Singular Vectors on Fractals

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Dirichlet's Theorem

Theorem (Dirichlet)

For every $\mathbf{x} \in \mathbb{R}^d$ *and for all* $N \geq 1$, *there exist* $\mathbf{0} \neq (p, \mathbf{q}) \in \mathbb{Z} \times \mathbb{Z}^d$ *such that*

$$\|\mathbf{q} \cdot \mathbf{x} - p\| \leqslant 1/N, \qquad \|\mathbf{q}\| \leqslant N^{1/d}.$$

- For $0 < \varepsilon < 1$, $\mathbf{x}$ is ε-**Dirichlet improvable** $(\mathrm{DI}_\varepsilon)$ if for all $N \gg 1$, there is $\mathbf{0} \neq (p, \mathbf{q}) \in \mathbb{Z} \times \mathbb{Z}^d$ such that

$$\begin{cases} \|\mathbf{q} \cdot \mathbf{x} - p\| \leqslant \varepsilon/N, \\ \|\mathbf{q}\| \leqslant N^{1/d}. \end{cases}$$

- $\mathbf{x}$ is **singular** if $\mathbf{x} \in \bigcap_{\varepsilon > 0} \mathrm{DI}_\varepsilon =: \mathrm{Sing}(d)$.

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Dani's Correspondence

- $G = \mathrm{SL}(d+1, \mathbb{R})$, $\Gamma = \mathrm{SL}(d+1, \mathbb{Z})$,

$$g_t = \begin{pmatrix} e^{dt} & \mathbf{0} \\ \mathbf{0} & e^{-t}\mathrm{I}_d \end{pmatrix}, \qquad u(\mathbf{x}) = \begin{pmatrix} 1 & \mathbf{x} \\ \mathbf{0} & \mathrm{I}_d \end{pmatrix}.$$

- Dani's Correspondence:

$$\mathbf{x} \in \mathrm{Sing}(d) \iff (g_t u(\mathbf{x})\Gamma)_{t \geqslant 0} \text{ diverges in } G/\Gamma.$$

The question

- Ergodicity of $g_t \Longrightarrow \mathrm{Sing}(d)$ has 0 Lebesgue measure.
- **Y. Cheung** (Annals '11): singular vectors in $\mathbb{R}^2$ have dimension $4/3$.
  - **Cheung-Chevallier** (Duke '16): singular vectors in $\mathbb{R}^d$ have dimension $d^2/(d+1)$.

Question (Buguead, Cheung, Chevallier)

What is $\dim_H(\mathrm{Sing}(2) \cap \mathcal{K})$, *where* $\mathcal{K} = \mathcal{C} \times \mathcal{C}$ *and* $\mathcal{C}$ *is Cantor's middle* $1/3$ *set?*

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A theorem - special case

Theorem (Khalil '19)

If $\mathcal{K}$ is a product of two copies of Cantor's middle thirds set, then

$$\dim_H(\mathrm{Sing}(2) \cap \mathcal{K}) \leqslant \frac{2}{3}\frac{\log 4}{\log 3}.$$

- Remark: $\dim_H(\mathcal{K}) = \frac{\log 4}{\log 3}$.

Fractals

- **Iterated Function Systems** (IFS): $\mathcal{F} = \left\{ f_i : \mathbb{R}^d \to \mathbb{R}^d : 1 \leq i \leq n \right\}$;

$$f_i = \rho_i O_i + b_i,$$
$$0 < \rho_i < 1, \quad O_i \in \mathrm{SO}(d, \mathbb{R}), \quad b_i \in \mathbb{R}^d.$$

- There is a unique $\mathcal{F}$-invariant compact set $\mathcal{K} = \bigcup_i f_i(\mathcal{K})$.
- Examples:
  1. $\mathcal{F} = \{x \mapsto x/3, x \mapsto (x+2)/3\} \longrightarrow \mathcal{K} =$ Cantor's middle thirds set.
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Self-similar Measures

- For a probability vector $(p_1, \ldots, p_n)$, a measure μ is **self-similar** if

$$\mu = \sum_i p_i (f_i)_* \mu.$$

Theorem (Moran 1945)

If $s = \dim_H(\mathcal{K})$, then s is the unique solution of $\sum_i \rho_i^s = 1$.
The self-similar measure for $p_i = \rho_i^s$ is the restriction of the Hausdorff measure to $\mathcal{K}$ (assuming the open set condition).

- Example: Cantor's set: $\rho = 1/3$, $s = \log 2/\log 3$, $p_1 = p_2 = 1/2$.

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General result

- If $\mathcal{L} \subset \mathbb{R}^d$ is a proper rational affine subspace, then $\mathcal{L} \subset \mathrm{Sing}(d)$.
- The main result can be thought of as a quantitative converse to this statement for fractals:

$$\dim_H \begin{pmatrix} \text{singular vectors in the support} \\ \text{of a fractal measure } \mu \end{pmatrix} \leqslant \begin{pmatrix} \text{non-concentration} \\ \text{parameters of } \mu \end{pmatrix}$$

The numbers $\alpha_\ell(\mu)$

- $\alpha_\ell(\mu)$ is the largest number:

$$\mu(\varepsilon \text{ neighborhood of } \mathcal{L}) \leqslant \varepsilon^{\alpha_\ell(\mu) - o(1)}$$

  for all affine subspaces $\mathcal{L}$ of dimension $d - \ell$.

