Random walks on planar maps and Liouville BM

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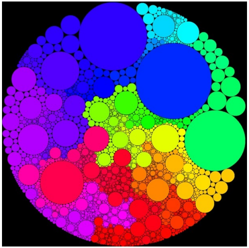


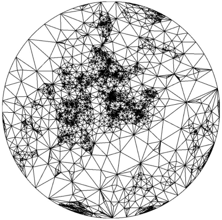
Horowitz seminar, Tel-Aviv (online)

* on leave from University of Cambridge

Motivation

Benjamini and Schramm 2001

Q: What does a random walk on a random planar map look like?



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UIPT (Angel–Schramm, 2003): local limit of random triangulations with n faces, $n \to \infty$.

Motivation

Ambjørn, Watabiki (nonrigourous): "Spectral dim of LQG" $= 2$ i.e., $p_t(x,x) \approx 1/t$.

Some remarkable results

Theorem (Benjamini and Schramm (2001))

Subject to bounded degree, **every** *local limit of a sequence of planar maps (rooted at a randomly chosen vertex) is recurrent.*

Theorem (Gurel-Gurevitch and Nachmias (2013))

Exponential tail on degree suffices. In particular, **UIPT** *is recurrent.*

See Nachmias' Saint Flour notes for recent survey.

This talk: scaling limits of random walks.

DDK Ansatz (cf. Duplantier–Sheffield)

Metric of the form

$$ds^2 = e^{\gamma h}(dx^2 + dy^2).$$

where $\gamma \in \mathbb{R} \leftrightarrow$ **central charge**; h is the **Gaussian free field**.

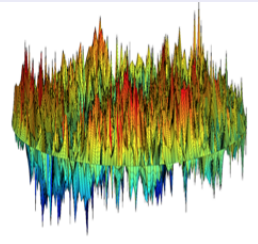
Gaussian free field

A random "function" $D \to \mathbb{R}$

$$(h, f) \sim \mathcal{N}(0, \sigma^2);$$

$$\sigma^2 = \iint G(x,y) f(x) f(y) dx dy$$

where $G =$ Green function on D.



Continuum theory ???

That gives $h(z)$, but what about $e^{\gamma h(z)}$?

Three objects:

A Riemannian **metric** on sphere $\mathbb{S}^2$ or domain $D \subset \mathbb{C}$	$e^{\gamma h(z)}(dx^2 + dy^2)$
A volume **measure**	$e^{\gamma h(z)} dz$
A **diffusion** (Brownian motion on surface)	$dZ_t = e^{-\frac{\gamma}{2} h(Z_s)} dB_s.$

Continuum theory !!!

Three objects:

A (Riemannian) **metric** on the sphere $\mathbb{S}^2$ or domain $D \subset \mathbb{C}$	$e^{\frac{\gamma}{d_\gamma} h(z)}(dx^2 + dy^2)$	V*
A volume **measure**	$e^{\gamma h(z)} dz$	V
A **diffusion** (Brownian motion on surface)	$dZ_t = e^{-\frac{\gamma}{2} h(Z_s)} dB_s.$	V

$*$: announced in 2019 by Gwynne–Miller x5; Dubédat–Falconnet–Gwynne–Pfeffer–Sun
For $\gamma = \sqrt{8/3}$, already known by Miller–Sheffield.

Continuum theory: volume measure

Consider a regularisation $h_\varepsilon(z)$, e.g., the circle average value of h.

Theorem (Kahane 1985; Duplantier–Sheffield 2010; Shamov 2017; B. 2017)

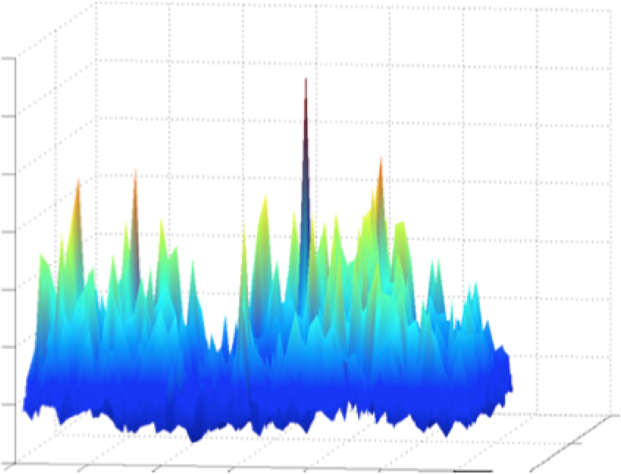
Define:

$$\mu_\gamma(S) = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \int_S e^{\gamma h_\varepsilon(z)} dz$$

exists in probability.

