ. Moreover, it is a group homomorphism called the Calabi homomorphism.

Recall that a non-empty subset U of M is called Hamiltonianly displaceable if there exists a Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}(M, \omega)$ such that $\varphi(U) \cap \mathrm{Closure}(U) = \emptyset$. The following two definitions were introduced in [15].

Definition 4.2.1. *A quasi-morphism on* $\widetilde{\mathrm{Ham}}(M, \omega)$ *coinciding with the Calabi homomorphism* $Cal_U \colon \widetilde{\mathrm{Ham}}_U(M, \omega) \to \mathbb{R}$ *on any open and Hamiltonianly displaceable set* U *is called a Calabi quasi-morphism.*

Definition 4.2.2. *A quasi-morphism* $r \colon \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$ *is said to be Lipschitz with respect to Hofer's metric if there exists a constant* $K > 0$ *so that*

$$|r(\varphi_H) - r(\varphi_F)| \leq K \cdot \|H - F\|_{C^0}$$

For the relation of $\|H - F\|_{C^0}$ to the Hofer distance between the corresponding Hamiltonian diffeomorphisms φ_H and φ_H see e.g. [15].

4.3 The Quantum homology of our main examples

4.3.1 The quantum homology algebra

In this section we briefly recall the definition of the quantum homology ring of (M^{2n}, ω). We refer the readers to [35] for a detailed exposition on this subject. Let M be a closed rational strongly semi-positive symplectic manifold of dimension $2n$. By abuse of notation, we shall write $\omega(A)$ and $c_1(A)$ for the integrals of ω and c_1 over $A \in \pi_2(M)$. Let Γ be the abelian group

$$\Gamma = \pi_2(M) \ / \ (\ker c_1 \cap \ker \omega). \tag{4.1}$$

We denote by Λ the Novikov ring

$$\Lambda = \Big\{ \sum_{A \in \Gamma} \lambda_A q^A \mid \lambda_A \in \mathbb{Q}, \ \#\{A \in \Gamma \mid \lambda_A \neq 0, \ \omega(A) > c\} < \infty, \ \forall c \in \mathbb{R} \Big\}. \tag{4.2}$$

This ring comes with a natural grading defined by $\deg(q^A) = 2c_1(A)$. We shall denote by Λ_k all the elements of Λ with degree k. Note that $\Lambda_k = \emptyset$ if k is not an integer multiple of $2N_M$, where N_M is the minimal Chern number of M defined by $c_1(\pi_2(M)) = N_M \mathbb{Z}$.

As a module, the quantum homology ring of (M,ω) is defined as

$$QH_*(M) = QH_*(M,\Lambda) = H_*(M,\mathbb{Q}) \otimes \Lambda.$$

A grading on $QH_*(M)$ is given by $\deg(a \otimes q^A) = \deg(a) + 2c_1(A)$, where $\deg(a)$ is the standard degree of the class a in the singular homology of M. Next, we define the quantum product on $QH_*(M)$ (cf. [35], [52]). For $a \in H_i(M)$ and $b \in H_j(M)$ we define $a * b \in QH_{i+j-2n}(M)$ as

$$a * b = \sum_{A \in \Gamma} (a * b)_A \otimes q^{-A},$$

where $(a * b)_A \in H_{i+j-2n+2c_1(A)}(M)$ is determined by the requirement that

$$(a * b)_A \ \circ \ c = \Phi_A(a,b,c) \qquad \text{for all } c \in H_*(M).$$

Here $\circ$ is the usual intersection product on $H_*(M)$, and $\Phi_A(a,b,c)$ denotes the Gromov-Witten invariant that counts the number of pseudo-holomorphic curves representing the class A and intersecting with a generic representative of each of $a,b,c \in H_*(M)$. The product $*$ is extended to $QH_*(M)$ by linearity over the ring Λ. Note that the fundamental class $[M]$ is the unity with respect to the quantum multiplication.

It follows from the definitions that the zero-degree component of $a * b$ coincides with the classical cap-product $a \cap b$ in the singular homology. Moreover, there exists a natural pairing $\Delta\colon QH_k(M) \times QH_{2n-k}(M) \to \Lambda_0$ defined by

$$\Delta\left(\sum a_A \otimes q^A, \sum b_B \otimes q^B\right) = \sum_{c_1(A)=0} \left(\sum_B (a_{-B} \circ b_{B+A})\right) q^A.$$

The fact that the inner sums on the right hand side are always finite follows from the finiteness condition in (4.2). Moreover, the pairing Δ defines a Frobenius algebra structure i.e., it is non-degenerate in the sense that $\Delta(\alpha,\beta) = 0$ for all β implies $\alpha = 0$, and $\Delta(\alpha,\beta) = \Delta(\alpha * \beta, [M])$. Notice that Δ associates to each pair of quantum homology classes $\alpha, \beta \in QH_*(M)$ the coefficient of the class $P = [point]$ in their quantum product. We also define a non-degenerate $\mathbb{Q}$-valued pairing Π to be the zero order term of Δ, i.e.,

$$\Pi\left(\sum a_A \otimes q^A, \sum b_B \otimes q^B\right) = \sum_B (a_{-B} \circ b_B). \tag{4.3}$$

Note that $\Pi(\alpha,\beta) = \Pi(\alpha * \beta, [M])$ for every pair of quantum homology classes α and β. Furthermore, the finiteness condition in the definition of the Novikov ring (4.2) leads to a natural valuation function $val\colon QH_*(M) \to \mathbb{R}$ defined by

$$val\Big(\sum_{A\in\Gamma} a_A \otimes q^A\Big) = \max\{\omega(A) \mid a_A \neq 0\}, \quad \text{and} \quad val(0) = -\infty. \tag{4.4}$$

4.3.2 The case of $S^2 \times S^2$

Let $X_\lambda = S^2 \times S^2$ be equipped with the split symplectic form $\omega_\lambda = \omega \oplus \lambda\,\omega$, where $\lambda > 1$. In this subsection we discuss several issues regarding the quantum homology of the manifold X_λ and in particular we show that the quantum homology subalgebra $QH_4(X_\lambda) \subset QH_*(X_\lambda)$ is a field for every $\lambda > 1$.

Denote the standard basis of $H_*(X_\lambda)$ by $P = [\text{point}]$, $A = [S^2 \times \text{point}]$, $B = [\text{point} \times S^2]$ and the fundamental class $M = [X_\lambda]$. The quantum homology of X_λ is generated over the Novikov ring Λ by these elements. Since $\lambda > 1$, it follows that $\Gamma = \pi_2(X_\lambda)$, where the last is isomorphic to the free abelian group generated by A and B. From the following Gromov-Witten invariants (see e.g. [15], [35]):

$$\Phi_{A+B}(P,P,P) = 1, \quad \Phi_0(A,B,M) = 1, \quad \Phi_A(P,B,B) = 1, \quad \Phi_B(P,A,A) = 1,$$

one finds the quantum identities:

$$A * B = P, \quad A^2 = M \otimes q^{-B}, \quad B^2 = M \otimes q^{-A}. \tag{4.5}$$

Next, instead of the standard basis $\{A, B\}$ of Γ, we consider the basis $\{e_1, e_2\} = \{B - A, A\}$. Set $x = q^{e_1}$ and $y = q^{e_2}$. In this notation, the quantum product of the generators of $QH_*(X_\lambda)$ becomes

$$A * B = P, \quad A^2 = M \otimes x^{-1}y^{-1}, \quad B^2 = M \otimes y^{-1}. \tag{4.6}$$

It follows from the definition of the Novikov ring (4.2) that

$$\Lambda = \left\{ \sum \lambda_{\alpha,\beta} \cdot x^\alpha y^\beta \mid \lambda_{\alpha,\beta} \in \mathbb{Q} \right\},$$

where each sum satisfies the following finiteness condition:

$$\#\{(\alpha,\beta) \mid \lambda_{\alpha,\beta} \neq 0,\ \alpha(\lambda-1) + \beta > c\} < \infty,\ \forall\, c \in \mathbb{R}.$$

Taking into account the above mentioned grading of Λ we get

$$\begin{aligned}\Lambda_{4k} &= \Big\{\sum \lambda_{\alpha,\beta}\cdot x^{\alpha}y^{\beta} \ \in \Lambda \ \big| \ 4\beta = 2c_1(\alpha e_1+\beta e_2) = 4k\Big\} \\ &= \Big\{\sum \lambda_{\alpha}\cdot x^{\alpha}y^{k} \ \big| \ \#\{\alpha \mid \lambda_\alpha \neq 0,\ \alpha(\lambda-1) > d\} < \infty,\ \forall\, d\in\mathbb{R}\Big\}.\end{aligned}$$

The finiteness condition above implies that λ_α vanishes for large enough α's.

