Random Spanning Forests + Hyperbolic Symmetry.


Arboreal gas: $P > 0$, $G = (V,E)$ finite graph.

$$P_{P} [F] = P^{|F|}/Z_P \times P_{P}^{1 - |F|} \prod_{e \in F} (1-P_{P})$$

Three origin stories: 1) $P_{P}$ bond percolation, conditioned on the event that open edges are a forest.

2) $q$-random cluster model in the limit $q \downarrow 0$, $P_{P} = \beta q$

3) Graphical representation of $H_{2}$ spin model.
Main question: Take $G_n \uparrow G_{\infty}$. Is there a percolation phase transition as $\beta$ varies? If so, what are the percolative properties?

Today: $G_n \uparrow \mathbb{Z}^d$. Convention: translationally invariant subseq. limits

Classical observations:

1) $P_{\beta} \preceq P_{\beta}^{perc}$. Hence if $P_{\beta} < P_{c}(\mathbb{Z}^d)$, no $\infty$-tree, and $P_{\beta}[0 \leftrightarrow \infty] \leq C e^{-C|x|}$

2) The # of infinite trees is as 0 or 1.
Example: $G_n = K_n$, complete graph on $n$ vertices.

$P_\beta$ is Erdős-Rényi conditioned to be acyclic.

Thm: The arboreal gas has a transition on $K_n$ at $\beta = \frac{1}{\alpha}$:

$$E_{\frac{1}{\alpha} n} [|T_{01}|] \sim \begin{cases} c_2 \alpha n & \alpha > 1 \\ c_2 \alpha n^{\frac{1}{\alpha}} & \alpha = 1 \\ c_2 & \alpha < 1 \end{cases}$$

Due to Łuczak-Pittel; finer detail known, see Martin-Yeo.

Thm If $\alpha > 1$, 2nd largest component has size $= n^{\frac{1}{\alpha}}$

(compare w Erdős-Rényi: 2nd largest is size $= \log n$)
Thm (BCKS): On $\mathbb{Z}^2$ for any $\beta > 0$ percolation does not occur.

Moreover, $\Pr_{\beta} [0 \leftrightarrow x] \leq C 12^{-c(\beta)}$

Remark (Dej-Garoni-Sokal): can predict this from $p_c(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}$

for $q$-RCM; suggests $p_c(2) \sim \frac{1}{\sqrt{2}} \uparrow \infty$.

Challenge: Find a direct probabilistic proof of this theorem.

Rest of talk:
1) Conjectural behavior of the arrested gas on $\mathbb{Z}^d$

2) Outline of the key new tool used in our proof:
a magic formula for correlation probabilities.
Conjecture: On $\mathbb{Z}^2$, for all $\beta > 0$ there exists $\beta(\beta) > 0$ such that $P_\beta[0\leftrightarrow x] \leq e^{-m(\beta)1x1}$.

Remark: This says the transition at $\beta_c = \infty$ is sharp. Believe $m(\beta) \approx e^{-c\beta}$, much smaller than for approach to criticality for percolation.

Conjecture: On $\mathbb{Z}^d$, $d \geq 3$, there is a percolation transition: $\exists \beta_c < \infty$ such that $P_\beta[0\leftrightarrow x] = c_\beta > 0$ ($\beta > \beta_c$).

Moreover, $P_\beta[0\leftrightarrow x] - c_\beta \approx 1x1^{-d+2}$ ($\beta > \beta_c$).

Conjecture: On any graph, $P_\beta[e,f] \leq P_\beta[e]P_\beta[f]$ for any edge $e \neq f$. 

Very roughly, can attribute differences compared to bond percolation to very different ground states:

Percolation $\omega_p = 1$: all edges open, unique configuration.

Arboreal gas $\omega_p = \infty$: Uniform choice of a spanning tree. Exponentially many configurations.

Remark: Other families of $G_{\omega_p\infty}$ interesting too; expect similar differences to percolation.
Δ graph Laplacian:  \((Δf)(i) = \sum_{j \sim i} β(t_{ij} - f_i)\)

\[P_{\theta} \left[ 0 \rightarrow x \right] \propto \int e^{tx} e^{-\sum_j β \cosh(t_{ij} - t_0)} \left( \det \beta \right)^{3/2} \prod_{i=0}^{t_0} e^{-3β t_i} dt_i \]

Remark: If replace 3 with 1 the measure on the RHS is the 'magic formula' or 'mixing measure' for the VRSP!