- The smaller $\alpha_\ell(\mu)$ is, the more concentrated its support is near subspaces of dimension $d - \ell$.

- Frostman exponents of projections of μ: take limit in ε is for subspaces $\mathcal{L}$ in a given "direction".

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Singular vectors on fractals

Theorem (Khalil '19)

Let μ be the Hausdorff measure on a self-similar fractal $\mathcal{K}$. Then,

$$\dim_H(\mathrm{Sing}(d) \cap \mathcal{K}) \leqslant \dim_H(\mathcal{K}) - \min_{1 \leq \ell \leq d} \frac{(d-\ell+1)\alpha_\ell(\mu)}{d+1}.$$

Optimality

- IFS irreducible $\Longrightarrow \alpha_\ell(\mu) > 0$.
- Example: $\mathcal{K} = [0,1]^d$, $\alpha_\ell(Leb) = \ell$.

$$\dim_H(\mathrm{Sing}(d)) \leqslant \frac{d^2}{d+1} = \frac{d}{d+1}d \qquad \text{(recovers Cheung-Chevallier).}$$

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Proofs

Recall

- $G = \mathrm{SL}(d+1,\mathbb{R})$, $\Gamma = \mathrm{SL}(d+1,\mathbb{Z})$,

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- Dani's Correspondence:

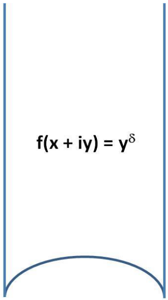
$$\mathbf{x} \in \mathrm{Sing}(d) \iff (g_t u(\mathbf{x})\Gamma)_{t\geqslant 0} \text{ diverges in } G/\Gamma.$$

- The plan: show orbits are **biased** to return to compact sets.

Bias via Margulis functions

Definition: a proper function f on G/Γ.

Example: $f : \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z}) \to \mathbb{R}_+$ is given by the y-coordinate in the upper half plane model.



Example: $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$

$$\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z}) \leftrightarrow \{\text{unimodular lattices in } \mathbb{R}^n\}$$

$$\begin{aligned} f(x) &= \frac{1}{\text{smallest covolume of a subgroup of } x} \\ &= \frac{1}{\text{smallest volume of a subtorus of } \mathbb{R}^n/x}. \end{aligned}$$

Bias $\cong$ height contraction

The idea of Kadyrov-Kleinbock-Lindenstrauss-Margulis (Lebesgue case):

- Build a Margulis function $f : G/\Gamma \to \mathbb{R}_+$:

$$\int_{B(0,1)} f(g_t u(\mathbf{x})y) \, d\mathrm{Leb}(\mathbf{x}) \leqslant e^{-\beta t} f(y) + b,$$

  for fixed β, t and b.

- Ultimately:

$$\dim_H(\mathrm{Sing}(d)) \leqslant d - \frac{\beta}{d+1}.$$

Genesis of β

1. $x \in G/\Gamma$:

   Covolume of a subgroup H of the lattice $x = \left\| \bigwedge \text{ basis of } H \right\|$.

2. An idea of Margulis reduces contraction inequality to proving:

$$\int_{B(0,1)} \|g_t u(\mathbf{x}) v\|^{-\gamma} \; d\mu(\mathbf{x}) \leqslant e^{-\gamma' t} \|v\|^{-\gamma}$$

   for good choices of γ and γ' and all $v \in \bigwedge^* \mathbb{R}^n$.

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The optimal β for linear contraction

Proposition (K '19)

For every $0 \neq v \in \bigwedge^{\ell} \mathbb{R}^{d+1}$,

$$\int \left\| g_t u(\mathbf{x}) v \right\|^{-\alpha_\ell(\mu)+o(1)} \, d\mu(\mathbf{x}) \leqslant C e^{-\beta t} \left\| v \right\|^{-\alpha_\ell(\mu)+o(1)},$$

$\beta = \frac{\alpha_\ell(\mu)(d-\ell+1)}{d+1} - o(1)$.

- The key point is that the exponent $\alpha_\ell(\mu)$ works for the representation $\bigwedge^{\ell} \mathbb{R}^d$.

The optimal β

$$\int \|g_t u(\mathbf{x})v\|^{-\alpha_\ell(\mu)+o(1)} \; d\mu \leqslant e^{-\beta t} \int \|\pi_+(u(\mathbf{x})v)\|^{-\alpha_\ell(\mu)+o(1)} \; d\mu$$

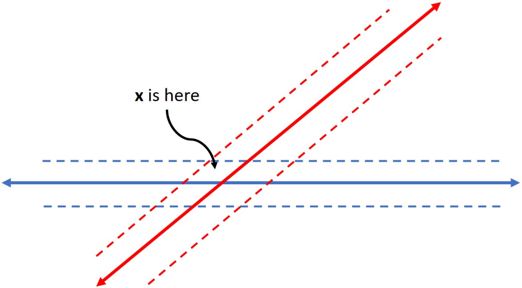
- In the standard basis, coordinates of $u(\mathbf{x})v$ are linear equations in $\mathbf{x}$.

