Random Walks on Circle Packings

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Tel-Aviv Horowitz Seminar, 23/03/20

Circle Packings

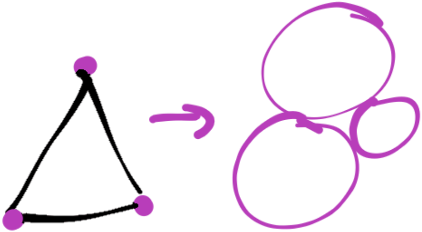
Is there a canonical way of drawing a planar graph?

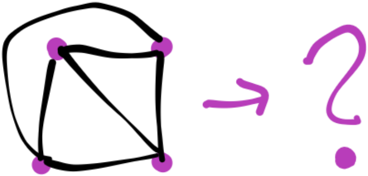
Definition

A **circle packing (CP)** is a collection of circles in the plane with disjoint interiors.

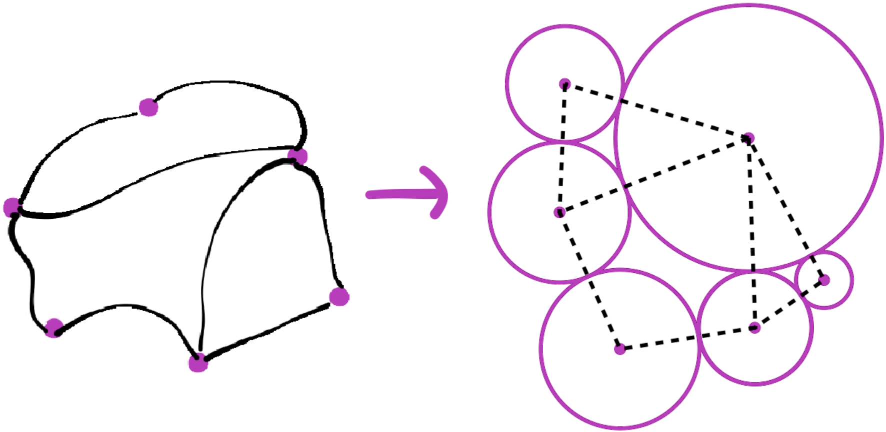
Definition

Given a circle packing, its **tangency graph** is the graph obtained by using the circles as vertices and connecting tangent circles by an edge.





Circle Packing: An Example



Note

A circle packing induces a drawing of its tangency graph in straight lines by mapping the vertices to the centers of their correspding circles.

The Circle Packing Theorem (CPT)

Theorem (Koebe-Andreev-Thurston, 1936)

- **(Existence)** Every finite planar graph is the tangency graph of a circle packing.
- **(Uniqueness)** If the graph is a triangulation (outer face included) then the circle packing representing it is unique up to Möbius transformations and reflections.

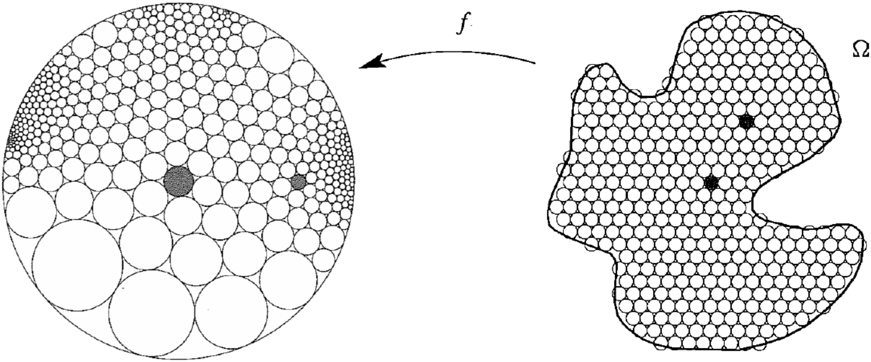
Corollary (Fáry-Wagner Theorem, 1936)

Every finite planar graph can be drawn in the plane in straight lines.

