Rough walks in random environment

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Random walks in random environment as rough paths

- Program: lifting invariance principles to rough path topology.
- Two aspects/schools:
  - RWRE / particle systems: richer picture of the model on large scales: limiting path, area anomaly.
  - S(P)DEs: Non-trivial noise approximations. Universality.

Rough paths

Universality of the Brownian motion?

Donsker's invariance principle. X *(simple) random walk*.

$$X_t^n = n^{-1} X_{\lfloor tn^2 \rfloor} \searrow$$

$$W \quad \text{standard Brownian motion}$$

$$\tilde{X}_t^n = X_t^n + (tn^2 - \lfloor tn^2 \rfloor)(X_{t+1/n^2}^n - X_t^n) \nearrow$$

Universality of the Brownian motion?

- Fun fact about SDE approximations [Wong-Zakai '65]: f nice real function, $f(0)=0$, stochastic integration yields a limit

$$Y^n_{t+1} = Y^n_t + f(Y^n_t)(X^n_{t+1} - X^n_t), t = k/n^2 \longrightarrow Y_t = \int_0^t f(Y_s)dW_s \text{ (Ito)}.$$

But the Riemann–Stieltjes integral has a different limit

$$\tilde{Y}^n_t = \int_0^t f(\tilde{Y}_s)d\tilde{X}^n(s) \longrightarrow \tilde{Y}_t = \int_0^t f(Y_s)dW_s + \frac{1}{2}\int_0^t f'(Y_s)f(Y_s)ds$$
$$=: \int_0^t f(Y_s)\circ dW_s \text{ (Stratonovich)}.$$

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- Observations
  - incomplete information: knowing W alone does not determine the SDE: Ito map $X \mapsto Y$ is not continuous in the Skorohod uniform topology.
  - However, true for every (nice) f

  $\rightsquigarrow$ some kind of universality, if we fix the notion of integration.

Rough paths

Definition (Lyons '98)

Let $\alpha \in (\frac{1}{3}, \frac{1}{2})$. A α-Hölder rough path is a pair $(Z, \mathbb{Z})$ defined by $(Z_{s,t}, \mathbb{Z}_{s,t}) \in \mathbb{R}^d \times \mathbb{R}^{d \otimes d}$ for $0 \le s < t \le T$ so that $Z_{s,t} = Z_t - Z_s$,

(i) $Z \in C^\alpha$, $\mathbb{Z} \in C^{2\alpha}$ and

(ii) $\mathbb{Z}_{s,t} - \mathbb{Z}_{s,u} - \mathbb{Z}_{u,t} = Z_{s,u} \otimes Z_{u,t}$ ("Chen's relation").

Norm by $C^\alpha \oplus C^{2\alpha}$.

Morally: $\mathbb{Z}_{s,t} = \int_s^t \int_s^{r_1} dZ_{r_2} \otimes dZ_{r_1}$. Set $\mathbf{Z} := (Z, \mathbb{Z})$.

Theorem (Lyons '98, Gubinelli '04)

Construction of rough integral; $(Y, Y', \mathbf{Z}) \mapsto \int_0^\cdot Y_s d\mathbf{Z}_s$ is continuous (if Y is controlled); Itô-Lyons map $\mathbf{Z} \mapsto Y_t = Y_0 + \int_0^t f(Y_s) d\mathbf{Z}_s$ is continuous.

In particular, if $(Z^n, \mathbb{Z}^n) =: \mathbf{Z}^n \to \mathbf{Z}$ in rough path topology then

$$Y_t^n = Y_0^n + \int_0^t f(Y_s^n) d\mathbf{Z}_s^n \to Y_t = Y_0 + \int_0^t f(Y_s) d\mathbf{Z}_s.$$

Second level perturbation $\rightsquigarrow$ new drift term

Kelly '16: Assume Stratonovich lift of semimartingales $(Z^n, \mathbb{Z}^n) \to (Z, \mathbb{Z})$ in rough path topology, where for some $\Gamma \in \mathbb{R}^{d\times d}$

$$\mathbb{Z}_{s,t} = \int_s^t \int_s^{r_1} dZ_{r_2} \otimes \circ dZ_{r_1} + (t-s)\Gamma.$$

Let $f \in C^1(\mathbb{R}, \mathbb{R}^d)$, then the solutions to

$$Y_t^n = Y_0^n + \int_0^t f(Y_s^n) \circ dZ_s^n$$

converge weakly in the rough path topology to the solution to

$$Y_t = Y_0 + \int_0^t f(Y_s) \circ dZ_s + \int_0^t \Gamma f(Y_s) \cdot f'(Y_s) ds,$$

where $\Gamma f(Y_s) \cdot f'(Y_s) = \sum_{i,j=1}^d \Gamma^{i,j} f_i'(Y_s) f_j(Y_s)$.

