Random Spanning Forests

Hyperbolic Symmetry

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Random Spanning Forests + hyperbolic Symmetry.


Arborescent gas: $\beta > 0$, $G = (V,E)$ finite graph.

$$P_\beta[F] \propto \frac{|F|}{2^{|E|}} \cdot \beta^{|\text{edges}|} \cdot (1 - P_\beta)^{|E\setminus F|}$$

Three origin stories:

1) $P_\beta$ bond percolation, conditioned on the event that open edges are a forest.

2) $q$-random cluster model in the limit $q \downarrow 0$, $p = \beta q$.

3) Graphical representation of the $d=2$ spin model.
Main question: Take $G_n \uparrow G_\infty$. Is there a percolation phase transition as $\beta$ varies? If so, what are the percolative properties?

Today: $G_n \uparrow \mathbb{Z}^d$. Convention: translationally invariant subseq. limits $\implies$ all results are quantitative + finite volume.

Classical observations:

1) $P_\beta \leq P_\beta^{perc}$. Hence if $P_\beta < P_c(\mathbb{Z}^d)$, no $\infty$-tree, and $P_\beta[0 \leftrightarrow \infty] \leq C e^{-c|x|}$

2) The # of infinite trees is as 0 or 1.
Example: $G_n = K_n$, complete graph on $n$ vertices.

$P_\beta$ is Erdős–Rényi conditioned to be acyclic.

$\beta = \frac{\lambda}{n}$

Thm: The arboreal gas has a transition on $K_n$ at $\beta = \frac{\lambda}{n}$:

$$EX_n \left[ |T_0| \right] \sim \left\{ \begin{array}{ll}
c_\alpha n & \alpha > 1 \\
c_\alpha n^{1/3} & \alpha = 1 \\
c_2 & \alpha < 1 \\
\end{array} \right.$$

Due to Łuczak-Pittel; finer detail known, see Martin-Yeo.

Thm: If $\alpha > 1$, 2nd largest component has size $\approx n^{1/3}$

(compare w/ Erdős–Rényi: 2nd largest is size $\approx \log n$)
Thm (BEKS) On $\mathbb{Z}^2$ for any $\beta > 0$ percolation does not occur.

Moreover, $P_{\beta}[0 \leftrightarrow x] \leq C(1 + e^{-c(\beta)})^{-L_0}$.

Remark (Dey-Garoni-Sokal): can predict this from $p_c(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}$ for $q$-RBM; suggests $p_c(q) \sim \frac{1}{q} \uparrow \infty$.

Challenge: Find a direct probabilistic proof of this theorem.

Rest of talk

1) Conjectural behavior of the anchored gas on $\mathbb{Z}^d$

2) Outline of the key new tool used in our proof:
   a) Magne formula for connection probabilities.
Conjecture: On \( \mathbb{Z}^2 \), for all \( \beta > 0 \), \( \exists m(\beta) > 0 \) s.t. \( \Pr_{\beta}[0 \leftrightarrow x] \leq e^{-m(\beta) \beta x} \).

Remark: This says the transition at \( \beta_c = \infty \) is sharp. Belive \( m(\beta) \approx e^{-\beta} \), much smaller than for approach to criticality for percolation.

Same behaviour as for 4D Yang-Mills (\( \$$\$$ \$$\$$).

Conjecture: On \( \mathbb{Z}^d \), \( d > 3 \), there is a percolation transition: \( \exists \beta_c < \infty \) s.t. \( \Pr_{\beta}[0 \leftrightarrow y] = c_{\beta} > 0 \) (\( \beta > \beta_c \)).

Moreover, \( \Pr_{\beta}[0 \leftrightarrow y] - c_{\beta} \approx 1 \times 1 - d + 2 \) (\( \beta > \beta_c \)) due to large non-giant trees.

Conjecture: On any graph, \( \Pr_{\beta}[e \leftrightarrow f] \leq \Pr_{\beta}[e] \Pr_{\beta}[f] \) \( e \neq f \).

Remark: \( \Pr[e \leftrightarrow f] \leq 2 \Pr_{\beta}[e] \Pr_{\beta}[f] \).
Very roughly, can attribute differences compared to bond percolation to very different ground states:

Perculation $\omega$ $p=1$: all edges open, unique configuration.

Arborescent gas $\omega$ $\beta=\infty$: Uniform choice of a spanning tree. Exponentially many configurations.

Remark: Other families of $G_{n,\beta}$ interesting too; expect similar differences to percolation.
Magic Formula

\[ (\Delta f)(i) = \sum_{j \sim i} \beta (f_j - f_i) \]

\[ P_{\beta} [0 \rightarrow x] \propto \int e^{tx} e^{-\sum_{j} \beta \cosh(t_i - t_j)} (\det \beta \delta) \left. \right|^{3/2}_{t=-\infty} \left[ e^{-3\beta t} \right]_{t=0} \]

**Remark:** If replace 3 with 1 the measure on the RHS is the ‘magic formula’ or ‘mixing measure’ for the VRSP!

ERRW: edge weights \( \chi + \chi^{-1} \). Diaconis - Coppermith proved

\[ P_{\text{ERRW}(\chi)} \propto \int P_{\text{SRW}(C)}(c) \, d\lambda(y(c)) \]

\[ \Rightarrow \text{Random environment} \]
Schematic proof of non-preservation on $\mathbb{Z}^2$