Some History: Riemann Mapping Approximation

Theorem (Rodin-Sullivan 87', conjectured by Thurston 85')

Let Ω be a bounded simply-connected domain. Then circle packings can be used to approximate a Riemann map from Ω to the unit disk.

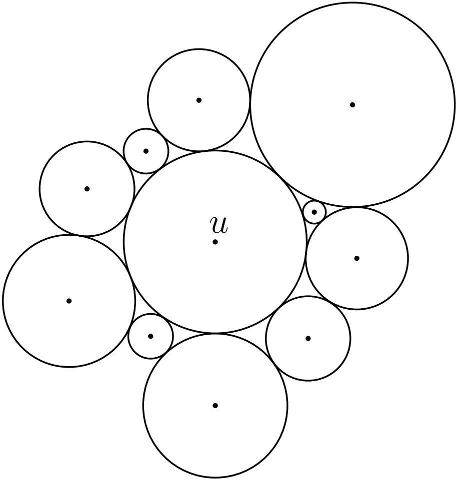


(image from Stephenson's book "Introduction to Circle Packing")

Some History: Riemann Mapping Approximation

Ring Lemma (Rodin-Sullivan 87')

For each $d \in \mathbb{N}$, there exists $r = r(d) > 0$ such that if a unit circle is surrounded by d circles forming a cycle externally tangent to it then their radii are larger than r.



Infinite Graphs can also be circle packed!

All graphs in this talk are assumed to be:

- connected.
- locally finite.
- Infinite triangulations are assumed to have no outer face.

Question

Given an infinite planar triangulation, can it be circle packed? If so, is the circle packing unique?

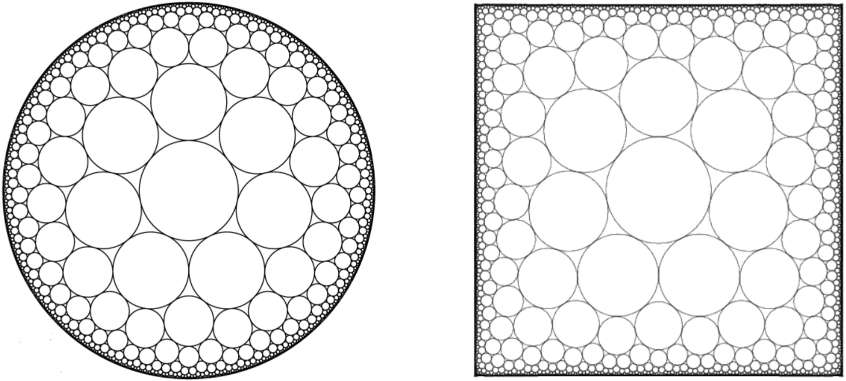
Claim

Every infinite planar triangulation can be circle packed.

Sketch of Proof

Take a sequence of finite subgraphs exhausting the graph and circle pack them using the CPT. Then use a compactness argument.

Uniqueness?



(image from Asaf Nachmias' Saint-Flour lecture notes)

The He-Schramm Theorem: Some Definitions

Definition

For a circle-packed infinite triangulation (CPIT), define its **carrier** $\Omega \subseteq \mathbb{R}^2$ to be the union of its induced drawing's (triangular) faces.

Definition

A graph is called **1-ended** if the deletion of any finite set of vertices leaves the graph with exactly one infinite component.

Examples

- $\mathbb{Z}^2$ (the square lattice) is 1-ended.
- $\mathbb{Z}$ is not 1-ended.

The He-Schramm Theorem

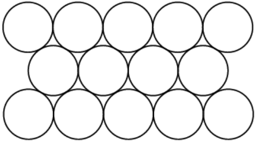
Theorem (He-Schramm, 95')

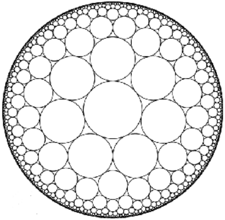
Let G be a **1-ended** infinite triangulation. Then exactly one of the two following holds:

1. G can be circle-packed with $\mathbb{R}^2$ as carrier.
2. G can be circle-packed with the open unit disk $\mathbb{D}$ as carrier.

