Bound-displacement non-equivalence in Substitution tilings

Abstract: Given two Delone sets Y, Z in $\mathbb{R}^d$ we study the existence of a bounded-displacement (BD) map between them, namely a bijection f from Y to Z so that the quantity $\|y - f(y)\|$, y in Y, is bounded. This notion induces an equivalence relation on collections X of Delone sets and we study the cardinality of $BD(X)$, a set of distinct BD-class representatives. In this talk we focus on set X of point sets that correspond to tilings in a substitution tiling space. We provide a sufficient condition under which $|BD(X)| = 2^{\aleph_0}$. In particular, we show that, in the context of primitive substitution tilings $|BD(X)|$ can be greater then one.

Bound-displacement non-equivalence in Substitution tilings

• **Def.** A point set $\Lambda \subseteq \mathbb{R}^d$ is called a <u>Delone set</u> if it is

(i) Uniformly discrete: $\exists r>0 \ \forall x \neq y \in \Lambda \ \|x-y\| \geq r$.

(ii) Relatively dense: $\exists R>0 \ \bigcup_{x\in\Lambda} B(x,R) = \mathbb{R}^d$.

<u>Examples:</u>

(I) Lattices / translated lattices.

(II) Perturbed lattices: move each point by at most M (keeping uniform discreteness).

(III) <u>More constructions</u>: - Random point processes.

- Cut-and-project sets.
- Sets of 'return times' to a section in an $\mathbb{R}^d$-action on a manifold.
- τ = tiling $\rightsquigarrow \Lambda_\tau$ pick a point in each tile (keeping uniform discreteness).

- Note that there is a correspondence

tilings $\longleftrightarrow$ Delone sets

[finitely many tile types with bounded diameter and inradius]

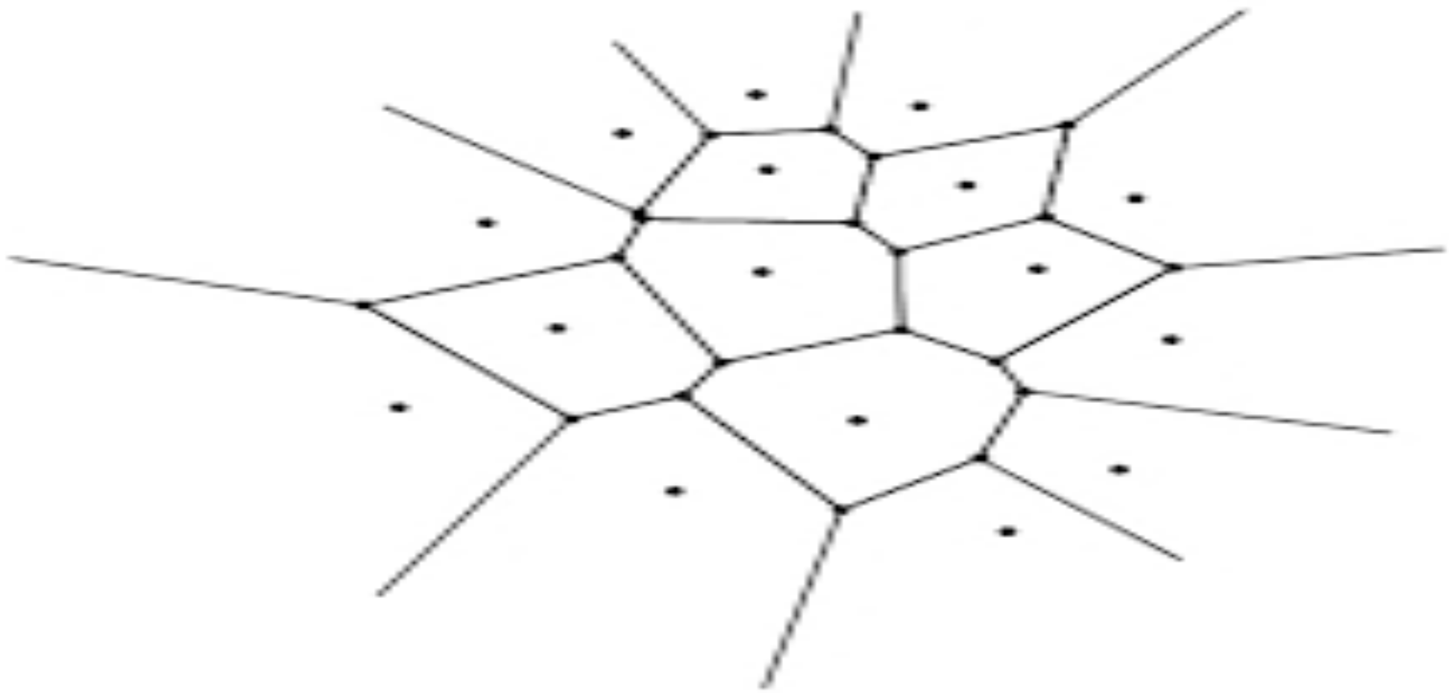
$\longrightarrow$: $\mathcal{T}$ tiling $\mapsto \Lambda_{\mathcal{T}}$ point from each tile.