Arboresal gas $\xrightarrow{\text{non-linear}}$ matrix-tree thm

$\mathbb{H}^{12}$ spin model $\xrightarrow{\text{Dimensional reduction}}$ $\mathbb{H}^{24}$ spin model $\xrightarrow{\text{Horospherical coordinates and Gaussian integrals}}$ Magic formula

Rmk: `magic' is the two vertical steps: same magic as for URSP/ERBM
Matrix tree theorem and \( H^{12} \)

Uniform spanning tree = determinantal = free fermion

\[ s_i, \ldots, s_m, \eta_i, \ldots, \eta_n \text{ anti-commuting variables:} \]

\[ s_i s_j = -s_j s_i, \quad \eta_i \eta_j = -\eta_j \eta_i, \quad s_i s_j = s_j s_i \]

Grassmann algebra \( IR \)-linear combinations, e.g., \( 3 \xi_1 \eta_2 + 7 + \xi_i \xi_j \eta_5 \)

Derivatives: \( F \) a monomial not containing \( s_i \)

\[ 2s_i (s_i F) = F, \quad 2s_i F = 0 \]

\( 2s_i \) antiderivative

Extend \( 2s_i \) to full algebra by linearity.

Exercise: For any \( n \times n \) matrix \( A \),

\[ \det A = 2s_1 2s_2 \ldots 2s_n 2s_n \exp \left( \sum_{i<j} A_{ij} s_i \eta_j \right) \]

\( \text{fuzzy integral} \quad \text{fuzzy GFF} \)
A positive definite, \( \frac{(2\pi)^n}{\text{det} A} = \int_{\mathbb{R}^n} e^{-\langle \varphi, A \varphi \rangle} \, d\varphi \)

Matrix-tree theorem: take \( A = -\beta \Delta + h \)

\[
\sum_{(F, r)} \prod_{(T, r)} \beta^{\# T} h^r
\]

Feynman integral = derivatives

Uniform spanning trees: take \( h = \text{all zero} \), \( \# \text{Forests with one root} = \# \text{spanning trees} \)
Fundamental observation (Caracciolo et al.) if $h = 1$

$$\int e^{-\langle m, A g \rangle} - \sum_{j} \beta \langle \xi_i, \xi_j \rangle = \sum_{F} \beta^{\mid F \mid}$$

Reformulation $H^{012}_{\text{model}}$

$z_i = 1 - \xi_i \eta_i$

$u_i = (\xi_i, \eta_i, z_i)$ 'spin at $i$'

$u_i \cdot u_j = -\xi_i \xi_j - \xi_j \eta_i - z_i z_j$

Check: $u_i \cdot u_i = -1$ (like the hyperbolic model of hyperbolic space in $H^{2+1}$)

Moreover: $P_{\beta} [i \leftrightarrow j] = -\langle u_i \cdot u_j \rangle$
Step 2: Dim. Reduction: $H^{2,14}$ has spins $u=(x,y,z_1,z_2,n_1,n_2)$

with $x^2+y^2-z_1^2-2z_2n_1-2z_2n_2=-1,2\pi$,

where $x,y\in\mathbb{R}$, $z_1,n_1,z_2,n_2$ anticommute.

Formally, working with algebra generated by $u$ with coefficients smooth functions of $x,y$.

Berezin integral: $\int F = \int dx dy \int_{H^{0,14}} \frac{1}{2} F.$

On finite graph $G$, $H^{2,14}$ model defined as $H^{0,12}$. Integrate on $(H^{2,14})^*$. 

Thm (dimensional reduction/SUSY localization)

$$\langle u_i \cdot u_j \rangle_{H^{2,14}} = \langle u_i \cdot u_j \rangle_{H^{0,12}}$$
Step 3: Horospherical coordinates

The $H^2$ model has $u = (x, y, z)$ as

\[ x^2 + y^2 - z^2 = -1, \quad z > 0 \]

\[ u_i \cdot u_j = x_i x_j + y_i y_j - z_i z_j \]

Good parametrization of 2-dim ntille $H^2$ is given by:

\[ x = \sinh t - \frac{1}{2} s^2 e^t \]

\[ y = s e^t \quad \text{s.t.} \in \mathbb{R} \]

\[ 2 = \cosh t + \frac{1}{2} s^2 e^t \]

Then

\[ u_i \cdot u_j = \cosh (t_i - t_j) + \frac{1}{2} s_i s_j e^{t_i + t_j} - 1 \]

Similarly good coordinates for $H^2/\Gamma_4$, also get terms quadratic in $\xi$s and $\eta$s.
Step 4: Mermin-Wagner. This is a classical argument that says long-range order is incompatible with continuous symmetries in $d=1$ and $d=2$.

LRO $\propto$ 1 fluctuating are Gaussian,

but GFF has unbounded fluctuations in $d=1,2$.

Made rigorous by Mermin-Wagner, many others.
We adopt a version of Sabot for VRSP ($\alpha=1$)