Lemma 4.3.1. *For any $\lambda > 1$, the subalgebra $QH_4(X_\lambda) \subset QH_*(X_\lambda)$ is a field.*

Proof of Lemma Let $0 \neq \gamma \in QH_4(X_\lambda)$. Since $QH_4(X_\lambda) = H_4(X_\lambda)\otimes\Lambda_0 + H_0(X_\lambda)\otimes\Lambda_4$, it follows that

$$\gamma = M \otimes \sum \lambda_{\alpha_1}\, x^{\alpha_1} + P\otimes y \sum \lambda_{\alpha_2}\, x^{\alpha_2},$$

where λ_{α_1} and λ_{α_2} vanish for large enough α_1 and α_2 respectively. Next, let $\beta = P\otimes y$ be a formal variable. From the above multiplicative relations (4.6), we see that $\beta^2 = M\otimes x^{-1}$. Hence, we can consider the following ring identification:

$$QH_4(X_\lambda) \simeq \mathcal{R}[\beta] \ / \ \mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by $\beta^2 - x^{-1}$ and $\mathcal{R} = \mathbb{Q}[[x]]$ is the ring of Laurent series $\sum \alpha_j x^j$, with coefficients in $\mathbb{Q}$, and all α_j vanish for j greater than some large enough j_0. Note that for any Laurent series $\Phi(x) \in \mathcal{R}$, the maximal degree of $\Phi^2(x)$ is either zero or even. Therefore $\mathcal{R}$ does not contain a square root of x^{-1} and hence $\mathcal{I}$ is a maximal ideal. Thus, we conclude that $QH_4(X_\lambda)$ is a two-dimensional extension field of $\mathcal{R}$. This completes the proof of the lemma.

Remark 4.3.2. Note that the above statement no longer holds in the monotone case where $\lambda = 1$, since $QH_4(X_1)$ contains zero divisors (see e.g. [15], [35]).

4.3.3 The case of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$

Here we study the quantum homology algebra of $Y_\mu = (\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\mu)$, which is the symplectic one-point blow-up of $\mathbb{C}P^2$ introduced in Section 4.1. We will show that the quantum homology subalgebra $QH_4(Y_\mu)$, which plays a central role in the proof of Theorem 4.1.1, is semi-simple. It is worth mentioning (see Remark 4.3.4 below) that the algebraic structure of $QH_4(Y_\mu)$ turns out to be dependent on μ.

We denote by E the exceptional divisor and by L the class of the line $[\mathbb{C}P^1]$. Recall that for $0 < \mu < 1$, ω_μ is a symplectic form on Y_μ with $\omega_\mu(E) = \mu$ and $\omega_\mu(L) = 1$. Denote the class of a point by $P = [point]$ and set $F = L - E$. The elements P, E, F and the fundamental class $M = [Y_\mu]$ form a basis of $H_*(Y_\mu)$.

The following description of the multiplicative relations for the generators of $QH_*(Y_\lambda)$ can be found in [33].

$$\begin{array}{ll} P * P = (E+F) \otimes q^{-E-F}, & E * P = F \otimes q^{-F}, \\ P * F = M \otimes q^{-E-F}, & E * E = -P + E \otimes q^{-E} + M \otimes q^{-F}, \\ E * F = P - E \otimes q^{-E}, & F * F = E \otimes q^{-E}. \end{array}$$

Consider the rational non-monotone case where $\frac{1}{3} \neq \mu \in \mathbb{Q} \cap (0,1)$. Note that in this case $\Gamma \simeq \mathbb{Z} \otimes \mathbb{Z}$. As in the previous example of $S^2 \times S^2$, we apply a unimodular change of coordinates and consider the following basis of Γ

$$\Gamma \simeq \begin{cases} \mathrm{Span}_{\mathbb{Z}}\{F - 2E,\ E\}, & 0 < \mu < \frac{1}{3} \\ \\ \mathrm{Span}_{\mathbb{Z}}\{2E - F,\ E\}, & \frac{1}{3} < \mu < 1 \end{cases}$$

Denote $e_1 = F - 2E$, $e_2 = E$ when $0 < \mu < \frac{1}{3}$ and $e_1 = 2E - F$, $e_2 = E$ when $\frac{1}{3} < \mu < 1$. Set $x = q^{e_1}$ and $y = q^{e_2}$. From the definition of the Novikov ring (4.2) we have

$$\Lambda = \left\{ \sum \lambda_{\alpha,\beta} x^\alpha y^\beta \mid \lambda_{\alpha,\beta} \in \mathbb{Q} \right\},$$

where each sum satisfies the following finiteness condition:

$$\#\left\{(\alpha,\beta) \mid \lambda_{\alpha,\beta} \neq 0,\ \alpha|3\mu - 1| + \beta\mu > c\right\} < \infty,\ \forall\, c \in \mathbb{R}.$$

The graded Novikov ring has the form

$$\begin{aligned} \Lambda_{2i} &= \left\{ \sum \lambda_{\alpha,\beta} \cdot x^\alpha y^\beta \ \in \Lambda \mid 2\beta = 2c_1(\alpha e_1 + \beta e_2) = 2i \right\} \\ &= \left\{ \sum \lambda_\alpha \cdot x^\alpha y^i \mid \ \#\{\alpha \mid \lambda_\alpha \neq 0,\ \alpha|3\mu - 1| > d\} < \infty,\ \forall\, d \in \mathbb{R} \right\}. \end{aligned}$$

Next we present the quantum product of $QH_*(Y_\mu)$ with respect to the above basis of Γ.

$$\begin{aligned} P * P &= (E+F) \otimes x^{\kappa} y^{-3}, & E * P &= F \otimes x^{\kappa} y^{-2}, \\ P * F &= M \otimes x^{\kappa} y^{-3}, & E * E &= -P + E \otimes y^{-1} + M \otimes x^{\kappa} y^{-2}, \\ E * F &= P - E \otimes y^{-1}, & F * F &= E \otimes y^{-1}, \end{aligned}$$

where $\kappa = \mathrm{sgn}(3\mu - 1)$ i.e., $\kappa = 1$ for $\frac{1}{3} < \mu < 1$, and $\kappa = -1$ for $0 < \mu < \frac{1}{3}$.

Lemma 4.3.3. *The subalgebra $QH_4(Y_\mu) \subset QH_*(Y_\mu)$ is semi-simple.*

Proof of Lemma 4.3.3: Since $QH_4(Y_\mu) = H_4(Y_\mu) \otimes \Lambda_0 + H_2(Y_\mu) \otimes \Lambda_2 + H_0(Y_\mu) \otimes \Lambda_4$, it follows that for every $0 \neq \delta \in QH_4(Y_\mu)$

$$\begin{aligned} \delta &= M \otimes \sum \lambda_{\alpha_1} x^{\alpha_1} + E \otimes y \sum \lambda_{\alpha_2} x^{\alpha_2} \\ &+ F \otimes y \sum \lambda_{\alpha_3} x^{\alpha_3} + P \otimes y^2 \sum \lambda_{\alpha_4} x^{\alpha_4}, \end{aligned}$$

where λ_{α_i} vanish for large enough α_i for $i = 1, 2, 3, 4$. Next, put $\beta_1 = E \otimes y$, $\beta_2 = F \otimes y$ and $\beta_3 = P \otimes y^2$. From the above multiplication table, we see that

$$\begin{cases} \beta_1^2 = -\beta_3 + \beta_1 + x^{\kappa}, \\ \beta_2^2 = \beta_1, \\ \beta_3^2 = x^{\kappa}(\beta_1 + \beta_2) \\ \beta_1 \cdot \beta_2 = \beta_3 - \beta_1, \\ \beta_2 \cdot \beta_3 = x^{\kappa}, \\ \beta_1 \cdot \beta_3 = x^{\kappa} \beta_2. \end{cases}$$

Thus, we have the following ring identification:

$$QH_4(Y_\mu) \simeq \mathcal{R}[\beta_1, \beta_2, \beta_3] \;/\; \mathcal{I},$$

where $\mathcal{R} = \mathbb{Q}[[x]$ is the ring of Laurent series $\sum \alpha_j x^j$ and $\mathcal{I}$ is the ideal generated by the above relations. It is easy to check that the sixth equation follows immediately from the second and the fifth equations and hence, it can be eliminated. Moreover, by isolating β_3 and β_1 from the first and the second equations respectively, we conclude that the above system is equivalent to the following one:

$$\begin{cases} (\beta_2^2 - \beta_2^4 + x^{\kappa})^2 = x^{\kappa}(\beta_2^2 + \beta_2) \\ \beta_2^3 = -\beta_2^4 + x^{\kappa}, \\ \beta_2 \cdot (\beta_2^2 - \beta_2^4 + x^{\kappa}) = x^{\kappa}, \end{cases} \tag{4.7}$$

Moreover, we claim that in fact

$$QH_4(Y_\mu) \;\simeq\; \mathcal{R}[\beta_1,\beta_2,\beta_3] \,/\, \mathcal{I} \;\simeq\; \mathcal{R}[\beta_2] \,/\, \mathcal{J}$$

where $\mathcal{J}$ is the ideal generated by $\beta_2^4+\beta_2^3-x^\kappa$. Indeed, the first equation in (4.7) is obtained by multiplying the third equation by $\beta_2^2+\beta_2$ and assigning the second equation. The third equation is obtained from the second after multiplying it by β_2-1. Next, note that the polynomial $\beta_2^4+\beta_2^3-x^\kappa$ does not share a common root with its derivative since the roots of the derivative are 0 and $-3/4$. Thus, it has no multiple roots in $\mathcal{R}$ and hence the quantum homology subalgebra $QH_4(Y_\mu)$ is semi-simple as required.

Remark 4.3.4. Strangely enough, it follows from the above lemma that the algebraic structure of the quantum homology subalgebra $QH_4(Y_\mu)$ depends on μ. More precisely, it can be shown that the polynomial $\beta_2^4+\beta_2^3-x^\kappa$ is irreducible over $\mathcal{R}$ for $\kappa=1$ while reducible for $\kappa=-1$. Thus, $QH_4(Y_\mu)$ is a field when $\frac{1}{3}<\mu<1$, while for $0<\mu<\frac{1}{3}$, it is a direct sum of fields. We omit here the technical details because for our purpose, it is sufficient that $QH_4(Y_\mu)$ is semi-simple.