Random walks in random environment

Random walks in random environment on $\mathbb{Z}^d$

- Environment $\omega \in \Omega$: for every $x \in \mathbb{Z}^d$

$$\omega_x(y) \geq 0, \quad \sum_{y:|x-y|=1} \omega_x(y) = 1.$$

- For fixed ω, let $(X_n)_{n\geq 0}$ be a Markov chain on $\mathbb{Z}^d$ starting at the origin, with the **quenched** law:

$$P_\omega(X_{n+1} = y | X_n = x) = \omega_x(y), \text{ for every } |x-y| = 1, \ n \geq 0.$$

- For a probability P on Ω the **annealed** law $\mathbb{P}$ on the random walk is

$$\mathbb{P}(\cdot) = \int_\Omega P_\omega(\cdot) \mathrm{d}P(\omega).$$

Ballistic RWRE: the measure P on Ω satisfies

- $(\omega_x)_{x\in\mathbb{Z}^d}$ is i.i.d. and uniformly elliptic ($\mathbb{P}(\kappa \leq \omega \leq 1-\kappa) = 1$ for some $\kappa > 0$) and
- Sznitman's T' ballisticity condition holds

  (ballisticity means $\frac{X_n}{n} \to v \neq 0$ $\mathbb{P}$-a.s).

Random conductance model:

- environment coming from random (ergodic and elliptic) weights (conductances) on (sometimes long range) edges of $\mathbb{Z}^d$.
- reversibility, no ballisticity: $v = 0$.

Regeneration structure, strong LLN, functional CLT

Sznitman-Zerner '99:

- Regeneration structure. $\mathbb{P}$- a.s. $\exists$ times $0 =: \tau_0 < \tau_1 < \tau_2 < ... < \infty$ so that τ_k have all moments and

$$\left(\{X_{\tau_k,\tau_k+m}\}_{0\le m\le \tau_{k+1}-\tau_k}, \tau_{k+1}-\tau_k\right)_{k\ge 1} \text{ are i.i.d..}$$

- Strong LLN. $\frac{X_n}{n} \to \frac{\mathbb{E}[X_{\tau_1,\tau_2}]}{\mathbb{E}[\tau_2-\tau_1]} =: v$ $\mathbb{P}$-a.s.

Sznitman '00: Let

$$X^n(t) := \frac{1}{n}(\bar{X}_{\lfloor n^2 t\rfloor} + (n^2 t - \lfloor n^2 t\rfloor)(\bar{X}_{\lfloor n^2 t\rfloor+1} - \bar{X}_{\lfloor n^2 t\rfloor}), \quad \bar{X}_n := X_n - nv$$

be the centering & rescaling.

- Functional CLT. Under $\mathbb{P}$

$$X^n(\cdot) \Rightarrow B, \text{ a Brownian motion, in } C([0,T],\mathbb{R}^d).$$

Covariance: $\Sigma^2_{i,j} = \frac{\mathbb{E}[\bar{X}^i_{\tau_1,\tau_2}\bar{X}^j_{\tau_1,\tau_2}]}{\mathbb{E}[\tau_2-\tau_1]}$.

Homogenization phenomenon

- Environment non-homogeneous.
- Averaging effect: on large scales walk's fluctuations are approaching a (Gaussian) limit.
- However, generally covariance is different than in homogenous case.

Ballistic RWRE as rough paths

Ballistic RWRE: convergence & area anomaly

$(X^n_t, \mathbb{X}^n_{s,t})$ - centred & rescaled linearly interpolated lift.

Theorem (Lopusanschi - O '18)

Ballistic RWRE, $d \geq 2$. Under $\mathbb{P}$:

$$(X^n, \mathbb{X}^n) \Rightarrow (B, \mathbb{B}^{Str} + \Gamma \cdot)$$

in the α-Hölder rough path topology, for every $\alpha < 1/2$.

