Massless phases for the Villain model in $d \geq 3$

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joint work with Wei Wu (University of Warwick)

Abelian spin model

- Ising model
- XY model
- Villain model
- $\mathbb{Z}_n$ (clock) model
- Abelian Higgs model
- Abelian lattice gauge theory
- etc.

The classical XY model

- $\Lambda \subset \mathbb{Z}^d$, $S_j = (\cos\theta_j, \sin\theta_j) \in \mathbb{S}^1$,

$$\boxed{\mathbb{P}_{\Lambda}^{XY}(dS) \sim e^{\beta \sum_{i\sim j} S_i \cdot S_j} \prod_{j\in\Lambda} d\theta_j \sim e^{\beta \sum_{i\sim j} \cos(\theta_i - \theta_j)} \prod_{j\in\Lambda} d\theta_j.}$$

- Modeling for liquid helium, magnetic insulator, superconductor etc.
- Infinite volume limit well-defined
  - Ginibre;
  - Bricmont-Fontaine-Landau;
  - Messager-Miracle-Sole-Pfister.

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The Villain model

- For $\beta \gg 1$, using

$$e^{\beta\left(\cos\left(\theta_i-\theta_j\right)-1\right)} \approx \underbrace{\sum_{m\in\mathbb{Z}} e^{-\frac{\beta}{2}\left(\theta_i-\theta_j+2\pi m\right)^2}}_{\text{Heat kernel on } \mathbb{S}^1},$$

obtain the following Villain approximation of the XY model:

$$\boxed{\mathbb{P}_\Lambda^{Vil} \sim \prod_{e=(i,j)} \sum_{m\in\mathbb{Z}} e^{-\frac{\beta}{2}\left(\theta_i-\theta_j+2\pi m\right)^2} \prod_{j\in\Lambda} d\theta_j.}$$

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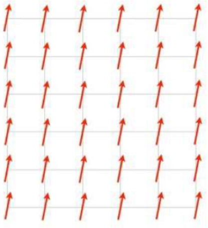
Phase transitions

Phase transition is often accompanied with broken of symmetry.



Ferromagnet

The XY/Villain model

- For $d = 2$, continuous symmetry is not broken at any temperature. Phase transition characterized by *topological defects*. Berezinskii-Kosterlitz-Thouless transition.
- For $d \geq 3$, symmetry breaking at low temperature. Macroscopic excitation/fluctuations are given by "spin wave function".

Dimension higher than 3: Gaussian domination

Theorem: Long-range order (Fröhlich-Simon-Spencer 1976)

For any $d \geq 3$, let us define the long-range order parameter

$$c = \lim_{x \to \infty} \underbrace{\langle S_0 \cdot S_x \rangle_{XY}}_{\text{two-point function}},$$

then we have

$$c \geq 1 - \frac{2G(0)}{\beta} > 0 \quad \text{(if } \beta \text{ is large)}$$

Question: Asymptotic behavior of the two-point function?

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Low temperature phase: Spin-wave approximation

For $\beta \gg 1$, use the **Gaussian approximation:**

- $\beta \sum_{i\sim j} \cos\left(\theta_i - \theta_j\right) \approx Const - \frac{\beta}{2} \sum_{i\sim j} (\theta_i - \theta_j)^2.$
- $\mathbb{P}_\Lambda^{XY/Vil} \sim e^{-\frac{\beta}{2}\sum_{i,j}\left(\theta_i - \theta_j\right)^2} \prod d\theta_j \implies$ GFF!
- $\left\langle S_0 \cdot S_x \right\rangle_{XY/Vil} = \left\langle e^{i(\theta_0 - \theta_x)} \right\rangle_{XY/Vil} \approx e^{-(G(0)-G(x))/\beta}.$

This gives in $d \geq 3$, using $G(x) \approx \frac{C_1}{|x|^{d-2}}$,

$$\boxed{\left\langle S_0 \cdot S_x \right\rangle_\beta \approx e^{-G(0)/\beta} + \frac{C_1}{\beta |x|^{d-2}}.}$$

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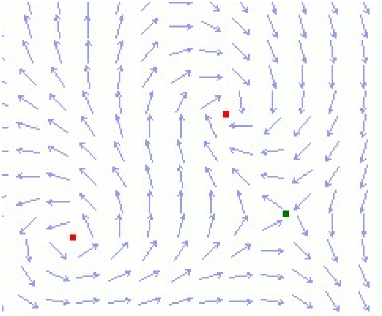
The spin-wave approximation ignores the vortices (only heuristic!)