4.4 Preliminaries on Floer homology

In this section we give a brief review of Floer homology. In particular we present some definitions and notions which will be relevant for the proof of our main results. We refer the readers to [54] or [35] for a more detailed description.

Let (M,ω) be a closed, connected and strongly semi-positive symplectic manifold. Let $J=\{J_t\}_{0\le t\le 1}$ be a periodic family of ω-compatible almost complex structures. We denote by $\mathcal{L}$ the space of all smooth contractible loops $x\colon\, S^1=\mathbb{R}/\mathbb{Z}\to M$. Consider a covering $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ whose elements are equivalence classes $[x,u]$ of pairs (x,u), where $x\in\mathcal{L}$, u is a disk spanning x in M, and where

$$(x_1,u_1)\sim(x_2,u_2)\quad\text{if and only if } x_1=x_2 \;\text{ and }\; \omega(u_1\#u_2)=c_1(u_1\#u_2)=0.$$

The group of deck transformations of $\widetilde{\mathcal{L}}$ is naturally identified with the group Γ (4.1), and we denote by

$$[x,u]\mapsto[x,u\#A],\quad A\in\Gamma$$

the action of Γ on $\widetilde{\mathcal{L}}$. Moreover, we denote by $\mathcal{H}$ the set of all the zero-mean normalized Hamiltonian functions i.e.,

$$\mathcal{H} = \Big\{ H \in C^\infty(S^1 \times M) \mid \int_M H_t \, \omega^n = 0, \text{ for all } t \in [0,1] \Big\}.$$

For $H \in \mathcal{H}$, the symplectic action functional $\mathcal{A}_H \colon \widetilde{\mathcal{L}} \to \mathbb{R}$ is defined as

$$\mathcal{A}_H([x,u]) := -\int_u \omega + \int_{S^1} H(x(t),t)dt.$$

Note that

$$\mathcal{A}_H([x, u\#A]) = \mathcal{A}_H([x,u]) - \omega(A).$$

Let $\mathcal{P}_H$ be the set of all contractible 1-periodic orbits of the Hamiltonian flow generated by H. Denote by $\widetilde{\mathcal{P}_H}$ the subset of pairs $[x,u] \in \widetilde{\mathcal{L}}$ where $x \in \mathcal{P}_H$. It is not difficult to verify that $\widetilde{\mathcal{P}_H}$ coincides with the set of critical points of $\mathcal{A}_H$. We define the *action spectrum* of H, denoted by $Spec(H)$, as

$$Spec(H) := \Big\{ \mathcal{A}_H(x,u) \in \mathbb{R} \mid [x,u] \in \widetilde{\mathcal{P}_H} \Big\}.$$

Recall that the action spectrum is either a discrete or a countable dense subset of $\mathbb{R}$ [38].

We now turn to give the definition of the filtered Floer homology group. For a generic $H \in \mathcal{H}$ and $\alpha \in \{\mathbb{R} \setminus Spec(H)\} \cup \{\infty\}$ define the vector space $CF_k^\alpha(H)$ to be

$$CF_k^\alpha(H) = \Big\{ \sum_{[x,u] \in \widetilde{\mathcal{P}}(H)} \beta_{[x,u]}[x,u] \mid \beta_{[x,u]} \in \mathbb{Q}, \ \mu([x,u]) = k, \ \mathcal{A}_H([x,u]) < \alpha \Big\}, \tag{4.8}$$

where each sum satisfies the following finiteness condition:

$$\# \Big\{ [x,u] \in \widetilde{\mathcal{P}_H} \mid \beta_{[x,u]} \neq 0 \text{ and } \mathcal{A}_H([x,u]) > \delta \Big\} < \infty, \ \text{for every } \delta \in \mathbb{R}. \tag{4.9}$$

Here $\mu([x,u])$ denotes the Conley-Zehnder index $\mu \colon \widetilde{\mathcal{P}_H} \to \mathbb{Z}$ (see e.g. [54]) which satisfies $\mu([x,u\#A]) - \mu([x,u]) = 2c_1(A)$. In particular, the Conley-Zehnder index of an element $x \in \mathcal{P}_H$ is well-defined modulo $2N_M$, where N_M is the minimal Chern number of (M,ω). The complex $CF_k^\infty(H)$ is a module over the Novikov ring Λ (4.2), where the scalar multiplication of $\xi \in CF_k^\infty(H)$ with $\lambda \in \Lambda$ is given by

$$\sum_A \sum_{[x,u]} a_A \cdot \alpha_{[x,u]}[x, u\#A].$$

For each given $[x,w]$ and $[y,v]$ in $\widetilde{\mathcal{P}_H}$, let $\mathcal{M}(H,J,[x,u],[y,v])$ be the moduli space of Floer connecting orbits from $[x,w]$ to $[y,v]$ i.e., the set of solutions $u\colon\ \mathbb{R}\times S^1\to M$ of the system

$$\left\{\begin{array}{c} \partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0, \\ \lim_{s\to-\infty} u(s,t) = x(t), \quad \lim_{s\to\infty} u(s,t) = y(t), \\ w\#u\#v \text{ represent the trivial class in } \Gamma. \end{array}\right\}$$

It follows from the assumption of strongly semi-positivity and from Gromov's compactness theorem [20] that for a generic choice of J the moduli spaces $\mathcal{M}([x,u],[y,v])$, for $\mu([x,u])-\mu([y,v])=1$, are compact.

The Floer boundary operator $\partial\colon\ CF_k^\alpha(H)\to CF_{k-1}^\alpha(H)$ is defined by

$$\partial([x,w]) = \sum n\big([x,w],[y,v]\big)\ [y,v],$$

where the sum runs over all the elements $[y,v]\in\widetilde{\mathcal{P}_H}$ such that $\mu[y,v]=k-1$ and $n\big([x,w],[y,v]\big)$ denotes counting the (finitely many) un-parameterized Floer trajectories with a sign determined by a coherent orientation. As proved by Floer in [17], the boundary operator ∂ is well defined, satisfies $\partial^2=0$ and preserves the subspaces $CF_*^\alpha(H)$ (see [23]). Therefore, defining the quotient group by

$$CF_*^{[a,b)}(H,J) = CF_*^b(H,J)\ /\ CF_*^a(H,J) \qquad (-\infty < a \le b \le \infty),$$

the boundary map induces a boundary operator $\partial\colon\ CF_*^{(a,b]}(H)\to CF_*^{(a,b]}(H)$, and we can define the Floer homology group by

$$HF_*^{(a,b]}(J,H) = (CF_*^{(a,b]}(H),\partial).$$

We will use the convention $HF_*(H,J)=HF_*^{(-\infty,\infty]}(H,J)$ and $HF_*^a(H,J)=HF_*^{(-\infty,a]}$. The graded homology $HF_*(H,J)$ is a module over the Novikov ring Λ, since the boundary operator is linear over Λ. Note that these homology groups have been defined for generic Hamiltonians only. However, one can extend the definition to all $H\in\mathcal{H}$ using a continuation procedure (see e.g [15]). A key observation is that the Floer homology groups are independent of the almost complex structure J and the Hamiltonian H used to define them. Moreover, if two Hamiltonian functions $H_1,H_2\in\mathcal{H}$ generate the same element $\varphi\in\widetilde{\mathrm{Ham}}(M,\omega)$, then $Spec(H_1)=Spec(H_2)$ (see [40] and [55]) and the spaces $HF_*^{(a,b]}(J,H_1)$ and $HF_*^{(a,b]}(J,H_2)$ can be canonically identified. Therefore, we shall drop the notation J and H in $HF_*(H,J)$ and denote $HF_*(\varphi) = HF_*(J,H)$ where $\varphi \in \widetilde{\mathrm{Ham}}(M,\omega)$ is generated by H.

We denote by $\pi_\alpha\colon HF_*(\varphi) \to HF^{(\alpha,\infty]}(\varphi)$ the homomorphisms induced by the natural projection $CF_\infty(H) \to CF_\infty(H)/CF_\alpha(H)$ of Floer complexes and by $i_\alpha\colon HF_\alpha(\varphi) \to HF_*(\varphi)$ the homomorphism induced by the inclusion map $i_\alpha\colon CF_*^\alpha(H) \to CF_*^\infty(H)$. Note that the homology long exact sequence yields Kernel π_α = Image i_α. There exists a natural ring structure on the Floer homology groups named *Pair-of-pants product* (see e.g. [50])

$$*_{\mathrm{pp}}\colon HF_\alpha(\varphi) \times HF_\beta(\psi) \to HF_{\alpha+\beta}(\varphi\psi).$$

In [50], Piunikhin, Salamon and Schwarz constructed a homomorphism between the Quantum homology groups $QH_*(M)$ and the Floer homology groups $HF_*(M)$. Furthermore, they showed that the homomorphism $\Phi\colon QH_*(M) \to HF_*(H)$ is an isomorphism which preserves the grading and intertwines the quantum product on $QH_*(M)$ with the pair-of-pants product on $HF_*(H)$ i.e., $\Phi\left(i_{\alpha+\beta}(\xi *_{pp} \eta)\right) = \Phi\left(i_\alpha(\xi)\right) * \Phi\left(i_\beta(\eta)\right)$, for every $\xi \in HF_\alpha(\varphi)$, $\eta \in HF_\beta(\psi)$. In what follows, we will refer to the isomorphism Φ as the PSS isomorphism. We refer the reader to [50] and [35] for the precise definition and further details on the isomorphism Φ. In what follows we include a short discussion on the PSS isomorphism in the case most relevant to this work i.e., the case of a closed strongly semi-positive symplectic manifold (M^{2n},ω) which is also rational and non-monotone.