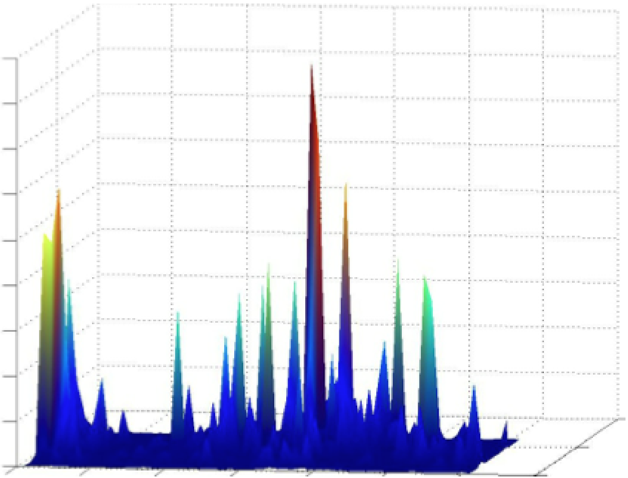
μ_γ *is* **Gaussian multiplicative chaos** *associated to GFF.*

Visualisation of Liouville measure



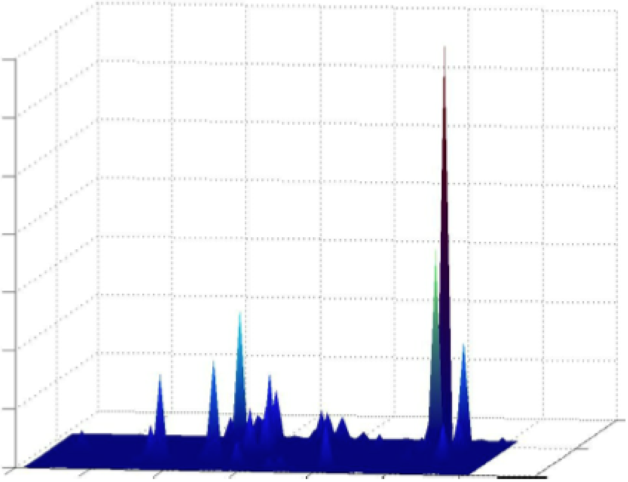
$\gamma = 0.2$

Visualisation of Liouville measure



$\gamma = 1$

Visualisation of Liouville measure



$\gamma = 1.8$

Liouville Brownian motion

Question:

How to define a canonical Brownian motion in this surface?

- In Riemannian case, metric defines smooth "connection"
- This induces a Laplace–Beltrami operator Δ.
- Brownian motion on manifold defined in terms of Δ.

Problem

Here none of these tools apply. We have to invent something else!

Liouville Brownian motion

Instead we have to do a regularisation procedure again.

Theorem (B. 2015, Garban–Rhodes–Vargas 2018)

The ε-regularised Brownian motion converges to a process, **Liouville Brownian motion**.

ε-regularised Liouville Brownian motion:

$$dZ_s^\varepsilon = \varepsilon^{\gamma^2/4} e^{-\frac{\gamma}{2} h_\varepsilon(Z_s)} dB_s$$

In other words,

$$Z_s^\varepsilon = B_{\phi_\varepsilon^{-1}(t)}; \phi_\varepsilon(t) = \varepsilon^{\gamma^2/2} \int_0^t e^{\gamma h_\varepsilon(B_s)} ds$$

Then $\lim_{\varepsilon \to 0} Z_s^\varepsilon$ exists (i.e., $\lim_{\varepsilon \to 0} \phi_\varepsilon(t)$ exists).

Liouville Brownian motion

Properties (Garban, Rhodes, Vargas, B., Jackson)

• Continuous; does not stay stuck

• μ_h is a.s. an invariant measure for Liouville Brownian motion given h.

• Spends all its time in a set of measure zero.

• for $\gamma > \sqrt{2}$, the trajectory has zero derivative at almost every time.

