Random Spanning Forests
+
Hyperbolic Symmetry

(joint work w/ R. Bauerschmidt, N. Crawford, A. Swan)

arXiv: 1912.04854

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joint w/ R. Bauerschmidt, N. Crawford, + A. Swan (arXiv:1912.04854)

Arboreal gas: $\beta > 0$, $G = (V, E)$ finite graph.

$$\mathbb{P}_\beta[F] \equiv \beta^{|F|} / Z_\beta \propto p_\beta^{|F|} (1 - p_\beta)^{|E \setminus F|}$$

#edges $\qquad$ $\beta/\beta+1$

Three origin stories:

1) p_β bond percolation, conditioned on the event that open edges are a forest.

2) q-random cluster model in the limit $q \downarrow 0$, $p = \beta q$

3) graphical representation of $\mathbb{H}^{d|2}$ spin model.

Later

Main question: Take $G_n \uparrow G_\infty$. Is there a percolation phase transition as β varies? If so, what are the percolative properties?

<u>Today</u>: $G_n \uparrow \mathbb{Z}^d$. Convention: translationally invariant subseq. limits

$\longrightarrow$ all results are quantitative + finite volume

<u>Classical observations</u>: 1) $\mathbb{P}_\beta \lesssim \mathbb{P}^{perc}_{p_\beta}$. Hence if $p_\beta < p_c(\mathbb{Z}^d)$, no ∞-tree,

and $\mathbb{P}_\beta[0 \longleftrightarrow x] \leq Ce^{-c|x|}$

Holley $\leq$

2) The # of infinite trees is as. 0 or 1.

Burton-Keane

Example: $G_n = K_n$, complete graph on n vertices.

$\mathbb{P}_\beta$ is Erdős-Rényi conditioned to be acyclic.

$\beta = \alpha/n$

Thm: The arboreal gas has a transition on K_n at $\beta = 1/n$:

$$\mathbb{E}_{\alpha/n}[|T_0|] \sim \begin{cases} c_\alpha n & \alpha > 1 \\ c_\alpha n^{1/3} & \alpha = 1 \\ c_\alpha & \alpha < 1 \end{cases}$$

tree containing 0

Due to Łuczak-Pittel; finer detail known, see Martin-Yeo.

Thm If $\alpha > 1$, 2nd largest component has size $\asymp n^{2/3}$

critical tree size

(compare in Erdős-Rényi: 2nd largest is size $\asymp \log n$)

subcritical size

Thm (BCHS) On $\mathbb{Z}^2$ for any $\beta > 0$ percolation does not occur.

Moreover, $\mathbb{P}_\beta[0 \longleftrightarrow x] \leq C|x|^{-c(\beta)}$

$\hookrightarrow c(\beta) > 0$

Rmk (Deny-Garoni-Sokal): can predict this from $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ for q-RCM; suggests $\beta_c(q) \sim \frac{1}{\sqrt{q}} \uparrow \infty$.

$\hookleftarrow$ thm $q \approx 1$

Challenge: Find a direct probabilistic proof of this theorem

Rest of talk

1) Conjectural behavior of the arboreal gas on $\mathbb{Z}^d$

2) Outline of the key new tool used in our proof: a magic formula for connection probabilities.

Conjecture: On $\mathbb{Z}^2$, for all $\beta > 0$ $\exists\, m(\beta) > 0$ s.t. $\mathbb{P}_\beta[0 \longleftrightarrow x] \lesssim e^{-m(\beta)|x|}$.

Rmk: This says the transition at $\beta_c = \infty$ is sharp. Believe $m(\beta) \simeq e^{-c\beta}$, much smaller than for approach to criticality for percolation.

Same behaviour as for 4D Yang-Mills ($\$\$\$$)

Conjecture On $\mathbb{Z}^d$, $d \geq 3$, there is a percolation transition: $\exists\, \beta_c < \infty$

s.t. $\mathbb{P}_\beta[0 \longleftrightarrow x] = c_\beta > 0 \qquad (\beta > \beta_c)$

Moreover, $\mathbb{P}_\beta[0 \longleftrightarrow x] - c_\beta \simeq |x|^{-d+2} \qquad (\beta > \beta_c)$

due to large non-giant trees

prove this is expected

Brunder-Hatch

Conjecture: On any graph, $\mathbb{P}_\beta[e, f] \leq \mathbb{P}_\beta[e]\,\mathbb{P}_\beta[f] \qquad e \neq f$.