$\longleftarrow$: Using "approx. Voronoi cells":

Λ Delone set. $\forall\, y \in \Lambda$ define

$$T_y = \bigcup \{ C \mid \forall x \in \Lambda \ \operatorname{dist}(y, C) \leq \operatorname{dist}(x, C) \}$$

cubes with vertices in $\frac{r}{2}\mathbb{Z}^d$



- If τ_Λ is a tiling obtained from Λ then Λ and Λ_{τ_Λ} differ by moving each point a bounded distance.

- **Def:** Two point sets $\Lambda, \Gamma \subseteq \mathbb{R}^d$ are <u>bound displacement (BD) equivalent</u> if $\exists$ a BD-map between them, that is a bijection $\varphi: \Lambda \to \Gamma$ with

$$\sup_{x \in \Lambda} \{ \| x - \varphi(x) \| \} < \infty$$

- similarly, tilings τ, η are BD-equiv. if Λ_τ and Λ_η are.

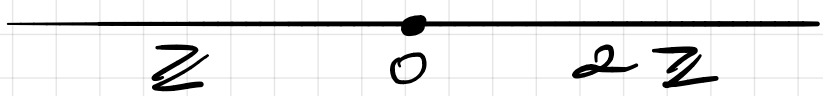
Observations:

- The definition induces an equivalence relation on collections of Delone sets.

- Every two lattices of the same covolume are BD equivalent.

- Every tiling of $\mathbb{R}^d$ by tiles of equal volume α is BD to $\alpha\mathbb{Z}^d$.

- $\exists$ Delone sets that are not BD to any lattice, e.g.



Previous works

Thm [Laczkovich '91]: TFAE for $\alpha > 0$ and a discrete set $\Lambda \subseteq \mathbb{R}^d$:

① $\exists$ BD-map $\varphi : \Lambda \to \alpha^{-1/d} \mathbb{Z}^d$.

② $\exists C > 0 \ \forall A \in \mathcal{Q}_d$ [= Finite unions of lattice cubes]

$$\left| \#(\Lambda \cap A) - \alpha \mathrm{Vol}(A) \right| \leq C \cdot \mathrm{Vol}_{d-1}(\partial A)$$

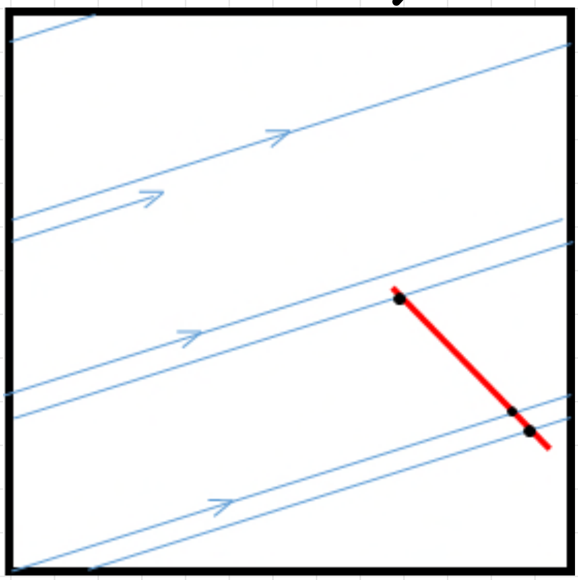
②' For tilings τ: $\exists C > 0 \ \forall A \in \mathcal{Q}_d$

$$\left| \#\{\text{tiles that intersects } A\} - \alpha \mathrm{Vol}(A) \right| \leq \ldots$$