• Spectral dimension $d_s = 2$ a.s. ($\rightarrow$ Ambjørn).

• ...

Connection between discrete and continuum theories?

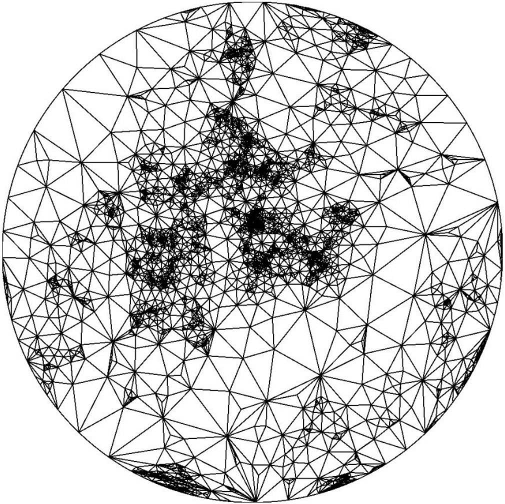
• Two stories should be "two sides of same coin".

• Need good embedding of planar maps in Euclidean space:

• Riemann mapping theorem, or Circle Packing theorem of Koebe–Andreev–Thurston; or Tutte embedding *.

$*$ two cases: infinite maps / finite maps with boundary.

Tutte embedding



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A big conjecture.

Let T_n = uniform random triangulation,
$\psi : T_n \to \mathbb{C}$ some nice embedding (circle packing, Tutte)
Let μ_n = measure putting mass $1/n$ at each centre.

Conjecture

μ_n converges in distribution to Liouville measure with $\gamma = \sqrt{8/3}$.
Moreover, if X = SRW on T_n then $\psi(X)$ converges to Liouville Brownian motion.

Main result

The punchline

Jointly with Ewain Gwynne we prove the first such result for a class of planar maps called **mated-CRT planar maps**.

Mated-CRT planar maps

- Nice discretisations of LQG;
- Coarse-grained versions of more natural models of planar maps (such as **UIPT** for $\gamma = \sqrt{8/3}$).

Mated-CRT maps

Two flavours

Finite; with boundary $\rightarrow$ disc topology;
or infinite; no boundary $\rightarrow$ spherical topology (=whole plane)

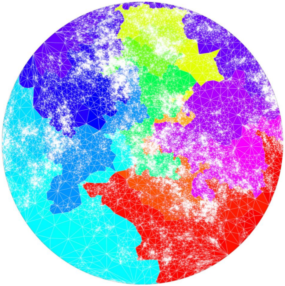
Each case has two equivalent descriptions:

- using **SLE/LQG theory**
- or as **topological gluing of a pair of CRTs**.

(Equivalence: Duplantier–Miller–Sheffield.)

Description 1 (full plane)

Fix $\gamma \in (0,2)$.
Let $h =$ **quantum cone**: roughly, $h = GFF + \gamma \log(1/|z|)$ in $\mathbb{C}$.
Let $\eta =$ space-filling $\mathbf{SLE}_\kappa$ where $\kappa = 16/\gamma^2 \in (4,\infty)$.
For any $\varepsilon > 0$, break $\mathbb{C}$ into cells $\eta([t_n, t_{n+1}])$ of μ_h-mass ε.



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$\mathcal{G}^\varepsilon$: adjacency graph.

Description 2

Let (L, R) be a pair of correlated two-sided Brownian motions;

$$\mathrm{Cov}(L_t, R_t) = -\cos\left(\frac{4\pi}{\gamma^2}\right)|t|.$$

Glue the associated CRTs to one another...

Bijections (Sheffield, Bernardi, Holden–Sun,...)

Natural models of planar maps close to Description 2

Statement of result

Let $X^{z,\epsilon}$ = RW from z on $\mathcal{G}^\varepsilon$. Rescaling

$$m_\epsilon := \left(\text{median exit time of } \eta(X^{0,\epsilon}) \text{ from } B_{1/2}\right)$$

We can show $m_\varepsilon \asymp \varepsilon^{-1}$.

Theorem (B.–Gwynne 2020)

$\forall z \in \mathbb{C}$, conditional law of $(\eta(X^{z,\epsilon}_{m_\epsilon t}))_{t\geq 0}$ given $(h,\eta))$ converges in probability to rescaled law of γ-LBM from z associated with h.