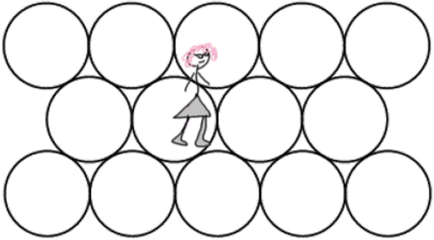
Furthermore, if G has **bounded degrees** then

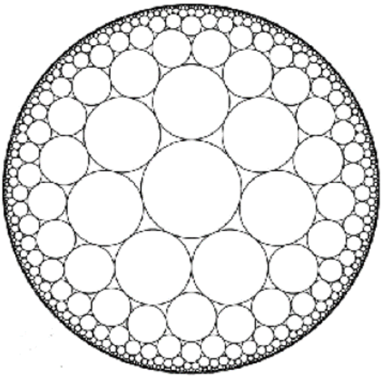
1. iff G is recurrent.
2. iff G is transient.





The He-Schramm Theorem





Removing the 1-Ended Assumption

Without 1-endedness of the graph, the resulting carrier might have holes.

Definition

A domain $\Omega \subseteq \mathbb{R}^2$ is called **parabolic** if for every $x \in \Omega$ and an open $U \subseteq \Omega$, Brownian motion started at x and killed upon hitting $\partial\Omega$ hits U with probability 1. Otherwise, it is called **hyperbolic**.

Theorem (Gurel-Gurevich, Nachmias, Souto 2017')

Let G be a (planar) infinite triangulation with bounded degrees, then:

G is recurrent $\iff$ some CP of G has parabolic carrier

$\iff$ any CP of G has parabolic carrier.

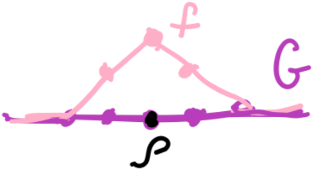
Parabolicity <-> Recurrence

Equivalent Condition for Parabolicity

A domain $\Omega \subseteq \mathbb{R}^2$ is parabolic iff for every $K \subseteq \Omega$ compact and $\varepsilon > 0$ there exists a Lipschitz function $0 \leq \phi \leq 1$ compactly supported in Ω with $\phi\mid_K \equiv 1$ such that $\iint_\Omega \|\nabla \phi\|^2 \, dA < \varepsilon$.

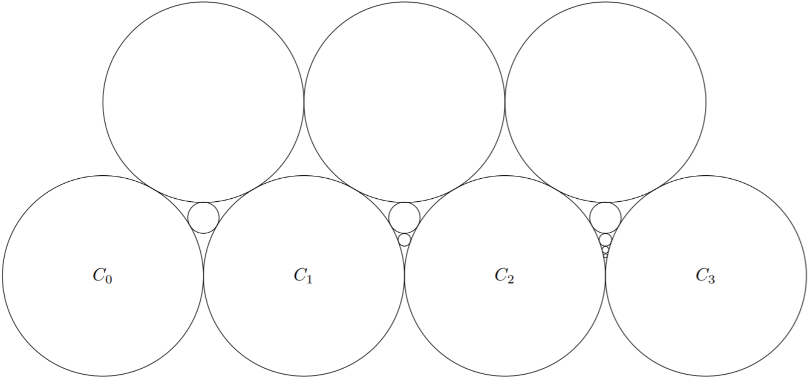
Equivalent Condition for Recurrence

Let $G = (V, E)$ be a graph and fix some $\rho \in V$. Then G is recurrent iff for every $\varepsilon > 0$ there exists a finitely supported function $f : V \to \mathbb{R}$ with $f(\rho) = 1$ and $\sum_{uv \in E} (f(v) - f(u))^2 < \varepsilon$.



The Bounded Degree Assumption

Without bounded degrees, the He-Schramm theorem might fail:
The circles added to the hexagonal lattice create a rightward drift and thus transience. However, the carrier remains the entire plane (i.e. parabolic).