Let $f : M \to \mathbb{R}$ be a Morse function such that the negative gradient flow of f is a Morse-Smale with respect to the metric $g = \omega(\cdot, J\cdot)$, where J is a compatible almost complex structure. We denote by $HM_*(f,\Lambda) = HM_*(f,\mathbb{Q}) \otimes \Lambda$ the Morse-Witten homology of f with coefficients in the Novikov ring Λ (4.2) (see e.g. [35] for the precise definition). It is known that there is a natural isomorphism between the quantum homology of M and the Morse-Witten homology $QH_*(M,\Lambda) \simeq HM_*(M,\Lambda)$ (see [35] and the references within).

Next, we consider the moduli space of spiked discs (or mixed trajectories) $\mathcal{M}^{PSS}(x,\widetilde{y}) = \mathcal{M}^{PSS}(x,\widetilde{y},J,H,f,g)$ of all the pairs (γ,u) where $\gamma : (-\infty,0] \to M$ and $u : \mathbb{R} \times S^1 \to M$ which satisfy

$$\left\{\begin{array}{c} \partial_s u + J(u)(\partial_t u - \beta(s) X_{H_t}(u)) = 0, \quad \dot{\gamma} + \nabla^g f \circ \gamma = 0 \\ \lim_{s\to-\infty} \gamma(s) = x, \quad \lim_{s\to-\infty} u(s,t) = \gamma(0), \quad \lim_{s\to\infty} u(s,t) = y(t) \\ \int_{\mathbb{R}} \int_{S^1} |u_s|^2 ds dt < \infty, \end{array}\right\}$$

where $x \in \mathrm{Crit}(f)$ is a critical point of f, H is a generic time-dependent Hamiltonian function, $\widetilde{y} = [y, u_y] \in \widetilde{\mathcal{P}_H}$, and $\beta : R \to [0;1]$ is a smooth cut-off function such that $\beta(s) = 0$ for $s \leq \frac{1}{2}$, $\beta'(s) \geq 0$ for $s \in \mathbb{R}$, and $\beta(s) = 1$ for $s \geq 1$.

For a generic choice of H, J, f, and g, the moduli space $\mathcal{M}^{PSS}(x, \widetilde{y})$ is a smooth manifold of dimension $\mu_{CZ}(\widetilde{y}) + \mu_{Morse}(x) - n$, where μ_{CZ} is the Conley-Zehnder index and μ_{Morse} is the standard Morse index. In the strongly semi-positive case the 0-dimensional component of $\mathcal{M}^{PSS}(x, \widetilde{y})$ is compact and we shall denote by $n(x, \widetilde{y})$ its cardinality. Following [50] we define a chain map $\Phi : CF_*(H) \to CM_*(f, \Lambda)$ between the Floer and the Morse-Witten complexes by

$$\Phi(\widetilde{y} \otimes q^A) = \sum_{\mu_{Morse}(x) = n - \mu_{CZ}(\widetilde{y})} n(x, \widetilde{y}) \cdot x \otimes q^{A \# u_y}.$$

A universal energy bound ensures that this map is a well defined Λ-module homomorphism which respects the grading. Moreover, standard arguments from Floer theory imply that it descends to a Λ-module homomorphism $\Phi : HF_*(H) \to HM_*(f, \Lambda)$ on the homology level as well (see [50], [35] for the precise details).

Remark 4.4.1. Throughout we shall use the same notation for maps between chain complexes and maps on the corresponding homology level.

In [50] it was shown that Φ is in fact an isomorphism. This was shown by constructing an explicit inverse $\Psi : HM_*(f, \Lambda) \to HF_*(H)$ which is also a Λ-module homomorphism which respect the grading and then showing that the maps $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are chain homotopic to the identity. The precise definition of Ψ and the proof of the above mentioned arguments can be found in [50] (see also [2], and [35]). In particular, both in [50] (Pages $189 - 192$) and in [35] (Figures 2 and 3 on Page 462) there are several pictures that illustrate the ideas behind the arguments that the maps $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are chain homotopic to the identity. We wish to emphasize that while the proof that $\Phi \circ \Psi = \mathbb{1}$ is based on standard techniques from Floer theory the other direction is slightly more delicate. However, in the rational non-monotone case, using abstract dimension reasons it is sufficient to show that $\Phi \circ \Psi = \mathbb{1}$ i.e.,

Claim 4.4.2. *Let (M, ω) be a rational non-monotone strongly semi-positive symplectic manifold. Then*

$$\Phi \circ \Psi = \mathbb{1}_{HM_*(f,\Lambda)} \Rightarrow \Psi \circ \Phi = \mathbb{1}_{HF_*(H)}$$

Proof of Claim 4.4.2: Denote by $\Phi^k : HF_k(H) \to HM_k(f, \Lambda)$ and $\Psi^k : HM_k(f, \Lambda) \to HF_k(H)$ the restrictions of the maps Φ and Ψ to the corresponding graded components. It follows from the assumption of the claim and from the fact that both Ψ and Φ respect the grading that $\Phi^k \circ \Psi^k = \mathbb{1}_{HF_k(M,\Lambda)}$. Moreover, in the rational non-monotone case we have that both $HF_k(H)$ and $HM_k(f, \Lambda)$ are **finite** dimensional vector spaces over Λ_0 of the **same** dimension. Note that in the rational non-monotone case Λ_0 can be identified as the field of Laurent series with coefficients in $\mathbb{Q}$. In order to check that both spaces have the same dimension over Λ_0 one should take $H = \varepsilon f$ where $\varepsilon > 0$ is sufficiently small (see e.g. [35]). Thus we conclude that Ψ^k is a monomorphism over Λ_0 between two spaces of the same dimension over Λ_0, while Φ^k is an epimorphism between them. Thus both are isomorphisms, inverse one to another. Since $HF_*(H)$ and $HM_*(f, \Lambda)$ are both finitely generated Λ-modules it follows that Ψ and Φ are isomorphisms which are inverse one to another as well. The proof of the claim is now complete.

4.5 The Existence of a Calabi Quasi-morphism

Let (M^{2n}, ω) be a closed connected rational strongly semi-positive symplectic manifold. Following the works of Viterbo [57], Schwarz [55], and Oh [39], we recall the definition of a spectral invariant c which plays a central role in the proof of Theorem 4.1.3. We refer the readers to [39] and [35] for complete details of the construction and proofs of the general properties of this spectral invariant. A brief description of Floer homology and the PSS isomorphism was also given in the above Section 4.4.

We define the spectral invariant $c\colon QH_*(M) \times \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$ as follows. For the elements $0 \neq a \in QH_*(M)$ and $\varphi \in \widetilde{\mathrm{Ham}}(M, \omega)$, we set

$$c(a, \varphi) = \inf \{\alpha \in \mathbb{R} \mid \Phi(a) \in \text{Image } i_\alpha\},$$

where $\Phi\colon QH_*(M) \to HF_*(\varphi)$ is the PSS isomorphism between the quantum homology and the Floer homology, and $i_\alpha\colon HF_\alpha(\varphi) \to HF_*(\varphi)$ is the natural inclusion in the filtered Floer homology. The non-trivial fact that $-\infty < c(a, \varphi) < \infty$ is proved in [39]. Moreover, $c(a, \varphi)$ has the following properties [39], [35]: For every $a, b \in QH_*(M)$ and every $\varphi, \psi \in \widetilde{\mathrm{Ham}}(M)$

(P1) $c(a * b, \varphi\psi) \leq c(a, \varphi) + c(b, \psi)$, (P2) $c(a, \mathbb{1}) = val(a)$,

(P3) $c(a, \varphi) = \sup_m c(a^{[m]}, \varphi)$,

(P4) $c(aq^A, \varphi) = c(a, \varphi) + \omega(A)$, for every $q^A \in \Lambda$,

where $a^{[m]}$ is the grade-m-component of a, $\mathbb{1}$ is the identity in $\widetilde{\mathrm{Ham}}(M, \omega)$ and $val(\cdot)$ is the valuation function (4.4) defined in Subsection 4.3.1.

The following lemma, which can be considered as a Poincaré duality type lemma, enables us to compare the spectral invariants of φ and φ^{-1}. It is the analogue of Lemma 2.2 from [15] in the rational non-monotone case .

Lemma 4.5.1. *For every $0 \neq \gamma \in QH_*(M)$ and every $\varphi \in \widetilde{\mathrm{Ham}}(M, \omega)$*

$$c(\gamma, \varphi) = -\inf \left\{ c(\delta, \varphi^{-1}) \mid \Pi(\delta, \gamma) \neq 0 \right\},$$

where $\Pi(\cdot, \cdot)$ is the $\mathbb{Q}$-valued pairing (4.3) *defined in Subsection 4.3.1.*

The proof of the lemma is given in Section 4.8 below. In order to prove Theorem 4.1.3 we will also need the following proposition. Assume that the subalgebra $QH_{2n}(M) \subset QH_*(M)$ is semi-simple over the field Λ_0 and let $QH_{2n}(M) = QH^1_{2n}(M) \oplus \cdots \oplus Q^k_{2n}(M)$ be a decomposition of $QH_{2n}(M)$ into a direct sum of fields. Then we have

Proposition 4.5.2. *There exists a positive constant $C \in \mathbb{R}$ such that for every $0 \neq \gamma \in QH^1_{2n}(M)$*

$$val(\gamma) + val(\gamma^{-1}) \leq C.$$

Postponing the proof of the above proposition we first present the proof of Theorem 4.1.3 and Theorem 4.1.1. In the proof of Theorem 4.1.3 we follow the strategy of the proof used by Entov and Polterovich in [15].