Rmk: $\mathbb{P}[e, f] \leq 2\, \mathbb{P}_\beta[e]\, \mathbb{P}_\beta[f]$

Very roughly, can attribute differences compared to bond percolation to very different ground states:

Percolation w/ $p=1$: all edges open, unique configuration.

Arboreal gas w/ $\beta=\infty$: Uniform choice of a spanning tree. Exponentially many configurations.

Rmk: Other families of $G_n \uparrow G_\infty$ interesting too; expect similar differences w/ percolation.

Magic Formula

Δ graph Laplacian: $(\Delta f)(i) = \sum_{j \sim i} \beta(f(j) - f(i))$

$$\boxed{\mathbb{P}_\beta[0 \longleftrightarrow x] \propto \int_{\mathbb{R}^{n-1}} e^{\overbrace{t_x}^{\in \mathbb{R}}} e^{-\sum_{i\sim j} \beta \cosh(t_i - t_j)} (\det \beta\Delta)^{3/2} \prod_{i \neq 0} e^{-3t_i} \, dt_i \quad \leftarrow t_0 = 0}$$

<u>Rmk</u> If replace 3 with 1 the measure on the RHS is the 'magic formula' or 'mixing measure' for the VRJP!

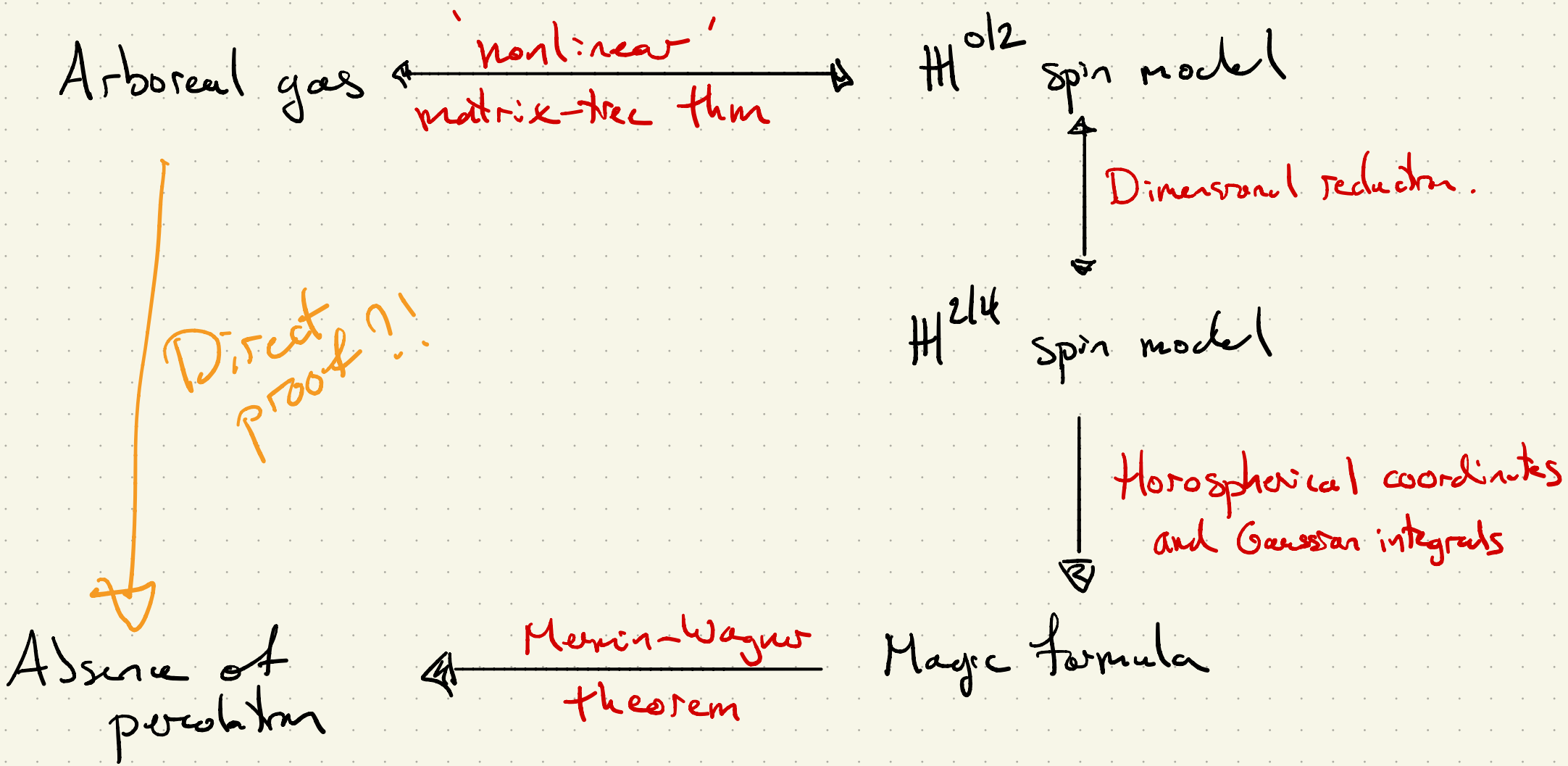
<u>ERRW</u>: edge weights $\{\gamma + n_e\}_{e \in E}$. Diaconis-Coppersmith proved