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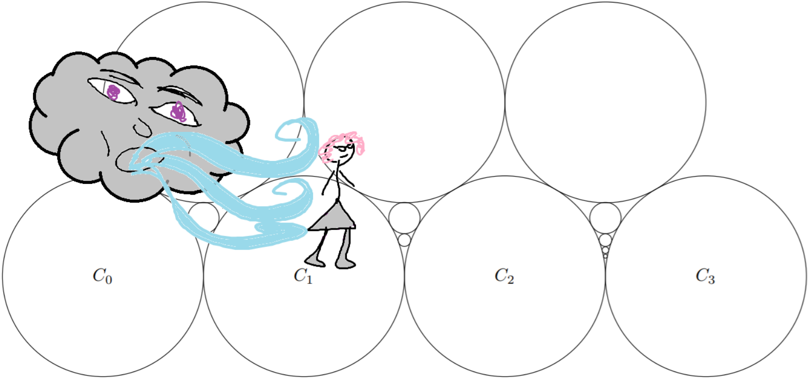


(image from Asaf Nachmias' Saint-Flour lecture notes)

Weighted Random Walk

Idea: Use weighted random walk

- Random walker chooses an edge with prob. proportional to its weight.
- Edge weights introduced by Duffin (1968) and Dubejko (1995).

Properties of the Weighted Random Walk

- The sequence of centers visited by the random walker is a martingale! (Dubejko, 1995).
- Weights bounded from above by a (universal) constant.
- Under bounded degree assumptions, weights also bounded from below by a constant.

Theorem (Gurel-Gurevich, S. 2019)

The weighted random walk on a CPIT with parabolic carrier is recurrent.

The Weights: Definition

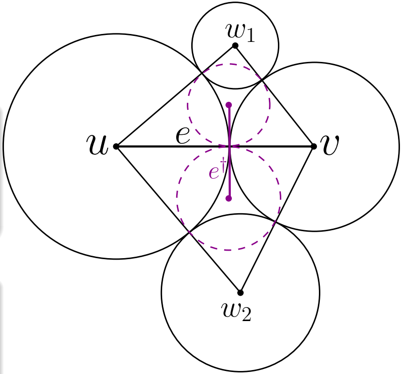
A circle-packed triangulation induces a circle packing of the dual graph by considering the incircles of the triangular faces.

Definition

The **weight** of an edge $e \in E$ is the length ratio $\frac{|e^\dagger|}{|e|}$.

Theorem (Dubejko, 95')

The sequence of centers of circles traversed in the weighted random walk is a martingale.



The Weights: Proof of Martingale Property

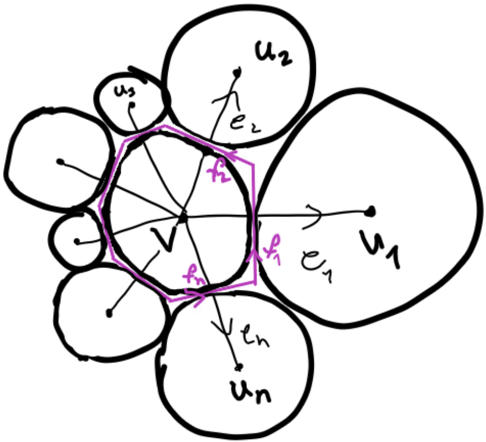
Proof

To show: $\sum_{i=1}^{n} c_{vu_i} \overrightarrow{e_i} = 0$.
Indeed, Let R be a 90° rotation, then:

$$R\left(\sum_{i=1}^{n} c_{vu_i} \overrightarrow{e_i}\right) =$$

$$\sum_{i=1}^{n} \frac{\left|\overrightarrow{f_i}\right|}{\left|\overrightarrow{e_i}\right|} R\left(\overrightarrow{e_i}\right) =$$

$$\sum_{i=1}^{n} \overrightarrow{f_i} = 0.$$



Special case of converse: bounded carrier

Proposition

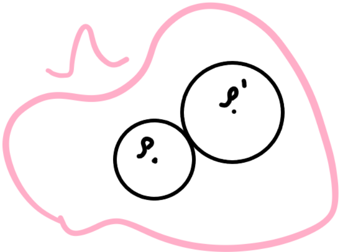
The weighted random walk $(X_n)_{n\in\mathbb{N}}$ on a circle-packed infinite triangulation with a bounded carrier is transient.