- *B - Brownian motion, $\mathbb{B}^{Str}$ - its iterated integral w.r.t. Stratonovich integration*
- *Γ is a deterministic $d \times d$ matrix with the explicit form*

$$\Gamma = \frac{\mathbb{E}[\mathrm{Antisym}(\mathbb{X}^1_{\tau_1,\tau_2})]}{\mathbb{E}[\tau_2 - \tau_1]},$$

where $\mathrm{Antisym}(A)^{i,j} = \frac{1}{2}(A^{i,j} - A^{j,i})$.

Ballistic RWRE: remarks

- The solution to an SDE approximated by $\bar{X}$ converges to a solution of an SDE with an explicit correction in terms of Γ.
- Our proof is general to any process on $\mathbb{R}^d$ having a regeneration structure and enough moments, where moments translates to regularity.
- Example for $\nexists$ arbitrary large moments: Random walks in Dirichlet environments with a large trap parameter.

- Area anomaly has a geometric interpretation: Γ is the expected signed stochastic area of $\bar{X}$, the re-centered walk, on a regeneration interval, normalized by its expected size:

$$\Gamma = \frac{\mathbb{E}[\mathrm{Antisym}(\mathbb{X}^1_{\tau_1,\tau_2})]}{\mathbb{E}[\tau_2 - \tau_1]}.$$

Ballistic RWRE: the measure P on Ω satisfies

- $(\omega_x)_{x \in \mathbb{Z}^d}$ is i.i.d. and uniformly elliptic and
- Sznitman's T' ballisticity condition holds.

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- environment coming from random (ergodic and elliptic) weights (conductances) on (sometimes long range) edges of $\mathbb{Z}^d$.
- reversibility, no ballisticity: $v = 0$.

Random walk in random conductances

- Random conductances (discrete time):
  $\Omega = \{\omega_{x,y} = \omega_{y,x} : x, y \in \mathbb{Z}^d, |x - y| = 1\}$; *for fixed* ω
  $$P^{\omega}(X_{n+1} = y | X_n = x) = \frac{\omega_{x,y}}{\sum_z \omega_{x,z}}.$$
- *Functional CLT:* $X^n(t) = n^{-1} X_{n^2 t}$. *Assume conductances i.i.d with values in* $[c, C] \subset (0, \infty)$ *(uniform ellipticity). Under* $\mathbb{P}$: X^n *converges weakly in* $D([0, T], \mathbb{R}^d)$ *to a Brownian motion with a covariance matrix* $\Sigma = \Sigma(\mathrm{law}(\omega))$ *[Künnemann '83].*
- Again, homogenization: on large scales walk feels the environment in an "averaged" sense, covariance is different than in homogenous case.

Random walk in random conductances - two routes

- Annealed (or in probability) results: additive functionals of MCs: Kozlov 79, Papanicolaou-Varadhan 81, Kipnis-Varadhan 86, De Masi, Ferrari, Goldstein and Wick 89 (very partial list).
- Quenched results for the continuous time process with variable speed using cocycle spaces: Sidoravicius-Snitzman 04, Berger-Biskup 07, Mathieu-Piatnitski 07 (very partial list).
- Variety of conditions. Extends to stationary ergodic conductances, with some regularity conditions on the conductances. Typically some positive and negative moment condition for $\mu = \sum_y \omega(0, y)$ the total rates at the origin and regularity of the jumps (e.g. second moment).

Random conductances and rough paths

Strategy 1. Additive functionals of Markov processes as rough paths

Kipnis-Varadhan theory in rough path topology

Theorem (Deuschel - O - Perkowski '19)

X Markov with generator $\mathcal{L}$, μ stationary and ergodic for $\mathcal{L}$,$\mathcal{L}^*$. $F : E \to \mathbb{R}^d$ bounded and measurable with $\int F d\mu = 0$ and $Z^n_t = n^{-1} \int_0^{n^2 t} F(X_s)ds$. Assume $\mathcal{H}^{-1}$ condition. Then

$$(Z^n, \mathbb{Z}^n) \to \left(B, \mathbb{B}^{Str} + \cdot \lim_{\lambda \to 0} \mathbb{E}[\Phi_\lambda \otimes \mathcal{L}_A \Phi_\lambda] \right)$$

in (p-variation) rough path topology (for all $p > 2$), where B is a Brownian motion with covariance

$$\langle B, B \rangle_t = 2t \lim_{\lambda \to 0} \mathbb{E}[\Phi_\lambda \otimes (-\mathcal{L}_S)\Phi_\lambda].$$

Note: correction vanishes if $\mathcal{L} = \mathcal{L}^*$.