3D Results and Conjectures

- Conjecture: For $d \geq 3$, there exists $\beta_c(d) \in (0, \infty)$, such that for $\beta > \beta_c(d)$,

$$\boxed{\langle S_0 \cdot S_x \rangle = \langle S_0 \rangle^2 + \frac{c_{eff}}{|x|^{d-2}} + o\left(\frac{1}{|x|^{d-2}}\right).}$$

- Fröhlich-Spencer 84: For $\beta \gg 1$,

$$\exp\left(-\frac{1}{2\beta}(G(0) - G(x))\right) \geq \langle S_0 \cdot S_x \rangle$$
$$\geq \exp\left(\left(-\frac{1}{2\beta} + o\left(\frac{1}{\beta}\right)\right)(G(0) - G(x))\right).$$

- Bricmont-Fontaine-Lebowitz-Lieb-Spencer 81: For $\beta \gg 1$,

$$\frac{c_2}{|x|^{d-2}} \leq \langle \sin\theta(0) \sin\theta(x) \rangle \leq \frac{c_1}{|x|^{d-2}}.$$

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Theorem (D.-Wu 20)

For $\beta \gg 1$, there exists a constant $c_{eff}(d, \beta) \in \mathbb{R}$ and an exponent $\alpha(d) > 0$ such that

$$\langle S_0 \cdot S_x \rangle_{Vil} = \langle S_0 \rangle_{Vil}^2 + \frac{c_{eff}}{|x|^{d-2}} + O\left(\frac{1}{|x|^{d-2+\alpha}} \right).$$

Remark

The constant c_{eff} is **not** equal to the constant C_1/β:

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Vector calculus in dimension 3

Two types of objects:

- Functions: $f : \mathbb{Z}^3 \to \mathbb{R}$

$$\begin{cases} \nabla f : E(\mathbb{Z}^3) \to \mathbb{R}, \\ \quad e = (x, y) \mapsto f(x) - f(y). \end{cases}$$

- Vector fields: two possibilities
  - Functions defined on the edges of $\mathbb{Z}^3$ and valued in $\mathbb{R}$;
  - Functions defined on $\mathbb{Z}^3$ and valued in $\mathbb{R}^3$.

$$\Longleftrightarrow \qquad \vec{F}(x) = \begin{pmatrix} \vec{F}(e_1) \\ \vec{F}(e_2) \\ \vec{F}(e_3) \end{pmatrix} \in \mathbb{R}^3.$$

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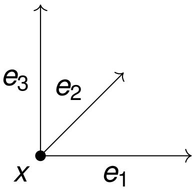

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This notations allows to define the standard differential operators in $\mathbb{R}^3$:

- The gradient: Functions $\to$ Vector fields;
- The divergence: Vector fields $\to$ Functions;
- The curl: Vector fields $\to$ Vector fields;
- The Laplacian:
  - Functions $\to$ Functions: $\Delta f(x) = \sum_{y \sim x} f(y) - f(x)$;
  - Vector fields $\to$ Vector fields:

$$\vec{F} = (F_1, F_2, F_3) \longmapsto \Delta \vec{F} = (\Delta F_1, \Delta F_2, \Delta F_3).$$

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A standard identity:

- For vector fields $\operatorname{div} \operatorname{curl} = 0$.

Helmholtz-Hodge decomposition

Any vector field $\vec{v}$ which is in $L^2\left(\mathbb{Z}^3\right)$ can be decomposed

$$\vec{v} = \underbrace{\nabla \phi}_{\text{potential field}} + \underbrace{\vec{\psi},}_{\operatorname{div} \vec{\psi} = 0}$$

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The Villain model and Coulomb gas

$$Z = \int \prod_e \sum_{m \in \mathbb{Z}} \exp\left(-\frac{\beta}{2}(\nabla\theta(e) - 2\pi m)^2\right) \prod_x d\theta(x)$$

$$= \sum_{\vec{m} \in \vec{V}(\mathbb{Z}^d)} \int \prod_e \exp\left(-\frac{\beta}{2}(\nabla\theta(e) - 2\pi \vec{m}(e))^2\right) \prod_x d\theta(x)$$

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Split into "vortex charge"

$$\sum_{\vec{m} \in \vec{V}(\mathbb{Z}^d)} = \sum_{\vec{q} : \operatorname{div} \vec{q} = 0} \ \sum_{\vec{m} : \operatorname{curl} \vec{m} = \vec{q}}$$