$$\frac{\gamma + n_2}{3\gamma + n_1 + n_2 + n_3} \quad \mathbb{P}_0^{ERRW(\gamma)} \propto \int \mathbb{P}_0^{SRW(C)} \, d\mu_\gamma(C)$$

RW

Random environment

Schematic proof of non-percolation on $\mathbb{Z}^2$



<u>Rmk</u>: 'magic' is the two vertical steps: <u>same</u> magic as for URSP/ERRW

Matrix-tree theorem and $\mathbb{H}^{0|2}$

Uniform spanning tree = determinental
= free fermion

$\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n$ anticommuting variables:

$$\xi_i \eta_j = -\eta_j \xi_i, \quad \eta_i \eta_j = -\eta_j \eta_i, \quad \xi_i \xi_j = -\xi_j \xi_i, \qquad \xi_i^2 = \eta_i^2 = 0$$

Grassmann algebra $\mathbb{R}$-linear combinations, e.g., $3\xi_1\eta_2 + 7 + \xi_1\xi_3\eta_5$

Derivatives: F a monomial not containing ξ_i.

$$\partial_{\xi_i}(\xi_i F) = F, \quad \partial_{\xi_i} F = 0.$$

∂_{ξ_i} antiderivation

Extend ∂_{ξ_i} to full algebra by linearity.

Taylor expand (truncates)

Exercise: For any $n \times n$ matrix A,

$$\det A = \underbrace{\partial_{\eta_1}\partial_{\xi_1} \cdots \partial_{\eta_n}\partial_{\xi_n}}_{\text{fermion integration}} \underbrace{\exp\Big(\sum_{i,j} A_{ij} \overbrace{\xi_i \eta_j}^{\text{commuting}}\Big)}_{\text{fermion GFF}}$$

Cf A positive definite, $\frac{(2\pi)^n}{\det A} = \int_{\mathbb{R}^{2n}} e^{-(\varphi, A\varphi)} d\varphi$

Matrix-tree theorem: take $A = -\beta \underline{\Delta} + h$ ← diagonal matrix

$$\underbrace{\int e^{-(\xi, A\xi)}}_{\text{Fermionic integral} = \text{derivatives}} = \sum_{(F,R)} \prod_{(T,r)} \beta^{|T|} h_r$$

same

(F,R): rooted spanning forests $\neq$ arborial gas

h_r: root weight

β: per edge.

Uniform spanning trees: take $h = \mathbb{1}_{i=0}$. {Forests with one root} $\simeq$ {spanning trees}.