Proof of Theorem 4.1.3: Let $QH_{2n}(M) = QH^1_{2n}(M) \oplus \cdots \oplus Q^k_{2n}(M)$ be a decomposition of $QH_{2n}(M)$ into a direct sum of fields. Consider the map $\widetilde{r}\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ defined by:

$$\widetilde{r}(\varphi) = -\mathrm{vol}(M) \cdot \lim_{n \to \infty} \frac{c(e_1, \varphi^n)}{n},$$

where e_1 is the unit element of $QH^1_{2n}(M)$. This is a standard homogenization of the map $c(e_1, \cdot)\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$. We claim that $\widetilde{r}$ is a Lipschitz homogenous Calabi quasi-morphism. The proof of the Calabi property and the Lipschitz property of $\widetilde{r}$ goes along the same lines as the proof of Propositions 3.3 and 3.5 in [15] with the notations suitably adapted. Thus, we will omit the details of the proof of these properties and concentrate on proving that $\widetilde{r}$ is a quasi-morphism. We will show that $c(e_1, \cdot)$ is a quasi-morphism, this immediately implies that its homogenization $\widetilde{r}$ is also a quasi-morphism.

Notice that the upper bound follows easily from the triangle inequality (P1):

$$c\left(e_1, \varphi\psi\right) = c\left(e_1 * e_1, \varphi\psi\right) \leq c\left(e_1, \varphi\right) + c\left(e_1, \psi\right).$$

Next, it follows from (P1) and Lemma 4.5.1 that

$$c\left(e_1, \varphi\right) \;\leq\; c\left(e_1, \varphi\psi\right) + c(e_1, \psi^{-1}) = c\left(e_1, \varphi\psi\right) - \inf_{a:\Pi(a,e_1)\neq 0} c\left(a, \psi\right).$$

From the definition of the intersection pairing Π (4.3) we have that

$$\{a \mid \Pi(a, e_1) \neq 0\} = \{a \mid \Pi(a^{[0]}, e_1) \neq 0\} = \{a \mid \Pi(a^{[0]} * e_1, M) \neq 0\}$$

Combining this with the above property (P3) we may further estimate

$$c(e_1, \varphi) \leq c(e_1, \varphi\psi) - \inf_{a:\Pi(a^{[0]}*e_1,M)\neq 0} c(a^{[0]}, \psi). \tag{4.10}$$

Our next step is to find a lower bound for the term $c(a^{[0]}, \psi)$ provided that $\Pi(a^{[0]} * e_1, M) \neq 0$. For this, we shall first "shift" and then "project", roughly speaking, the element $a^{[0]} \in QH_0(M)$ to the field $QH^1_{2n}(M)$. More precisely, since we assumed that the minimal Chern number N_M divides n, there exists an element q^A in the Novikov ring Λ such that $a^{[0]}q^A \in QH_{2n}(M)$. Thus, it follows from properties (P1) and (P4) that

$$c(a^{[0]}, \psi) = c(a^{[0]}q^A, \psi) - \omega(A) \geq c(e_1 * a^{[0]}q^A, \psi) - c(e_1, \mathbb{1}) - \omega(A). \tag{4.11}$$

Moreover, it follows from the assumption $\Pi(a^{[0]} * e_1, M) \neq 0$ and from the definition of the element e_1, that $e_1 * a^{[0]}q^A \in QH^1_{2n}(M) \setminus \{0\}$. Hence, since $QH^1_{2n}(M)$ is a field, $e_1 * a^{[0]}q^A$ is an invertible element inside it. Using the triangle inequality (P1) once again we get

$$c(e_1, \psi) \leq c(e_1 * a^{[0]}q^A, \psi) + c\big((e_1 * a^{[0]}q^A)^{-1}, \mathbb{1}\big).$$

Here $(e_1 * a^{[0]}q^A)^{-1}$ is the inverse of $e_1 * a^{[0]}q^A$ inside QH^1_{2n}. Next, by substituting this in the above inequality (4.11) and applying (P2) we can conclude

$$c(a^{[0]}, \psi) \geq c(e_1, \psi) - val\big((e_1 * a^{[0]}q^A)^{-1}\big) - val(e_1) - \omega(A).$$

By assigning this lower bound of $c(a^{[0]},\psi)$ into (4.10) we further conclude

$$c(e_1,\varphi) \leq c(e_1,\varphi\psi) - c(e_1,\psi) + \sup_{a:\Pi(a^{[0]}*e_1,M)\neq 0} val\left((e_1 * a^{[0]}q^A)^{-1}\right) + C',$$

where C' is the value $val(e_1)+\omega(A)$. The last step of the proof is to find a universal upper bound for $val\big((e_1 * a^{[0]}q^A)^{-1}\big)$ provided that $\Pi(a^{[0]} * e_1, M) \neq 0$. Note that the condition $\Pi(a^{[0]} * e_1, M) \neq 0$ implies that $val\left(e_1 * a^{[0]}\right) \geq 0$ and hence $val\left(e_1 * a^{[0]}q^A\right) \geq -\omega(A)$. Therefore, it follows from Proposition 4.5.2 that $val\big((e_1 * a^{[0]}q^A)^{-1}\big) < C + \omega(A)$. We have shown that $c(e_1,\cdot)$ is a quasi-morphism, the proof of the theorem is thus complete.

Proof of Theorem 4.1.1: Let M be one of the manifolds $(X_\lambda,\omega_\lambda)$ or (Y_μ,ω_μ) listed in the theorem. It follows from Lemma 4.3.1 and Lemma 4.3.3 that the subalgebra $QH_4(M)$ is semi-simple. Moreover, the minimal Chern number of $(X_\lambda,\omega_\lambda)$ and (Y_μ,ω_μ) is 2 and 1 respectively. Thus, it follows from Theorem 4.1.3 that there exists a Lipschitz homogeneous Calabi quasi-morphism $r\colon \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ as required.

Remark 4.5.3. As mentioned in Remark 4.3.4 above, in the case of (Y_μ,ω_μ) where $0 < \mu < \frac{1}{3}$, the subalgebra $QH_4(Y_\mu)$ splits into direct sum of two fields. Thus, using the units of these fields alternately, Theorem 4.1.3 implies the existence of two Calabi quasi-morphisms. We do not know whether they are equivalent or not.

We return now to the proof of Proposition 4.5.2. We will follow closely Lemma 3.2 in [15].

Proof of Proposition 4.5.2: From the definition of the graded Novikov ring it follows that Λ_0 can be identified with the field $\mathcal{R} = \mathbb{Q}[[x]$ of Laurent series $\sum \alpha_j x^j$ with coefficients in $\mathbb{Q}$ and $\alpha_j = 0$ for large enough j's. Moreover, it is not hard to check that $QH_k(M)$ is a finite dimensional module over Λ_0. We denote by $\sigma\colon \mathcal{R} \to \mathbb{Z}$ the map which associates to a nonzero element $\sum \alpha_j x^j \in \mathcal{R}$ the maximal j, such that $\alpha_j \neq 0$. We set $\sigma(0) = -\infty$. For $\kappa \in \mathcal{R}$, put $|\kappa|_1 = \exp\,\sigma(\kappa)$. Thus, $|\cdot|_1$ is a non-Archimedean absolute value on $\mathcal{R}$ and moreover, $\mathcal{R}$ is complete with respect to $|\cdot|_1$. For preliminaries on non-Archimedean geometry we refer the readers to [3]. Since the field $QH^1_{2n}(M)$ can be considered as a finite dimensional vector space over $\mathcal{R}$, the absolute value $|\cdot|_1$ can be extended to an absolute value $|\cdot|_2$ on $QH^1_{2n}(M)$ (see [3]). Note that $|\cdot|_2$ induces a multiplicative norm $\|\cdot\|_2$ on $QH^1_{2n}(M)$. On the other hand, we can consider a different norm on $QH^1_{2n}(M)$ defined by $\|\gamma\|_3 = \exp\, val(\gamma)$. Since all the norms on a finite dimensional vector space are equivalent, there is a constant $C_1 > 0$ such that

$$\|\gamma\|_3 \leq C_1 \cdot \|\gamma\|_2, \text{ for every } 0 \neq \gamma \in QH^1_{2n}(M).$$

Hence, for $0 \neq \gamma \in QH^1_{2n}(M)$, we have

$$\|\gamma\|_3 \cdot \|\gamma^{-1}\|_3 \leq C_1^2 \cdot \|\gamma\|_2 \cdot \|\gamma^{-1}\|_2 = C_1^2.$$

Therefore, $val(\gamma) + val(\gamma^{-1}) \leq C$ where $C = 2 \log C_1$. This completes the proof of the proposition.