ERRW: edge weights \(\xi Y + n e^{\beta x}\). Diaconis—Coppersmith proved

\[P_{\theta}^{ERRW} \propto \int P_{\theta}^{SRW(C)} \, d\mu_x(C)\]
Schematic proof of non-perturbation on $\mathbb{Z}^2$

Arborescent gas $\xrightarrow{\text{non-linear}}$ matrix-trace thin $\xrightarrow{\text{H}^{12}}$ spin model $\xrightarrow{\text{Dimensional reduction}}$ $\xrightarrow{\text{H}^{214}}$ spin model $\xrightarrow{\text{Horospherical coordinates and Gaussian integrals}}$

Absence of perturbation $\xrightarrow{\text{Mermin-Wagner theorem}}$ Magic formula

Remark: 'magic' is the two vertical steps: same magic as for NLS/GPE
Matrix-tree theorem and \( \Theta \)^{12} Uniform spanning tree = determinantal = free fermion

\( \xi, \ldots, \xi_n, \eta_1, \ldots, \eta_n \) anticommuting variables:

\[ \xi_i \eta_j = -\eta_j \xi_i, \quad \eta_i \eta_j = -\eta_j \eta_i, \quad \xi_i \xi_j = \delta_{ij} \delta_{ij} \]

Grassmann algebra: IR-linear combinations, e.g., \( 3 \xi_1 \eta_2 + 7 + \xi_3 \xi_3 \eta_5 \)

**Derivatives**: \( F \) a monomial not containing \( \xi_i \)

\[ \partial_{\xi_i} (F \eta) = F, \quad \partial_{\xi_i} F = 0 \]

Extend \( \partial_{\xi_i} \) to full algebra by linearity.

**Exercise**: For any \( n \times n \) matrix \( A \)

\[ \text{cdet} \ A = \partial_{\eta_1} \partial_{\xi_1} \ldots \partial_{\eta_n} \partial_{\xi_n} \exp \left( \sum_{i,j} A_{ij} \xi_i \eta_j \right) \]
A positive definite, \( \frac{(2\pi)^n}{\det A} = \int_{\mathbb{R}^n} e^{-\langle \varphi, A\varphi \rangle} \, d\varphi \)

**Matrix-tree theorem**: take \( A = -\beta \Delta + h \)

\[
\int e^{-\langle \varphi, A\varphi \rangle} = \sum_{(F_i \in \mathcal{F})} \prod_{(T_i \in \mathcal{T}_i)} \beta^{|T_i|} h_i
\]

**Uniform spanning trees**: take \( h = -1 \). \( \mathcal{F} \) forests with one root \( \Rightarrow \{\text{spanning trees}\} \).
Fundamental observation (Caracciolo et al.) if $h = \Pi$

$$\int e^{-\langle n, A^2 \rangle} - \sum_j p \delta_{n_j, n} = \sum_F p^{1F}$$

Reformulation ($\mathcal{H}^{012}_{\text{model}}$)

$z_i \equiv 1 - \delta_{i, n_i}$

$u_i \equiv (\delta_{i, n_i}, z_i)$

$u_i \cdot u_j \equiv - \delta_{i, j} - \delta_{n_i, n_j} - 2 z_i z_j$

$$\int (\prod z_i) e^{-\frac{1}{2} p (u_i - \Delta u)} = \sum_F p^{1F}$$

Moreover: $P_{\beta} [i \leftrightarrow j] = - \langle u_i \cdot u_j \rangle$
Step 2: Dim. Reduction: \( H^{2|4} \) has spins \( u = (x,y,z_1,z_2,n_1,n_2) \)
with \( x^2 + y^2 - z^2 - 2z_1 n_1 - 2 z_2 n_2 = -1, 2, 3, \)
where \( x,y \in \mathbb{R}, \ z_1, n_1, z_2, n_2 \) anticommute.

Formally, working with algebra generated by \( \uparrow \) with coefficients smooth at \( x,y \).

Berezin integral: \( \int_{H^{2|4}} \int_{\mathbb{R}^2} dx \, dy \int_{H^{0|2}} \frac{1}{2} F \).

On finite graph \( G \), \( H^{2|4} \) model defined as \( H^{0|2} \). Integrate on \( (H^{2|4})^\uparrow \).

Thm (dimensional reduction/SUSY localization)

\[
\langle u_i \cdot u_j \rangle_{H^{2|4}} = \langle u_i \cdot u_j \rangle_{H^{0|2}}
\]
Step 3: Horospherical coordinates

$\mathbb{H}^2$ model has $u = (x, y, z)$ as:

\[ x^2 + y^2 - z^2 = -1, \quad z > 0 \]

\[ u_i \cdot u_j = x_i x_j + y_i y_j - z_i z_j \]

Good parametrization of 2-dim $\mathbb{H}^2$ wth $\mathbb{H}^2$ is given by:

\[ x = \sinh t - \frac{1}{2} s^2 e^t \]

\[ y = s e^t \quad \text{s.t.} \quad e^t \in \mathbb{R} \]

\[ z = \cosh t + \frac{1}{2} s^2 e^t \]

Then

\[ u_i \cdot u_j = \cosh (t_i - t_j) + \frac{1}{2} s_i s_j e^{t_i + t_j} - 1 \]

Similarly, good coordinates for $\mathbb{H}^{2/4}$ also yet terms quadratic in $s_i$ and $y_i$. 
Step 4: Mermin-Wagner. This is a classical argument that says long-range order is incompatible with continuous symmetries in $d=1$ and $d=2$:

$LRO \Rightarrow$ fluctuations are Gaussian,

but GFF has unbounded fluctuations in $d=1,2$.

Made rigorous by Mermin-Wagner, many others.

We adapt a version of Salt for VRSG ($\alpha=1$)