Fundamental observation (Caraccioli et al) if $h = \mathbb{1}$

$$\int e^{-(\eta, A\xi) - \sum_{i\sim j} \beta \xi_i \eta_i \xi_j \eta_j} = \sum_F \beta^{|F|}$$

0 commuting variables

2 anticommuting variables

roots at vertices

roots at edges ($\bar{\omega} - 1$)

$|V| - |E| = 1$

$\mathbb{H}^{0|2n}$

Reformulation ($\mathbb{H}^{0|2}$ model)

$z_i \equiv 1 - \xi_i \eta_i$

$u_i \equiv (\xi_i, \eta_i, z_i)$ 'spin at i'

Check: $u_i \cdot u_i = -1$ (like the hyperboloid model of hyperbolic space in $\mathbb{R}^{2+1}$)

$u_i \cdot u_j \equiv -\xi_i \eta_j - \xi_j \eta_i - z_i z_j$

$$\int \left(\prod \frac{1}{z_i}\right) e^{-\frac{1}{2}\beta(u, -\Delta u)} = \sum_F \beta^{|F|}$$

Haar measure

Moreover: $\mathbb{P}_\beta[i \longleftrightarrow j] = -\langle u_i \cdot u_j \rangle$

<u>Step 2: Dim. Reduction</u>: $\mathbb{H}^{2|4}$ has spins $u = (x, y, z, \xi_1, \eta_1, \xi_2, \eta_2)$

with $x^2 + y^2 - z^2 - 2\xi_1\eta_1 - 2\xi_2\eta_2 = -1$, $z \geq 0$,

where $x, y \in \mathbb{R}$, $\xi_1, \eta_1, \xi_2, \eta_2$ anticommute.

Formally, working with algebra generated by $\longrightarrow$ with coefficients smooth fns of x, y.

<u>Berezin integral</u>: $$\int_{\mathbb{H}^{2|4}} F \equiv \int_{\mathbb{R}^2} dx\, dy \int_{\mathbb{H}^{0|4}} \frac{1}{2} F .$$

On finite graph G, $\mathbb{H}^{2|4}$ model defined as $\mathbb{H}^{0|2}$. Integrate on $(\mathbb{H}^{2|4})^{\wedge}$.

<u>Thm</u> (dimensional reduction/SUSY localization)

$$\langle u_i \cdot u_j \rangle_{\mathbb{H}^{2|4}} = \langle u_i \cdot u_j \rangle_{\mathbb{H}^{0|2}}$$

Step 3: Horospherical coordinates

$\mathbb{H}^2$ model has $u = (x, y, z)$ w/

$$x^2 + y^2 - z^2 = -1, \quad z > 0$$

$$u_i \cdot u_j = x_i x_j + y_i y_j - z_i z_j$$

Good parametrization of 2-dim'l mfld $\mathbb{H}^2$ is given by:

$$x = \sinh t - \tfrac{1}{2} s^2 e^t$$

$$y = s e^t \qquad\qquad s, t \in \mathbb{R}$$

$$z = \cosh t + \tfrac{1}{2} s^2 e^t$$

then

$$\boxed{u_i \cdot u_j = \cosh(t_i - t_j) + \tfrac{1}{2} s_i s_j e^{t_i + t_j} - 1}$$

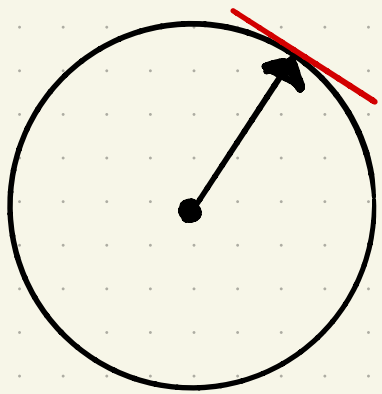
Similarly good coordinates for $\mathbb{H}^{2,1}$, also get terms quadratic in ξs and ηs.

Step 4: Mermin–Wagner

This is a classical argument that says long-range order is incompatible with continuous symmetries in $d=1$ and $d=2$:

LRO $\simeq$ $\perp$ fluctuations are Gaussian,

but GFF has unbounded fluctuations in $d=1,2$.



Made rigorous by Mermin–Wagner, many others.

We adapt a version of Sabot for VRSP ($a=1$)