②'' $\exists C > 0 \ \forall$ patch P in τ

$$\left| \# P - \alpha \cdot \mathrm{Vol}(\mathrm{supp}(P)) \right| \leq C \cdot \mathrm{Vol}_{d-1}(\partial P)$$

- Thm [Haynes, Kelly, Weiss '14]



$\mathbb{T}^n$, $n > d \geq 2$

$\mathcal{V} \cong \mathbb{R}^d$

S = Poincaré section
$n-d$ dimensional.

$$\Lambda_{S, x_0} = \{ v \in \mathcal{V} \mid v.x_0 \in S \}$$

- For a.e. $\mathcal{V}$, $\forall x_0$, and S "nice enough", Λ_{S,x_0} is BD to a lattice.
- $\exists$ a residual set of $\mathcal{V}$'s, $\forall x_0$, S "nice enough", Λ_{S,x_0} is not BD to a lattice.

Remark: For $\underbrace{\text{linear sections}}$ S, Λ_{S,x_0} is a cut-and-project set.

$S = \pi(K)$, $K \subseteq \mathcal{U}$ an $n-d$-dim subspace, transverse to $\mathcal{V}$, with non-empty interior in $\mathcal{U}$.

Substitution tilings:

- $\mathcal{F} = \{T_1, \dots, T_n\}$ finite set of tiles in $\mathbb{R}^d$, $\beta > 0$ is a fixed expansion constant, and each T_i has a tiling by elements of $\frac{1}{\beta}\mathcal{F} = \{\beta^{-1}T_1, \dots, \beta^{-1}T_n\}$.

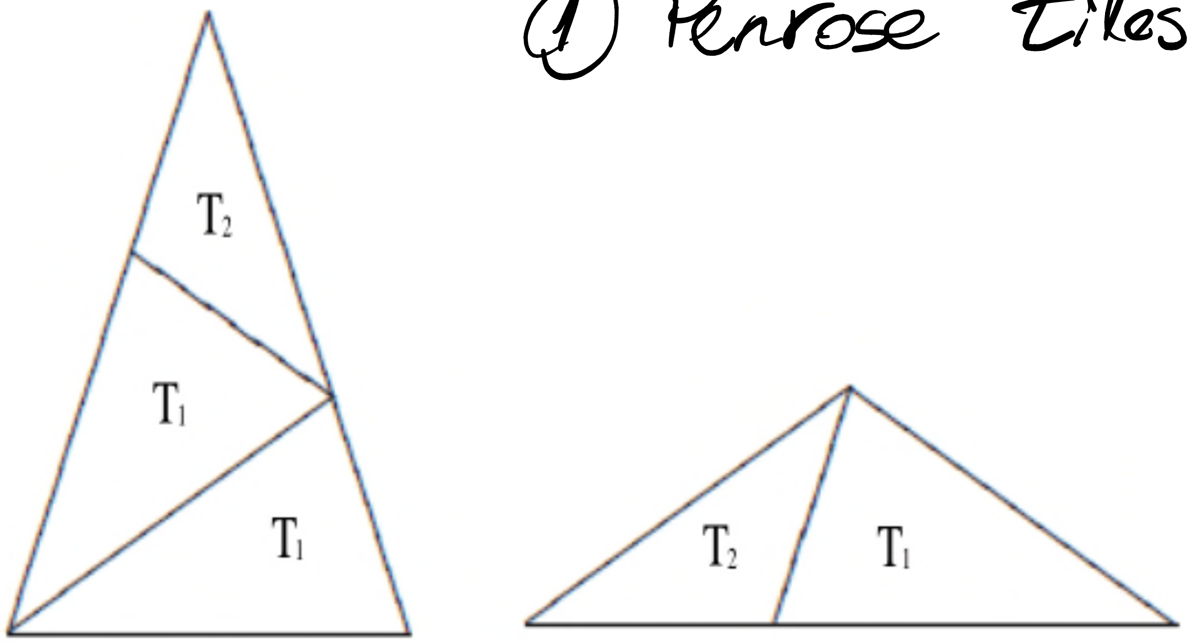
- The substitution rule σ is the operation of expanding by β and substitute by the given rule.