Proof

Fix two vertices $\rho, \rho' \in V$.
$(X_n)_{n\in\mathbb{N}}$ is a bounded martingale, so $X_n \xrightarrow{a.s} X$ for some RV X.
Suppose recurrence, then:

1. a.s. we have $X_n = \rho$ infinitely often $=> X \overset{a.s}{=} \rho$.
2. a.s. we have $X_n = \rho'$ infinitely often $=> X \overset{a.s}{=} \rho'$.

in contradiction.



Sketch of the Proof

Theorem (Gurel-Gurevich, S. 2019)

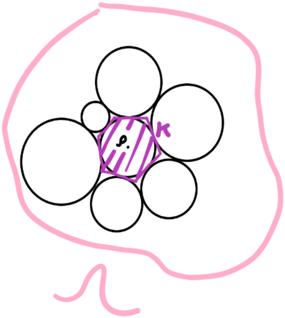
The weighted random walk on a circle-packed infinite triangulation $\{C_v\}_{v \in V}$ with parabolic carrier Ω is recurrent.

Sketch of Proof

Goal: find $f : V \to \mathbb{R}$ finitely supported, $f(\rho) = 1$ with small energy $\varepsilon(f) = \sum_{xy \in E} c_{xy} (f(y) - f(x))^2$.

Take $f = \phi|_V$ for $\phi : \Omega \to [0,1]$ of parabolicity:

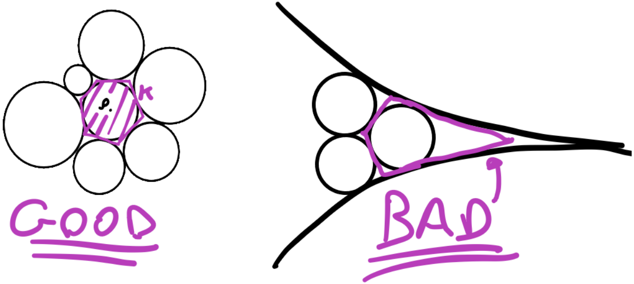
- $\phi|_K \equiv 1 \Rightarrow f(\rho) = 1$
- ϕ comp. supp. $\Rightarrow f$ finitely supported.
- $\iint_\Omega \|\nabla \phi\|^2 \, dA$ is small.



Sketch of the Proof

How to bound the discrete sum?

1. Discrete sum $\leq$ some 2D integral on polygons.
   - (fund. theorem, averaging, Cauchy-Schwarz, euclidean geometry)
2. For good-angled polygons: bound with continuous Dirichlet energy
   - (Harmonic analysis)
3. For bad-angled polygons: "fix" graph to bound in another way.
   - (Martingales, Markov chains, Coupling, Electric networks)



Two Transforms on a Markov Chain

Definition

A **network** is a pair (V, c), where $c : V \times V \to [0, \infty)$ are weights such that:

- *(Symmetric)* $c_{x,y} = c_{y,x}$ for all $x, y \in V$
- *(Normalizable)* $\pi(x) := \sum_{y \in V} c_{x,y} < \infty$ for all $x \in V$
- *(Connected)* The graph obtained by connecting x, y by an edge iff $c_{x,y} > 0$ is connected.

Definition

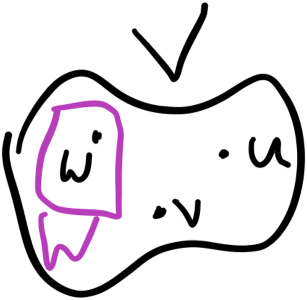
A Markov chain on state space V is said to be **represented by** the network (V, c) if its transition matrix P satisfies: $P_{x,y} = \frac{c_{x,y}}{\pi(x)}$.