(Prokhorov topology induced by local uniform metric on curves $[0,\infty) \to \mathbb{C}$.) In fact, convergence is uniform over z in any compact subset of $\mathbb{C}$.
Also true for Tutte embedding in the disc case.

Rough sketch of argument

1. Previous work

By work of Gwynne–Miller–Sheffield (2018), Tutte embedding ψ converges to LQG:

$$\sup_{x \in \mathcal{V}\mathcal{G}^{\varepsilon}} |\psi(x) - \eta(x)| \to 0$$

in probability.
Furthermore: up to parametrization, RW converges to BM.

So, "only" only need to deal with parametrisation.

2. Tightness

Let $B = B_x(r)$ be a Euclidean ball, τ^ε = exit time by RW of B.

Moments $\to$ **Green function** G^ε: eg,

$$\mathbb{E}(\tau^\varepsilon) = \sum_{y \in B} G_B^\varepsilon(x, y) \overset{?}{\asymp} \varepsilon^{-1} \int_B G_B(x, y) \mu_h(dy)?$$

Main issue: $G^\varepsilon(x, y) \asymp G(x, y)$ (uniformly if possible).

Already know (bounded) discrete harmonic functions converge to continuum harmonic functions.

GMS (2018) On-diagonal estimates: $G(x, x) \leq \varepsilon^{o(1)}$ uniformly whp.

Energy estimates

Electrical network theory (Grigoryan's lemma)
$+$ Harnack inequality: $G_\varepsilon(x,y)$ relates to R^ε_{eff}.

Hence find f^ε minimising Dirichlet energy $\mathcal{E}(f^\varepsilon, f^\varepsilon)$ subj. to boundary conditions.

Can use continuum guess to test!

Obtain $G^\varepsilon(x,y) \asymp G_B(x,y)$ for $|y-x| \geq \varepsilon^\beta$, for some small β.

Use naive diagonal bound in $B(x,\varepsilon^\beta)$ and *a priori uniform polynomial control* for GMC.

3. Characterisation of LBM

Suppose $\forall z$ we have a law P_z such that $z \mapsto P_z$ is continuous and:

- P_z is Markovian given h;
- P_z is a (random) time-change of BM from z;
- P leaves the Liouville measure $\mu = \mu_h$ associated to h invariant.

Theorem (B.–Gwynne 2020)

Then there is a (possibly random) constant c s.t. P_z = law of $(X^z_{ct}, t \geq 0)$, where X^z is LBM associated with h, starting from z.

Key idea: Revuz measure.

Definition (Positive Continuous Additive Functional (PCAF))

Let $\mathcal{A} = (\mathcal{A}_t(\omega); t \geq 0)$ be a functional on path space. $\mathcal{A}$ is a PCAF for the Markov process X if

- $\mathcal{A}$ is increasing in t
- $\mathcal{A}_t = \mathcal{A}_s + \mathcal{A}_{t-s} \circ \theta_s$ for every s, t.

In words $\mathcal{A}$ increases in a way that depends only on the future of the trajectory.

Ex: $F(t) = \int_0^t e^{\gamma h(B_s)} ds$ the Liouville clock (given h) is a PCAF for B.

Revuz measure of a PCAF

Definition

A measure μ is a Revuz measure for X is $\mu(A) = 0$ whenever X does not hit A a.s.

Theorem (Revuz; Fukushima)

For each PCAF of a Hunt process there exists a unique Revuz measure μ such that for all test functions f, g:

$$\int_{\mathbb{C}} \mathbb{E}_z \left[\int_0^t g(B_s) d\mathcal{A}_s \right] f(z) dz = \int_0^t \left[\iint f(z) p_s(z,w) \mu(dw) g(w) dz \right] ds.$$

Moreover μ determines $\mathcal{A}$ uniquely.

In words, $d\mathcal{A}_t =$ "$\mu(B_t)dt$" (in an integrated sense)

Here:

Let F be the time-change of $Z \sim P_z$ so that $B_t = Z_{F(t)}$.

Lemma

F is a PCAF for B.

Let $\mu =$ its Revuz measure. It **suffices** to show $\mu =$ Liouville μ_h !

One can check μ is necessarily invariant, but so is μ_h (assumption).

Z is strongly Feller so invariant measure is unique up to constants!