4.6 Restricting $\widetilde{r}$ to the fundamental group of $\mathrm{Ham}(M)$

In this section we discuss the restriction of the above mentioned Calabi quasi-morphism $\widetilde{r}\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$, where M is one of the manifolds listed in Theorem 4.1.1, to the abelian subgroup $\pi_1\big(\mathrm{Ham(M)}\big) \subset \widetilde{\mathrm{Ham}}(M)$. For this purpose, we follow [15] and use the Seidel representation $\Psi\colon\ \pi_1\big(\mathrm{Ham}(M)\big) \to QH^{\times}_{ev}(M,\mathbb{R})$ (see e.g. [56], [27]), where $QH^{\times}_{ev}(M,\mathbb{R})$ denotes the group of units in the even part of the quantum homology algebra of M with coefficients in a real Novikov ring. We start with the following preparation.

4.6.1 Hamiltonian fibrations over the two sphere

There is a one-to-one correspondence between homotopy classes of loops in $\mathrm{Ham}(M)$ and isomorphism classes of Hamiltonian fibrations over the two-sphere S^2 given by the following “clutching" construction (see e.g. [56], [27]). We assign to each loop $\varphi = \{\varphi_t\} \in \mathrm{Ham}(M)$ the bundle $(M,\omega) \to P_\varphi \to S^2$ obtained by gluing together the trivial fiber bundles $D^{\pm} \times M$ along their boundary via $(t,x) \mapsto (t, \varphi_t(x))$. Here we consider S^2 as $D^+ \cup D^-$, where $D^{\pm}$ are closed discs with boundaries identified with S^1. Moreover, we orient the equator $D^+ \cap D^-$ as the boundary of D^+. Note that this correspondence can be reversed.

As noted in [56], there are two canonical cohomology classes associated with such a fibration. One is the *coupling class* $u_\varphi \in H^2(P_\varphi,\mathbb{R})$ which is uniquely defined by the following two conditions: the first is that it coincides with the class of the symplectic form on each fiber, and the second is that its top power u_φ^{n+1} vanishes. The other cohomology class is the first Chern class of the vertical tangent bundle $c_\varphi = c_1(TP_\varphi^{\mathrm{vert}}) \in H^2(P_\varphi,\mathbb{R})$. We define an equivalent relation on sections of the fibration $P_\varphi \to S^2$ in the following way: First, equip S^2 with a positive oriented complex structure j, and P_φ with an almost complex structure J such that the restriction of J on each fiber is compatible with the symplectic form on it, and the projection $\pi\colon\ P_\varphi \to S^2$ is a (J,j)-holomorphic map. Next, two (J,j)-sections, ν_1 and ν_2, of $\pi\colon\ P_\varphi \to S^2$ are said to be Γ-*equivalent* if

$$u_\varphi[\nu_1(S^2)] = u_\varphi[\nu_2(S^2)], \quad c_\varphi[\nu_1(S^2)] = c_\varphi[\nu_2(S^2)].$$

It has been shown in [56] that the set $\mathcal{S}_\varphi$ of all such equivalent classes is an affine space modeled on the group Γ (4.1).

4.6.2 The Seidel representation

The following description of the Seidel representation, which is somehow different from Seidel's original work, can be found in [27]. For technical reasons, it will be more convenient to work in what follows with a slightly larger Novikov ring than in (4.2). More precisely, set $\mathcal{H}_{\mathbb{R}} := H_2^S(M,\mathbb{R}) \;/\; (\ker c_1 \cap \ker\omega)$, where $H_2^S(M,\mathbb{R})$ is the image of $\pi_2(M)$ in $H_2(M,\mathbb{R})$. We define the *real Novikov ring* as

$$\Lambda_{\mathbb{R}} = \Big\{ \sum_{A\in\mathcal{H}_{\mathbb{R}}} \lambda_A q^A \mid \lambda_A \in \mathbb{Q},\ \#\{A\in\Gamma \mid \lambda_A \neq 0,\ \omega(A) > c\} < \infty,\ \forall c\in\mathbb{R} \Big\},$$

and set $QH_*(M) := QH_*(M,\Lambda_{\mathbb{R}}) = H_*(M)\otimes\Lambda_{\mathbb{R}}$ to be the real quantum homology of M.

Next, let φ be a loop of Hamiltonian diffeomorphisms and ν be an equivalence class of sections of P_φ. Set $d = 2c_\varphi(\nu)$. We define a $\Lambda_{\mathbb{R}}$-linear map $\Psi_{\varphi,\nu}\colon\ QH_*(M) \to QH_{*+d}(M)$ as follows: for $a \in H_*(M,\mathbb{Z})$, $\Psi_{\varphi,\nu}$ is the class in $QH_{*+d}(M)$ whose intersection with $b \in H_*(M,\mathbb{Z})$ is given by

$$\Psi_{\varphi,\nu}(a)\cdot_M b = \sum_{B\in\mathcal{H}} n_{P_\varphi}(i(a), i(b); \nu + i(B))q^{-B},$$

where i is the homomorphism $H_*(M) \to H_*(P_\varphi)$, the intersection $\cdot_M$ is the linear extension to $QH_*(M)$ of the standard intersection pairing on $H_*(M,\mathbb{Q})$, and $n_{P_\varphi}(v,w;\mu)$ is the Gromov-Witten invariant which counts isolated J-holomorphic stable curves in P_φ of genus 0 and two marked points, such that each curve represents the equivalence class μ and whose marked points go through given generic representatives of the classes v and w in $H_*(P_\varphi,\mathbb{Z})$. When the manifold M is strongly semi-positive, these invariants are well defined. Moreover, it follows from Gromov's compactness theorem (see [20]) that for each given energy level k, there are only finitely many section-classes $\mu = \nu + i(B)$ with

$\omega(B) \leq k$ that are represented by a J-holomorphic curve in P_φ. Thus, $\Psi_{\varphi,\nu}$ satisfies the finiteness condition for elements in $QH_*(M)$.

For reasons of dimension, $n_{P_\varphi}(v, w; \mu) = 0$ unless $2c_\varphi(\mu) + \dim(v) + \dim(w) = 2n$. Thus,

$$\Psi_{\varphi,\nu}(a) = \sum a_{\nu,B}\, q^{-B}, \quad a_{\nu,B} \in H_*(M),$$

where $a_{\nu,B} \cdot_M b = n_{P_\varphi}(i(a), i(b); \nu + i(B))$, and

$$\dim(a_{\nu,B}) = \dim(a) + 2c_\varphi(\nu + i(B)) = \dim(a) + 2c_\varphi(\nu) + 2c_1(B).$$

Note also that $\Psi_{\varphi,\nu+A} = \Psi_{\varphi,\nu} \otimes q^A$. It has been shown by Seidel [56] (see also [27]) that $\Psi_{\varphi,\nu}$ is an isomorphism for all loops φ and sections ν.

Next, we use $\Psi_{\varphi,\nu}$ to define the Seidel representation $\Psi\colon\ \pi_1(\mathrm{Ham}(M)) \to QH_*(M, \Lambda_\mathbb{R})^\times$. In order to do so, we take a canonical section class ν_φ that (up to equivalence) satisfies the composition rule $\nu_{\varphi\psi} = \nu_\psi \# \nu_\varphi$, where $\nu_{\varphi\psi}$ denotes the obvious union of the sections in the fiber sum $P_{\psi\varphi} = P_\psi \# P_\varphi$. The section ν_φ is uniquely determined by the requirement that

$$u_\varphi(\nu_\varphi) = 0 \quad \textit{and} \quad c_\varphi(\nu_\varphi) = 0.$$

Moreover, it satisfies the above mentioned composition rule. Therefore, we get a group homomorphism

$$\rho\colon\ \pi_1(\mathrm{Ham}(M)) \to \mathrm{Hom}_{\Lambda_\mathbb{R}}(QH_*(M, \Lambda_\mathbb{R})).$$

It has been shown in [56] that for all $\varphi \in \pi_1(\mathrm{Ham}(M))$ we have $\rho(\varphi)(a) = \Psi_\varphi, \nu_\varphi([M]) *_M a$. The Seidel representation is defined to be the natural homomorphism

$$\Psi\colon\ \pi_1(\mathrm{Ham}(M)) \to QH_*(M, \Lambda_\mathbb{R})^\times,$$

given by $\varphi \mapsto \rho(\varphi)([M])$

4.6.3 Relation with the spectral invariant

Throughout, $\pi_1(\mathrm{Ham}(M))$ is considered as the group of all loops in $\mathrm{Ham}(M)$ based at the identity $\mathbb{1} \in \mathrm{Ham}(M)$. Let $\mathcal{L}$ be the space of all smooth contractible loops $x\colon\ S^1 = \mathbb{R}/\mathbb{Z} \to M$ and $\widetilde{\mathcal{L}}$ its cover introduced in Section 4.4 . Let φ be a loop of Hamiltonian diffeomorphisms. It is known (see e.g. [56]) that the orbits $\varphi_t(x)$ of φ are contractible. We consider the map $T_\varphi\colon \mathcal{L} \to \mathcal{L}$ which takes the loop $x(t)$ to $\varphi_t(x(t))$. In [56], Seidel showed that this action can be lifted (not uniquely) to $\widetilde{\mathcal{L}}$. In fact it is not hard to check that there is a one-to-one correspondence between such lifts of T_φ and equivalence classes of sections $\nu \in \mathcal{S}_\varphi$. We denote by $\widetilde{T}_{\varphi,\nu}$ the lift corresponding to $\nu \in \mathcal{S}_\varphi$. Next, let $\varphi \in \pi_1(\mathrm{Ham}(M))$ be a given loop generated by a normalized Hamiltonian $K \in \mathcal{H}$. The following formula, which can be found in [56] and [27], enables us to relate the Seidel representation with the spectral invariant c used to define the Calabi quasi-morphism $\widetilde{r}$:

$$(\widetilde{T}^*_{\varphi,\nu})^{-1}\mathcal{A}_H - \mathcal{A}_{K\sharp H} = -u_\varphi(\nu), \quad \text{for every } H \in \mathcal{H}. \tag{4.12}$$

It has been shown in [56] that the isomorphism in the quantum homology level described in Subsection 4.6.2, which is obtained by multiplication with $\Psi_\varphi, \nu_\varphi([M])$ corresponds, under the identification between the Floer and the quantum homology, to the isomorphism $i\colon HF_\alpha(H) \to HF_{\alpha+u_\varphi(\nu)}(K\#H)$ induced by the action of $(\widetilde{T}^*_{\varphi,\nu})$ on $\widetilde{\mathcal{L}}$. The following proposition can be found in [41] or [15].