- By applying σ repeatedly and take limits one obtains tilings of $\mathbb{R}^d$ - <u>substitution tilings</u>

- X_σ, the <u>tiling space</u>, is the collection of all these tilings, is usually minimal and uniquely ergodic w.r.t. the $\mathbb{R}^d$-action by translations.

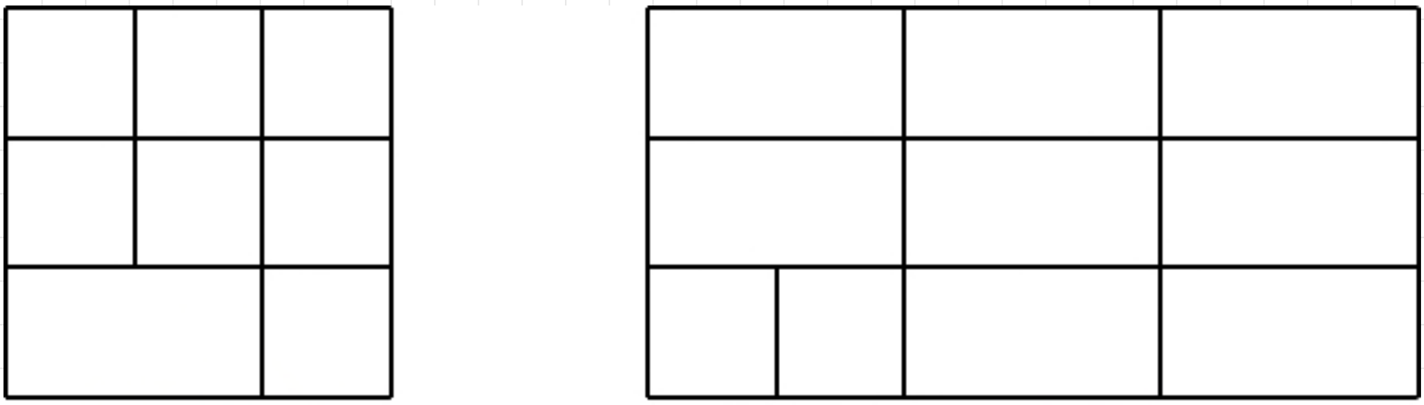
Examples

1 Penrose tiles



T_1 $\qquad$ T_2

2



T_1 $\qquad$ T_2

- Substitution matrix: $M_\sigma = (a_{ij})$

$$a_{ij} := \#\text{ tiles of type } i \text{ in } \sigma(T_j).$$

- σ is called primitive if M_σ is primitive.

For primitive substitution systems:

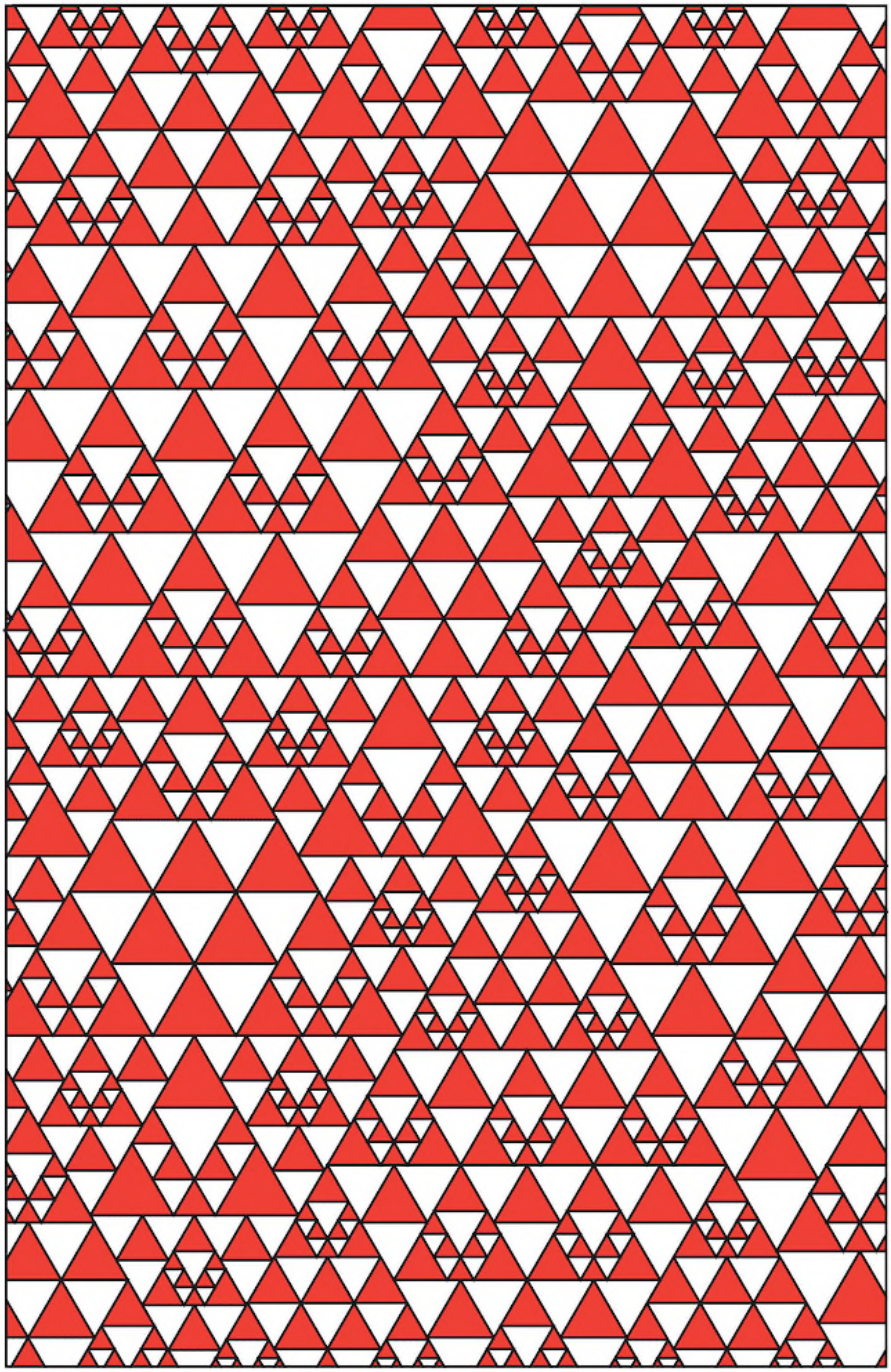
- Eigenvalues: $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$

Thm [S. '11] If $|\lambda_2| < 1$ then every $\mathcal{T} \in X_\sigma$ is BD to a lattice.

Thm [Aliste-Prieto, Coronel, Gambaudo '13]
If $|\lambda_2| < \lambda_1^{1/d} \Rightarrow \forall\, \mathcal{T} \in X_\sigma$ is BD to a lattice.

Thm [S. '14] (for polygonal tiles)
Let $t \geq 2$ be minimal s.t. $V_{\lambda_t} \notin \mathbb{1}^\perp$.
- $|\lambda_t| < \lambda_1^{d-1/d} \Rightarrow \forall\, \mathcal{T} \in X_\sigma$ is BD to a lattice.
- $|\lambda_t| > \lambda_1^{d-1/d} \Rightarrow \forall\, \mathcal{T} \in X_\sigma$ is <u>not</u> BD to a lattice.

__Thm__ [Smilansky, S. '20] Every incommensurable multiscale substitution tiling $\mathcal{T}$ is not BD to a lattice.



BD for non-lattices Λ_1, Λ_2

Thm [Frettlöh, Smilansky, S. '19] (non-BD condition)

Let Λ_1, Λ_2 be two Delone sets in $\mathbb{R}^d$ and assume $\exists$ a sequence of sets $(A_m)_{m=1}^{\infty}$ in Q_d s.t.

$$\frac{|\#(A_m \cap \Lambda_1) - \#(A_m \cap \Lambda_2)|}{Vol_{d-1}(\partial A_m)} \xrightarrow{m\to\infty} \infty$$

then Λ_1 is not BD to Λ_2.