The **Helmholtz-Hodge decomposition** implies

$$\left\{ \vec{m} \in \vec{V}(\mathbb{Z}^d) : \operatorname{curl} \vec{m} = \vec{q} \right\} = \left\{ \nabla \phi - \Delta^{-1} \operatorname{curl} \vec{q} \, : \, \phi \in F(\mathbb{Z}^d) \right\}$$

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For each vector field $\vec{m}$ such that $\operatorname{curl} \vec{m} = \vec{q}$, we have

$$\exp\left(-\frac{\beta}{2} \sum_e \left(\nabla\theta(e) - 2\pi \vec{m}(e) \right)^2 \right)$$
$$= \exp\left(\sum_e -\frac{\beta}{2} (\nabla\theta(e) + 2\pi \nabla\phi(e))^2 - \sum_e 2\pi^2 \beta \left(\Delta^{-1} \operatorname{curl} \vec{q}(e) \right)^2 \right)$$

After computation, we obtain

$$Z = \underbrace{\int \exp\left(-\frac{\beta}{2}(\nabla\phi, \nabla\phi)\right)\prod_x d\phi(x)}_{\text{Gaussian free field}}$$

$$\times \underbrace{\sum_{\vec{q}\in\vec{V}(\mathbb{Z}^3),\mathrm{div}\,\vec{q}=0} \exp\left(2\pi^2\beta\left(\vec{q}, \Delta^{-1}\vec{q}\right)\right)}_{\text{Coulomb gas}}$$

Factorizes into a Gaussian free field and a neutral Coulomb gas.

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Objective of the argument:

$$\text{Coulomb gas} = \langle \text{Complicated observable} \rangle_{\mu_\beta},$$

where the measure μ_β is a (complicated) vector-valued random surface.

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$$Z_C := \sum_{\vec{q} \in \vec{V}(\mathbb{Z}^3), \mathrm{div}\, \vec{q} = 0} \exp\left(2\pi^2 \beta \left(\vec{q}, \Delta^{-1} \vec{q}\right)\right)$$

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Similar representation for the two-point function

$$\boxed{\langle S_0 \cdot S_x \rangle_{\mathrm{Vil}} = \left\langle e^{i(\phi(x) - \phi(0))} \right\rangle_{GFF} \frac{Z_C(\vec{\sigma}_{0,x})}{Z_C},}$$

where $\vec{\sigma}_{0,x} := \vec{G} - \vec{G}(\cdot - x)$ and

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One step renormalization

Strategy: decompose the GFF $\beta^{1/2}\phi$ into two independent Gaussian fields:

- $\vec{\phi}_1$ which has covariance $\beta\left(\Delta^{-1} - \beta^{-\frac{1}{2}} Id\right)$;
- $\vec{\phi}_2$ which has covariance $\beta^{\frac{1}{2}} Id$.

$$Z_C(\vec{\sigma}_{0x}) = \sum_{\vec{q}\in\vec{V}(\mathbb{Z}^3),\mathrm{div}\,\vec{q}=0} \underbrace{e^{-\pi^2\beta^{1/2}(\vec{q},\vec{q})}}_{\text{is (very) small!}} \mathbb{E}_{\vec{\phi}_1}\left[e^{-2i\pi\left(\vec{q},\vec{\phi}_1+\vec{\sigma}_{0x}\right)}\right].$$

One step renormalization

Since most charges canceled out locally, one may perform a cluster expansion (following Bauerschmidt) to obtain

$$\frac{Z_C\left(\vec{\sigma}_{0x}\right)}{Z_C} = \frac{\mathbb{E}_{\vec{\phi}_1}\left[\exp\left(\sum_{\vec{q}} z(\beta, \vec{q}) \cos 2\pi\left(\vec{q}, \vec{\phi}_1 + \vec{\sigma}_{0x}\right)\right)\right]}{\mathbb{E}_{\vec{\phi}_1}\left[\exp\left(\sum_{\vec{q}} z(\beta, \vec{q}) \cos 2\pi\left(\vec{q}, \vec{\phi}_1\right)\right)\right]},$$

where $|z(\beta, q)| \leq \underbrace{e^{-c\beta^{1/2}(\vec{q},\vec{q})}}_{\text{is (very) small!}}$.