Proposition 4.6.1. *For every loop $[\varphi] \in \pi_1(\mathrm{Ham}(M)) \subset \widetilde{\mathrm{Ham}}(M)$ and every $a \in QH_*(M)$ we have*

$$c(a, [\varphi]) = val(a * \Psi([\varphi])^{-1}).$$

Proof of Proposition 4.6.1:

Let $K \in \mathcal{H}$ be the normalized Hamiltonian function generating the loop $[\varphi]$, and let $H \in \mathcal{H}$ be the zero Hamiltonian generating the identity. The proposition immediately follows from (4.12) applied to H and K.

4.7 Proof of Theorem 4.1.2

Recall that a homogeneous quasi-morphism on an abelian group is always a homomorphism (see e.g. [15]). Hence, in order to prove Theorem 4.1.2, we need to show that for the manifolds listed in the theorem, the restriction of the Calabi quasi-morphism $\widetilde{r}\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ on the fundamental group of $\mathrm{Ham}(M)$ is non-trivial. We will divide the proof into two parts.

4.7.1 The case of $S^2 \times S^2$.

Let $X_\lambda = S^2 \times S^2$ be equipped with the split symplectic form $\omega_\lambda = \omega \oplus \lambda\,\omega$, where $1 < \lambda$. As mentioned in Section 4.1, there is an element $[\varphi]$ of infinite order in the fundamental group of $\mathrm{Ham}(X_\lambda)$ (see [32]). This element can be represented by the following loop of diffeomorphisms

$$\varphi_t(z, w) = \big(z, \Upsilon_{z,t}(w)\big),$$

where $\Upsilon_{z,t}$ denotes the $2\pi t$-rotation of the unit sphere S^2 around the axis through the points z, $-z$. Seidel showed in [56] (see also [36]), by direct calculation, that

$$\Psi([\varphi])^{-1} = (A - B) \otimes q^{\alpha A + \beta B}\big(\sum_{j=0}^{\infty} q^{j(A-B)}\big),$$

where A and B in $H_*(X_\lambda)$ are the classes of $[S^2 \times \text{point}]$ and $[\text{point} \times S^2]$ respectively, and $\alpha, \beta \in \mathbb{R}$ were chosen such that $2c_1(\alpha A + \beta B) = 1$ and $\omega_\lambda(\alpha A + \beta B) = \frac{1}{2} + \frac{1}{6\lambda}$.

Lemma 4.7.1. *For every $n \in \mathbb{N}$ we have*

$$val(\Psi([\varphi])^{-2n}) = 1 + \frac{n}{3\lambda}.$$

Proof First note that $val\big((A-B)^{2n}\big) = \max\{val(A^k B^{2n-k})\}$, where $0 \le k \le 2n$. Next, set $\alpha_k = A^k B^{2n-k}$. It follows from the quantum multiplication relations (4.5) that $val(\alpha_{k+2}) = val(\alpha_k) + (\lambda - 1)$ for every $0 \le k \le 2n-2$. Thus,

$$val\big((A-B)^{2n}\big) = \max\{val(A^{2n}), val(A^{2n-1}B)\} = -n + 1.$$

Set $\Delta = q^{\alpha A + \beta B}\big(\sum_{j=0}^{\infty} q^{j(A-B)}\big)$. It follow immediately that

$$val(\Delta^{2n}) = val\big(q^{2n(\alpha A + \beta B)}\big) = 2n\Big(\frac{1}{2} + \frac{1}{6\lambda}\Big) = n + \frac{n}{3\lambda}.$$

This completes the proof of the Lemma.

It follows from Lemma 4.3.1 that the subalgebra $QH_4(X_\lambda)$ is a field. Thus, combining Proposition 4.6.1 and Lemma 4.7.1, we conclude that

$$\widetilde{r}([\varphi]) = -\mathrm{vol}(X_\lambda) \cdot \lim_{n\to\infty} \frac{val\,(\Psi([\varphi])^{-2n})}{2n} = -\frac{1+\lambda}{6\lambda} \neq 0.$$

We have shown that the restriction of the quasi-morphism $\widetilde{r}\colon \widetilde{\mathrm{Ham}}(X_\lambda) \to \mathbb{R}$ on the fundametal group of $\mathrm{Ham}(X_\lambda)$ is non-trivial. This concludes the proof of Theorem 4.1.2 for the above case.

4.7.2 The case of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$

Let $Y_\mu = \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ be the symplectic one-point blow-up of $\mathbb{C}P^2$ introduced in Section 4.1, equipped with the symplectic form ω_μ, where $\frac{1}{3} \neq \mu \in (0,1)$. We will use here the same notation as in Subsection 4.3.3. It has been shown by Abreu-McDuff in [1] that the fundamental group of $\mathrm{Ham}(Y_\mu)$ is isomorphic to $\mathbb{Z}$ with a generator given by the rotation

$$\varphi : \ (z_1, z_2) \to (e^{-2\pi i t} z_1, z_2), \quad 0 \le t \le 1.$$

The Seidel representation of φ was computed in [33], [36] to be

$$\Psi([\varphi])^{-1} = P \otimes q^{E/2+3F/4-\delta(F-2E)}, \quad \text{where} \ \ \delta = \frac{(1-\mu)^2}{12(1+\mu)(1-3\mu)}. \tag{4.13}$$

The following lemma can be immediately deduced from Lemma 5.1 and Remark 5.5 which both appear in [33].

Lemma 4.7.2. *Let $\frac{1}{3} \neq \mu \in (0,1)$. Then*

$$\lim_{k\to\infty} \frac{val(\Psi([\varphi])^{-k})}{k} = \begin{cases} -\delta\,\omega(F-2E), & \frac{1}{3} < \mu < 1 \\ \\ \frac{12-\delta}{12}\,\omega(F-2E), & 0 < \mu < \frac{1}{3}. \end{cases} \tag{4.14}$$

Proof: Denote by Q the element $P \otimes q^{E/2+3F/4}$ and consider its powers Q^k where $k \in \mathbb{N}$. It follows from the quantum multiplicative relations discussed in Subsection 4.3.2 that the only two possible cycles obtained by multiplication by Q are

$$P \otimes q^{E/2+3F/4} \to E \otimes q^{F/2} \to F \otimes q^{E/2+F/4} \to M \to P \otimes q^{E/2+3F/4},$$

and

$$P \otimes q^{E/2+3F/4} \to F \otimes q^{F/2} \to M \otimes q^{F/4-E/2} \to P \otimes q^{F}.$$

Thus, since the first cycle does not change the valuation, while the second cycle increases it by $\omega(F/4-E/2)$, we have that $val(Q^k)$ is either bounded as $k \to \infty$ when $\omega(F/4-E/2) < 0$ or linearly grows otherwise. Hence, we get that

$$val(\Psi([\varphi])^{-k}) = \begin{cases} C - \delta\, k\, \omega(F-2E), & \frac{1}{3} < \mu < 1 \\ \\ C + \frac{k}{3}\,\omega(F/4-E/2) - \omega(\delta(F-2E)), & 0 < \mu < \frac{1}{3}, \end{cases}$$

where C is some universal constant. This completes the proof.

A straightforward calculation shows that the above expression (4.14) is strictly negative for every $0 < \mu < 1$. Thus, it follows from Proposition 4.6.1 and the fact that $val(a * b) \leq val(a) + val(b)$ that

$$\widetilde{r}([\varphi]) = -\mathrm{vol}(Y_\mu) \lim_{k\to\infty} \frac{val\left(e_1 * \Psi([\varphi])^{-k}\right)}{k} \geq -\mathrm{vol}(Y_\mu) \lim_{k\to\infty} \frac{val\left(\Psi([\varphi])^{-k}\right)}{k} > 0$$

Hence, the restriction of the quasi-morphism $\widetilde{r}\colon \widetilde{\mathrm{Ham}}(Y_\mu) \to \mathbb{R}$ on the fundamental group of $\mathrm{Ham}(Y_\mu)$ is non-trivial. The proof of Theorem 4.1.2 is now complete.