Proposition (Khalil '19)

There are ℓ expanding coordinates of g_t in $\bigwedge^\ell \mathbb{R}^{d+1}$, depending on v, which are ***transversal!***

- Remark: dimension of expanding subspace is $\binom{d}{\ell-1}$.

Key idea 1: transversality



- If $\pi_+(u(\mathbf{x})v)$ is small, then $\mathbf{x}$ must be simulatnaeuously in a small neighborhood of ℓ transversal hyperplanes: i.e. in a neighborhood of a $(d-\ell)$-dimensional subspace.
- μ decays like $\alpha_\ell(\mu)$ near $(d-\ell)$-dimensional subspaces!

Contraction $\implies$ dimension saving

- Using $g_{(N+1)t}u(\mathbf{x}) = g_t u(e^{Nt}\mathbf{x})g_{Nt}$, you get

$$\fint_{B(\mathbf{x}_0, e^{-Nt})} f(g_{(N+1)t}u(\mathbf{x})y)\ d\mathrm{Leb} = \int_{B(0,1)} f(g_t u(\mathbf{x})\tilde{y})\ d\mathrm{Leb}$$

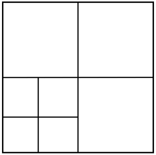
- An induction argument based on iterating the contraction inequality gives:

$$\mathrm{Leb}\begin{pmatrix} \text{orbits spending } \delta \text{ proportion} \\ \text{of } [0, T] \text{ in the cusp} \end{pmatrix} \lesssim e^{-\beta\delta T}.$$

Difficulties for fractals: iteration

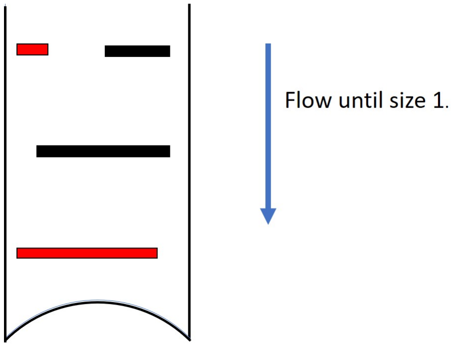
1. Lebesgue measure is invariant by translation and transforms precisely under scaling $\Longrightarrow$ facilitates scaling

2. When the IFS has different contraction ratios, the shape of the contraction inequality is **not** preserved under iteration.



- Loss in β by $\log \frac{\rho_{\max}}{\rho_{\min}}$.

Resolution: schematic



- top row: fractal at stage k.
- Black piece needs less time to become size 1 than red piece.

Resolution: variable propagation time

- Key idea 2: **Non-uniform** contraction inequality:

$$\int \rho(\mathbf{x}, k)^{-\beta} f(g_{\rho(\mathbf{x},k)} u(\mathbf{x}) y) \, d\mu(\mathbf{x}) \leqslant f(y) + b.$$

- $\rho(\mathbf{x}, k)$ is a cocycle given by the diameter of the piece of the fractal containing $\mathbf{x}$ at stage k.
- The induction argument needs to be reimagined ...

More Consequences

Consequences - Homogeneous Fractals

- An IFS $\mathcal{F}$ is **homogeneous** if each $f_i = \rho O + b_i$ (rotation and contraction are independent of i).

- A deep result of **Shmerkin** - in his work resolving Furstenberg's intersection conjecture - calculates the Frostman exponents of projections of μ in **every** direction when $\mathcal{F}$ is homogeneous on $\mathbb{R}^2$ with irrational rotation.

- Using prior ideas of Nazarov, Perez, Shmerkin:

Theorem (Khalil '19)

Suppose $\mathcal{F}$ is a homogeneous IFS on $\mathbb{R}^2$, with rotation part $\notin \mathbb{Q}\pi$. Then,

$$\alpha_1(\mu) = \text{the Frostman exponent in a.e. direction.}$$

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Consequences - Homogeneous Fractals

Corollary (Khalil '19)

If $\mathcal{F}$ is a homogeneous IFS on $\mathbb{R}^2$, with ergodic rotation part, then

$$\dim_H(\mathcal{K} \cap \mathrm{Sing}(2)) \leqslant \frac{2}{3}\dim_H(\mathcal{K}).$$

Consequences - Billiard Flows

Theorem (H. Masur (Duke '92))

The Hausdorff dimension of the set of directions $\mathbb{S}^1$ in which the straight line flow on a flat surface S is non-uniquely ergodic is at most $1/2$.

Our method shows

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The Hausdorff dimension of the set of directions in a fractal $\mathcal{K} \subseteq \mathbb{S}^1$ in which the straight line flow on a flat surface S is non-uniquely ergodic is at most $\dim_H(\mathcal{K})/2$.

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