Expanding the cosine $\cos 2\pi\left(q, \phi_1 + \sigma_{0x}\right)$ gives

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Gaussian Heuristics

A few remarks on the observable $Z(\vec{\sigma}_{0x})/Z(0)$:

For large β, the charges are essentially supported on dipoles $q = z(\beta)(\delta_x - \delta_{x+e_i})$. Use the heuristics

$$\sum_{e \,\text{edges}} z(\beta) \sin 2\pi(\nabla\phi(e)) \sin 2\pi(\sigma_{0x}(e))$$

$$\approx \sum_{e \,\text{edges}} z(\beta) 2\pi(\nabla\phi(e)) 2\pi(\sigma_{0x}(e)) \approx z(\beta)(\phi_0 - \phi_x).$$

If μ_β were Gaussian, this would imply

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- By a duality argument, we have proved that

$$\text{Villain} = \text{Gaussian free field} \times \text{Coulomb gas};$$

- By applying the sine-Gordon representation and a cluster expansion, we have

$$\text{Coulomb gas} = \langle \text{Complicated observable} \rangle_{\mu_\beta},$$

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Part 3: Quantitative homogenization

Strategy of the proof:

1. From Villain to Coulomb gas;
   - Vector calculus in dimension 3;
   - Duality argument (Fröhlich-Spencer);
2. From Coulomb gas to random surface;
   - Sine-Gordon representation;
   - Cluster expansion (low temperature, Bauerschmidt);
3. Quantitative homogenization of the random surface;
   - Helffer-Sjöstrand equation (Naddaf-Spencer 98);
   - Quantitative stochastic homogenization (Armstrong-Kuusi-Mourrat 2014-2020).

Answering the question:

How do we study quantitatively the large-scale behavior of random surfaces (with uniformly convex potential)?

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A simplified model: the $\nabla\phi$ model

Consider the probability measure on the space

$$\Omega := \left\{ \phi : \mathbb{Z}^d \to \mathbb{R} \right\} \implies \text{infinite-dimensional vector space}$$

defined by the formula

$$d\mu(\phi) := Const \times \exp\left(- \sum_{e \, \text{edges}} V\left(\nabla\phi(e)\right) \right) d\phi,$$

where $V : \mathbb{R} \to \mathbb{R}$ is $C^2(\mathbb{R})$ and uniformly convex $(0 < \lambda \leq V'' \leq \Lambda < \infty)$.

- Well-defined in infinite-volume by approximation (in $d \geq 3$, Funaki-Spohn);
- For the Gaussian free field, $V(x) = x^2$, and $\Delta^{-1} = G$ encodes covariance structure (and everything) about the field.

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The $\nabla \phi$ model

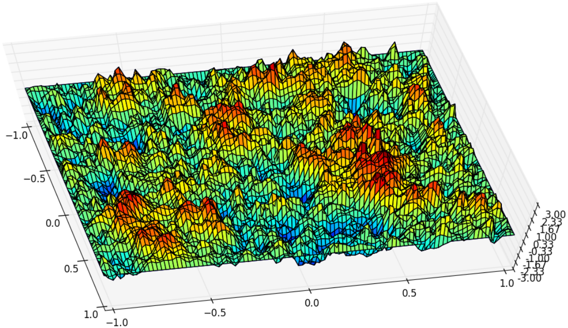

Figure: A realization of the random surface (by C. Gu).

The Witten Laplacian Δ_ϕ

How do we study this object?

- Defining a derivative: for each "suitable" function $f : \Omega \to \mathbb{R}$

$$\partial_x f(\phi) := \lim_{h \to 0} \frac{f(\phi + h \mathbb{1}_x) - f(\phi)}{h}.$$

- Defining the formal adjoint ∂_x^*: for any "suitable" pair of functions $f, g : \Omega \to \mathbb{R}$,

$$\int_\Omega \partial_x f(\phi) g(\phi) \mu(d\phi) = \int_\Omega f(\phi) \partial_x^* g(\phi) \mu(d\phi),$$

we have the explicit formula

$$\partial_x^* = -\partial_x + \left(\sum_{y \sim x} V'(\phi(y) - \phi(x)) \right) \partial_y.$$

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We can thus define **the Witten-Laplacian**

$$\Delta_\phi := \sum_{x \in \mathbb{Z}^d} \partial_x^* \partial_x$$

$$= - \sum_{x \in \mathbb{Z}^d} \partial_x^2 + \sum_{x \in \mathbb{Z}^d} \left(\sum_{y \sim x} V'(\phi(y) - \phi(x)) \right) \partial_x .$$

This operator satisfies, for any pair of functions $f, g : \Omega \to \mathbb{R}$,

$$\langle f \Delta_\phi g \rangle = \sum_{x \in \mathbb{Z}^d} \langle \partial_x f \partial_x g \rangle = \langle g \Delta_\phi f \rangle .$$

Helffer-Sjöstrand operator

For each edge $e = (x, y)$, we let $\mathbf{a}(\nabla\phi(e)) = V''(\nabla\phi(e))$.