4.8 Proof of the Poincaré duality lemma

Let (M, ω) be a closed, rational and strongly semi-positive symplectic manifold of dimension $2n$. Note that for rational symplectic manifolds the action spectrum is a discrete subset of $\mathbb{R}$, and thus there are only a finite number of critical values of the action functional $\mathcal{A}_H$ in any finite segment $[a, b] \subset \mathbb{R}$. Let $H(t, x) \in \mathcal{H}$ be a Hamiltonian function generating $\varphi \in \widetilde{\mathrm{Ham}}(M)$, and denote by $\widetilde{H}(t, x) = -H(-t, x)$ the Hamiltonian function generating the inverse symplectomorphism φ^{-1}. The set $\mathcal{P}_H$ of critical points of $\mathcal{A}_H$ is isomorphic to $\mathcal{P}_{\widetilde{H}}$ via $\widetilde{x}(t) = x(-t)$, and $[x, u] \in \widetilde{\mathcal{P}_H}$ corresponds to $[\widetilde{x}, \widetilde{u}] \in \widetilde{\mathcal{P}_{\widetilde{H}}}$ where

$$\widetilde{u}(s,t) = u(-s,-t), \quad \mu([\widetilde{x}, \widetilde{u}]) = 2n - \mu([x,u]) \ \text{ and } \ \mathcal{A}_H([x,u]) = -\mathcal{A}_{\bar{H}}([\widetilde{x}, \widetilde{u}]).$$

We define a pairing $L\colon CF_k(H) \times CF_{2n-k}(\widetilde{H}) \to \Lambda_0$ by

$$L\left(\sum \alpha_{[x,u]} \cdot [x,u], \sum \beta_{[\widetilde{y},\widetilde{v}]} \cdot [\widetilde{y}, \widetilde{v}]\right) = \sum_{A} \Big(\sum_{[x,u]} \alpha_{[x,u]} \cdot \beta_{[x,u\sharp-A]} \Big) q^A, \tag{4.15}$$

where the inner sum runs over all pairs $[x, u] \in \widetilde{\mathcal{P}_H}$ and the outer sum runs over all $A \in \Gamma$ with $c_1(A) = 0$. The pairing L is well defined. Indeed, consider first the inner sum, the finiteness condition in the definition of $CF_*(H)$ implies that it contains only finitely many elements. Secondly, it follows from the same reason that the power series on the right hand side of (4.15) satisfies the finiteness condition from the definition of the Novikov ring (4.2). It is not hard to check that the pairing L is linear over Λ_0 and that it is non-degenerate in the standard sense. Thus, since the vector spaces $CF_k(H)$ and $CF_{2n-k}(\widetilde{H})$

are finite dimensional over Λ_0, which is in our case a field, the pairing L determines an isomorphism

$$CF_k(H) \simeq \mathrm{Hom}_{\Lambda_0}\left(CF_{2n-k}(\widetilde{H}), \Lambda_0\right).$$

From the universal coefficient theorem we obtain a Poincaré duality isomorphism

$$HF_k(H) \simeq \mathrm{Hom}_{\Lambda_0}\left(HF_{2n-k}(\widetilde{H}), \Lambda_0\right).$$

In [50] it has been shown that the pairing determined by this isomorphism, which by abuse of notation we also denote by L, agrees with the intersection pairing $\Delta(\cdot,\cdot)$ on the Quantum homology $QH_*(M)$. More precisely, let $\Phi\colon QH_*(M) \to HF_*(H)$ be the PSS isomorphism described in Section 4.4. Then, for every $a \in HF_k(H)$ and $b \in QH_{2n-k}(M)$ we have

$$\Delta(\Phi^{-1}(a), b) = L\left(a, \Phi(b)\right). \tag{4.16}$$

Next, we consider the filtered Floer homology complexes $CF_k^{(-\infty,\alpha]}(H)$ and $CF_{2n-k}^{(-\alpha,\infty]}(\widetilde{H})$. Note that these spaces are no longer vector spaces over Λ_0 since they are not closed with respect to the operation of multiplication by a scalar. We define a $\mathbb{Q}$-valued pairing $L'\colon CF_k^{[-\infty,\alpha]}(H) \times CF_{2n-k}^{(\alpha,\infty]}(\widetilde{H}) \to \mathbb{Q}$ by

$$L'\left(\sum \alpha_{[x,u]}\cdot[x,u], \sum \beta_{[\widetilde{y},\widetilde{v}]}\cdot[\widetilde{y},\widetilde{v}]\right) = \sum_{[x,u]}(\alpha_{[x,u]}\cdot\beta_{[x,u]}).$$

This pairing is well defined since any element in $CF_{2n-k}^{(-\alpha,\infty]}(\widetilde{H})$ is a finite sum. It is straightforward to check that the pairing L' is non-degenerate in the standard sense and that it coincides with the zero term of L. In other words, denote by $\tau\colon \Lambda_0 \to \mathbb{Q}$ the map sending $\sum a_A q^A$ to a_0, then for any $a \in CF_k^{(-\infty,\alpha]}(H)$ and $b \in CF_{2n-k}(\widetilde{H})$ we have

$$\tau L\left(i_{\alpha,H}(a), b\right) = L'(a, \pi_{-\alpha,\widetilde{H}}(b)). \tag{4.17}$$

By abuse of notation, we also denote by L' the induced pairing in the homology level: $L'\colon HF_k^{(-\infty,\alpha]}(H) \times HF_{2n-k}^{(-\alpha,\infty]}(\widetilde{H}) \to \mathbb{Q}$. Next, consider the following diagram:

$$\begin{array}{ccccc}
QH_k(M) & \xleftarrow{\Phi^{-1}} & HF_k(H) & \xleftarrow{i_{\alpha,H}} & HF_k^{(-\infty,\alpha]}(H) \\
\times_\Pi & & \times_{\tau L} & & \times_{L'} \\
QH_{2n-k}(M) & \xrightarrow{\Phi} & HF_{2n-k}(\widetilde{H}) & \xrightarrow{\pi_{-\alpha,\widetilde{H}}} & HF_{2n-k}^{(-\alpha,\infty]}(\widetilde{H}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Q} & & \mathbb{Q} & & \mathbb{Q}
\end{array}$$

Combining equations (4.16) and (4.17) together we conclude that for every element $a \in HF_k^{(-\infty,\alpha]}(H)$ and $b \in QH_{2n-k}(M)$ we have

$$\Pi(\Phi^{-1} \circ i_{\alpha,H}(a), b) = \tau L(i_{\alpha,H}(a), \Phi(b)) = L'(a, \pi_{-\alpha,\widetilde{H}} \circ \Phi(b)) \tag{4.18}$$

We are now in a position to prove Lemma 4.5.1.

Proof of Lemma 4.5.1 We divide the proof into two steps.

1. Fix an arbitrary $\varepsilon > 0$ and put $\alpha = \varepsilon - c(\gamma, \varphi)$. It follows from the definition of the spectral invariant c that $\Phi(\gamma) \notin \text{Image } i_{-\alpha,\varphi}$. Note that $\text{Image } i_{-\alpha,\varphi} = \text{Kernel } \pi_{-\alpha,\varphi}$ and thus $\xi := \pi_{-\alpha,\varphi} \circ \Phi(\gamma) \neq 0$. Since the pairing L' is non-degenerate there exists $\eta \in HF_{2n-*}^{(-\infty,\alpha]}(\varphi^{-1})$ such that $L'(\eta, \xi) \neq 0$. From (4.18) we have that $\Pi(\delta_0, \gamma) \neq 0$, where $\delta_0 = \Phi^{-1} \circ i_{\alpha,\varphi^{-1}}(\eta)$. It follows from the definition that $c(\delta_0, \varphi^{-1}) \leq \alpha$ and hence
$$\inf_{\delta:\Pi(\delta,\gamma)\neq 0} c(\delta, \varphi^{-1}) \leq c(\delta_0, \varphi^{-1}) \leq \alpha = \varepsilon - c(\gamma, \varphi)$$
This inequality holds for every $\varepsilon > 0$, hence we conclude that
$$\inf_{\delta:\Pi(\delta,\gamma)\neq 0} c(\delta, \varphi^{-1}) \leq -c(\gamma, \varphi).$$

2. Fix an arbitrary $\varepsilon > 0$ and put $\alpha = -\varepsilon - c(\gamma, \varphi)$. From the definition of $c(\cdot, \cdot)$ it follows that $\Phi(\gamma) \in \text{Image } i_{-\alpha,\varphi} = \text{Kernel } \pi_{-\alpha,\varphi}$. Hence, $\xi := \pi_{-\alpha,\varphi} \circ \Phi(\gamma) = 0$. Assume by contradiction that there exists δ satisfying $\Pi(\delta, \gamma) \neq 0$ such that $c(\delta, \varphi^{-1}) < \alpha$. We observe that $\Phi(\delta) \in \text{Image } i_{\alpha,\varphi^{-1}}$. Let $\eta \in HF_{2n-*}^{(-\infty,\alpha]}(\varphi^{-1})$ be such that $\Phi(\delta) = i_{\alpha,\varphi^{-1}}(\eta)$. It follows from (4.18) that $\Pi(\delta, \gamma) = L'(\eta, \xi) = 0$. This contradicts the above assumption that $\Pi(\delta, \gamma) \neq 0$. Thus we must have $c(\delta, \varphi^{-1}) \geq \alpha$ for every δ satisfying $\Pi(\delta, \gamma) \neq 0$. Hence,
$$\inf_{\delta:\Pi(\delta,\gamma)\neq 0} c(\delta, \varphi^{-1}) \geq \alpha = -\varepsilon - c(\gamma, \varphi).$$
Again, since this inequality holds for every $\varepsilon > 0$ we conclude that
$$\inf_{\delta:\Pi(\delta,\gamma)\neq 0} c(\delta, \varphi^{-1}) \geq -c(\gamma, \varphi).$$

The proof is now complete.

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