Two Transforms on a Markov Chain

Definition

Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain on state space V:

1. Let $W \subsetneq V$ finite, set $V' := V \backslash W$. The **chain censored to** V' is the process obtained by deleting appearances of W from $(X_1, X_2, \ldots)$.
2. The **repetition-deleted chain** is the process obtained by deleting adjacent appearances of the same state from $(X_1, X_2, \ldots)$.



$(v, w, v, u, w \ldots)$

$\Downarrow$ ①

$(v, v, u, \ldots)$

$\Downarrow$ ②

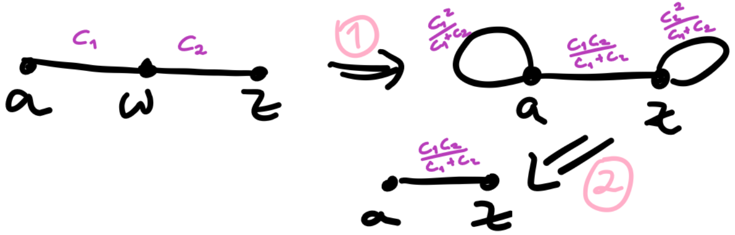
$(v, u, \ldots)$

Two Transforms on a Markov Chain

Proposition

Let (V, c) be a network and let (X_n) be a MC represented by (V, c). Then:

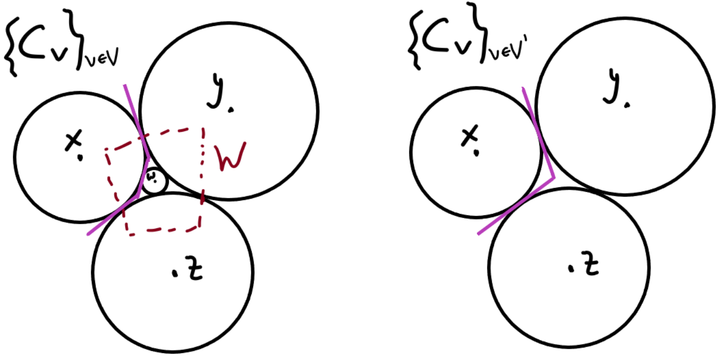
1. For $W = \{w\}$ and $V' := V \setminus W$, the chain censored to V' is represented by (V', c') where: $c'_{x,y} = c_{x,y} + \frac{c_{x,w} c_{w,y}}{\pi(w) - c_{w,w}}$.
2. The repetition-deleted chain is represented by (V, c') where:
$$c'_{x,y} = \begin{cases} c_{x,y}, & x \neq y \\ 0, & x = y \end{cases}$$



Coupling Lemma

Lemma

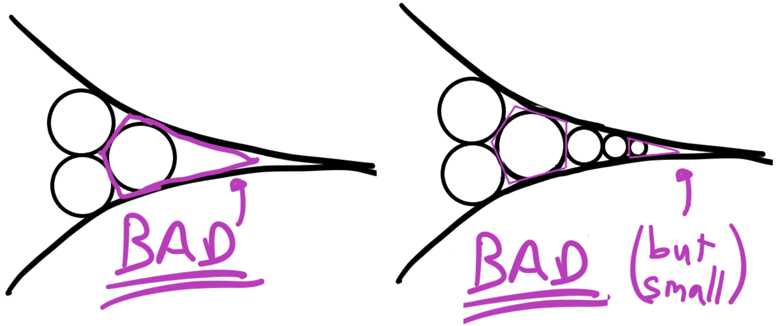
Let $\{C_v\}_{v\in V}$ be a CPIT and let C_x, C_y, C_z be mutually tangent such that $W := V \cap int\left(conv\left(\{x,y,z\}\right)\right)$ is finite.
Let $(X_n)_{n\in\mathbb{N}}$ be a weighted random walk on $\{C_v\}_{v\in V}$.
Then the chain obtained by censoring (X_n) to $V' := V\backslash W$ and then deleting repetitions is represented by the weights of $\{C_v\}_{v\in V'}$.