Definition (Helffer-Sjöstrand operator)

The Helffer-Sjöstrand operator is defined by the formula

$$\mathcal{L} := \Delta_\phi + \nabla \cdot \mathbf{a}\nabla$$

which acts on functions $f : \Omega \times \mathbb{Z}^d \to \mathbb{R}$.

- The operator Δ_ϕ is the Witten-Laplacian, it acts on the field variable (infinite-dimensional);
- The operator $\nabla \cdot \mathbf{a}\nabla$ is a uniformly elliptic operator, it acts on the space variable (dimension d).

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Proposition: Solvability of the Helffer-Sjöstrand equation

Given a function $g : \Omega \times \mathbb{Z}^d \to \mathbb{R}$, we can solve the equation

$$\mathcal{L}\mathcal{G} = g \text{ in } \Omega \times \mathbb{Z}^d,$$

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Helffer-Sjöstrand representation

Theorem: H.-S. representation (Naddaf-Spencer 98)

Given two random variables $F, G : \Omega \to \mathbb{R}$, we denote by:

- $f(x, \phi) = \partial_x F(\phi)$;
- $g(x, \phi) = \partial_x G(\phi)$;
- $\mathcal{G} : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ the solution of the equation

$$\mathcal{L}\mathcal{G} = g \text{ in } \Omega \times \mathbb{Z}^d,$$

then we have

$$\boxed{\mathrm{Cov}[F, G] = \sum_{x \in \mathbb{Z}^d} \left\langle f(x, \phi) \mathcal{G}(x, \phi) \right\rangle_\mu .}$$

Helffer-Sjöstrand representation

Objective: Understand the large-scale behavior of the solutions of the Helffer-Sjöstrand equation.

Homogenization heuristic

There exists a coefficient $\bar{\mathbf{a}} > 0$ such that, for any map $g : \Omega \times \mathbb{Z}^d \to \mathbb{R}$, if we consider the solution of the equation

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then we have

$$\left\| \mathcal{G}(x, \cdot) - \bar{G}(x) \right\|_{L^2(\mu)} \leq \frac{1}{|x|^{d-2+\alpha}} \text{ for some } \alpha > 0.$$

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Remark: Some results:

- Qualitatively: Naddaf-Spencer, Giacomin-Olla-Spohn;
- Quantitatively: Armstrong-Wu;
- Extend the techniques of (Armstrong-Kuusi-Mourrat) and (Armstrong-Wu) to the case of systems of equations and infinite range operators and elliptic systems;
- Requires mixing of the random field.

Homogenization for the dual Villain model

Some tools:

- Borrow the perturbative idea from Schauder theory and the fact that, for large β,

$$\mathcal{L}_{HS} := -\frac{1}{2\beta}\Delta + \mathcal{L}_{\mathrm{pert}}.$$

  Therefore the solution can be approximated by harmonic functions. A $C^{0,1-\epsilon}$ regularity is available, with $\epsilon \to 0$ as $\beta \to \infty$.
- Use quantitative ergodic theorem to study the associated subadditive energy quantities (like in A-K-M and A-W).
- To prove the mixing, take ∂_y to the H-S $\mathcal{L}_{HS}\mathcal{G} = g$ to obtain the second-order Helffer-Sjöstrand equation

$$-\mathcal{L}_{der}\partial_y \mathcal{G} = \partial_y f.$$

  A $C^{0,1-\epsilon}$ regularity is then available.

Open questions

- Massive scaling limit. Consider the XY model in the external field

$$Z_\Lambda^{XY} = \int e^{\beta \sum_{i \sim j} \cos(\theta_i - \theta_j) + h \cos(\theta_i)} \prod_{j \in \Lambda} d\theta_j.$$

  Conjectured for small h and $\beta > \beta_c$,

$$\langle S_0 \cdot S_x \rangle \sim \exp\left(- \sqrt{\frac{h}{\beta}} |x| \right).$$

  Lebowitz-Penrose obtained $\langle S_0 \cdot S_x \rangle \leq exp\,(-ch|x|)$ using Lee-Yang theorem.

- Prove the asymptotic two-point function for $d = 2$ and large β. (Kosterlitz-Thouless phase).

Thank you for your attention!