$\mathcal{H}^{-1}$ condition. For $(\lambda - \mathcal{L})\Phi_\lambda = F$: $\lambda \int |\Phi_\lambda|^2 d\mu + \int (\Phi_\lambda - \Phi_{\lambda'}) \otimes (-\mathcal{L})(\Phi_\lambda - \Phi_{\lambda'}) d\mu \to 0$.

Application to random conductances

Convergence in rough path topology and area anomaly

Theorem (Deuschel - O - Perkowski '19)

Assume i.i.d and uniformly elliptic conductances. For the Itô lift $(X^n, \mathbb{X}^n)$*, under* $\mathbb{P}$ *we have a rough path convergence*

$$(X^n, \mathbb{X}^n) \to \left(B, \int_0^{\cdot} B_s \otimes dB_s + \frac{1}{2} \langle B, B \rangle_{\cdot} + \cdot \Gamma \right),$$

where $\Gamma = -diag\,(\mathbb{E}[\omega(0, e_i)], i = 1, ..., d)$*.*

- Started from Itô, so would not expect Stratonovich integrals.
- For linear interpolations we get converges to Stratonovich (without anomaly!).

Strategy 2. Cocycle space and the corrector process

Classical quenched FCLT

- $\exists$ Hilbert space H of cocycle functions on $\Omega \times \mathbb{Z}^d$ with $H = H_{\text{sol}} \oplus H_{\text{pot}}$, $H \ni \Pi = \Psi + \chi$ for the position field $\Pi(\omega, x) = x$ and

$$X_t = \Pi(\omega, X_t) = \Psi(\omega, X_t) + \chi(\omega, X_t)$$

  is a decomposition to a martingale plus a "corrector".
- Analytical tools $\rightsquigarrow$ corrector is sublinear $\rightsquigarrow$ vanishes in limit (e.g. Sidoravicius and Sznitman '04).
- Quenched CLT for X then follows from martingale CLT for $\Psi(\omega, X_t)$.

$\|\Psi\|_H^2 = \mathbb{E}[\sum_x \omega(0,x)(\Psi(\omega,x))^{\otimes 2}]$.
Ψ is a cocycle if $\Psi(\omega, x) - \Psi(\omega, y) = \Psi(\tau_y \omega, x - y)$.
H_{pot} is a closure of gradient functions $\Phi(\omega, x) = \varphi(\tau_x \omega) - \varphi(\omega)$ so that φ is local.

Rough path treatment

- With Johaness Bäumler, Noam Berger, and Martin Slowik.
- Anealed result for general conditions.
- Remark: corrector $\chi(\omega, X_t)$ vanishes BUT its iterated integral converges to a the non-zero linear function $t\|\chi\|_H^2$.
- Compared to area anomaly we got using rough path Kipnis-Varadhan.

$\Pi \in H$, stationary ergodic, jump a.s. positive and has 1^+ moments.

Quenched - in progress

- Classical quenched invariance principle: Sidoravicius-Snitzman 04, Berger-Biskup 07, Mathieu-Piatnitski 07.
- $\mathbb{Z}^d$, $d \geq 3$, i.i.d nearest neighbor conductances in $\{0\} \cup [a, b]$
- Recent work by Paul Dario on moments of the corrector in space $\mathbb{E}|\chi(\omega, x)|^p < C_p$.
- Quenched Heat kernel bounds: Mathieu-Remy 04, Barlow 04.
- We deduce quenched moments of the corrector on the process uniformly in time
- $E_0^\omega[|\chi(\omega, X_t)|^p] < c_p(\omega)$ for all $p > 0$.
- Bounded jumps which enables transferring the estimates to the martingale part: $|\Delta\Psi(\omega, X_\cdot)| + |\Delta\chi(\omega, X_\cdot)| = 1$
- (after some work) get a quenched result.

Summary

- Program: invariance principle for rough walks in random environment.
  - Approximation of SDEs. Universality.
  - Richer understanding of path structure in RWRE models.
- Ballistic RWRE (non-reversible case).
  - Identification of area anomaly in terms of a stochastic area in regeneration interval.
- Kipnis-Varadhan theory in rough path topology.
  - No area anomaly if the process is reversible.
  - Application to random conductances: canonical limit for linear interpolations, correction for Itô rough path.
  - Method extends to many other models, not necessary reversible, e.g. periodic diffusions.
- Cocycle space for random conductances.
  - Area anomaly in Itô case, identified by the H norm of the corrector.
  - Holds quenched whenever the corrector has enough moments ($2+\epsilon$ should be enough).

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Thank you for your attention!