Why is the lemma useful?

Idea: Insert circles to "push" bad polygons

- Circle insertion gives control over the bad edges' weights.
- insert until all bad edges' weights are summable (with small sum).
- Use bound $c_{x,y}\left(f\left(y\right)-f\left(x\right)\right)^{2}\leq c_{x,y}$.
- Lemma => original network recurrent iff new network recurrent (same eff. resistance to inf).



Proof of Coupling Lemma

Notation

Chain	What is it?	Weights
(X_n)	weighted RW on $\{C_v\}_{v \in V}$	c
(Y_n)	censored & rep-deleted (X_n)	c'
(Z_n)	weighted RW on $\{C_v\}_{v \in V'}$	c''

Proof

To show:
$c' = c''$.

Note: for all
$uv \notin \{xy, xz, yz\}$
we have:
$c'_{uv} = c_{uv} = c''_{uv}$.

Proof of Coupling Lemma

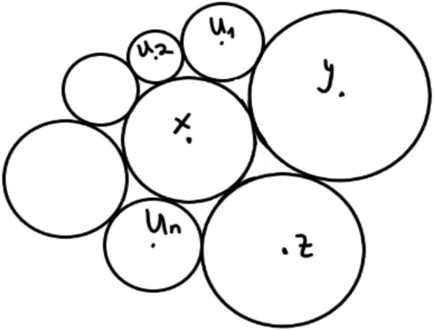
Proof

Set $Z_0 \equiv x$. Since (Z_n) is a martingale:

$$0 = \mathbb{E}\left[Z_1 - Z_0\right] = \sum_{i=1}^{n} \frac{c''_{x,u_i}}{\pi''(x)} \left(u_i - x\right) + \frac{c''_{x,y}}{\pi''(x)} \left(y - x\right) + \frac{c''_{x,z}}{\pi''(x)} \left(z - x\right)$$

So we get:

$$\sum_{i=1}^{n} c_{x,u_i} \left(u_i - x\right) + c''_{x,y} \left(y - x\right) + c''_{x,z} \left(z - x\right) = 0.$$

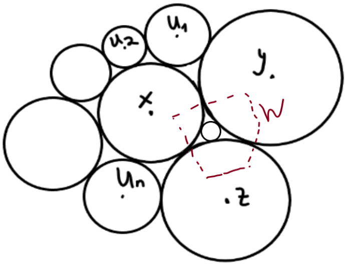


Proof of Coupling Lemma

Proof

Set $X_0 \equiv x$ and set stopping time $\tau = \inf\{t > 0 \mid X_t \notin W \cup \{x\}\}$.
Since (X_n) is a martingale:
$0 = \mathbb{E}[X_\tau - X_0] = \mathbb{E}[Y_1 - Y_0] =$
$\sum_{i=1}^{n} \frac{c'_{x,u_i}}{\pi'(x)}(u_i - x) + \frac{c'_{x,y}}{\pi'(x)}(y - x) + \frac{c'_{x,z}}{\pi'(x)}(z - x)$.

So we get:
$\sum_{i=1}^{n} c_{x,u_i}(u_i - x) + c'_{x,y}(y - x) + c'_{x,z}(z - x) = 0$.



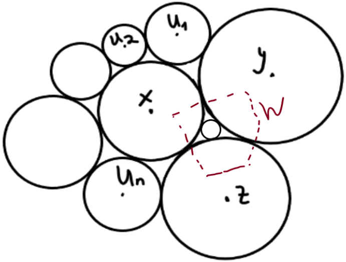
Proof of Coupling Lemma

Proof

Substracting the two equations gives:
$$\left(c'_{x,y} - c''_{x,y}\right)(y - x) + \left(c'_{x,z} - c''_{x,z}\right)(z - x) = 0.$$

$y - x$ and $z - x$ are edges of a triangle $=>$ linearly independent!

Thus: $c'_{x,y} - c''_{x,y} = c'_{x,z} - c''_{x,z} = 0.$



Thanks!