Thm [FSS '19]

$\exists$ mixed substitution system s.t. the tiling space contains continuously many distinct BD-classes.

Question: Can this happen in systems of Delone sets which are minimal (w.r.t. translations)?

Thm [Frettlöh, Garber '16]
$\exists$ cut-and-project set Λ whose hull contains a set Γ not BD to Λ.

For primitive substitution σ

- Recall, if $|\lambda_t| < \lambda_1^{d-1/d} \implies \#BD(X_\sigma)=1$
- For a patch P let $v(P) \in \mathbb{R}^n$ be s.t.

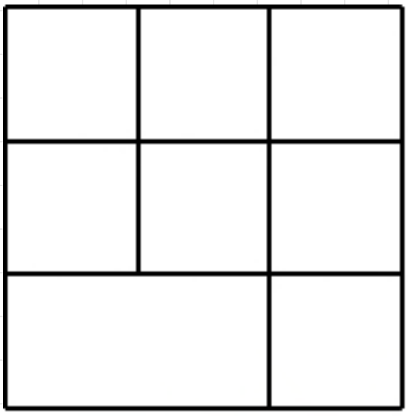
$$v(P)_i = \#\{\text{tiles of type } i \text{ in } P\}$$

Thm [S. '20]
Suppose $|\lambda_t| > \lambda_1^{d-1/d}$ and let $V_t \notin \mathbb{1}^{\perp}$ be an eigenvector of λ_t. Assume that $\exists$ patches P, Q s.t.

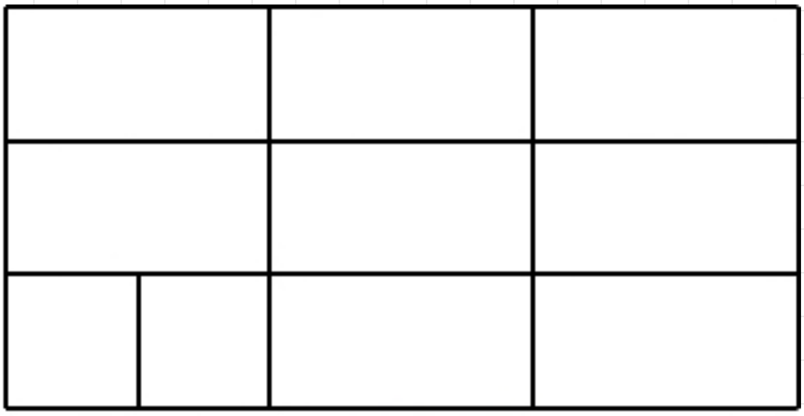
1. $\mathrm{supp}(P)$, $\mathrm{supp}(Q)$ differ by translation.
2. $v(P) - v(Q) \notin V_t^{\perp}$.

Then $|BD(X_\sigma)| = 2^{\aleph_0}$.

Example

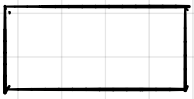
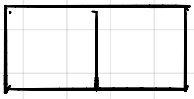


T_1



T_2

$$M_\sigma = \begin{pmatrix} 7 & 2 \\ 1 & 8 \end{pmatrix} \Rightarrow \lambda_1 = 9 \quad (\lambda_t =)\ \lambda_2 = 6$$

$P =$  , $Q =$  $\Rightarrow v_P = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad v_Q = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$v_P - v_Q = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = v_t$$

Idea of the proof:

- Assumptions ① + ② $\Rightarrow$ $|BD(X_\sigma)| \geq 2$.

- Recall: if $\exists$ a sequence of sets $(A_m)_{m=1}^{\infty}$ in Q_d s.t.

$$\underline{\underline{(*)}} \quad \frac{|\#(A_m \cap \Lambda_1) - \#(A_m \cap \Lambda_2)|}{Vol_{d-1}(\partial A_m)} \xrightarrow{m\to\infty} \infty$$

then Λ_1 is not BD to Λ_2.

- In the example: $A_m := supp(\sigma^m(P))$ or Q

$$\#\{\text{tiles in } \sigma^m(P)\} = \langle M_\sigma^m \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \rangle$$

$$\#\{\text{tiles in } \sigma^m(Q)\} = \langle M_\sigma^m \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \rangle$$

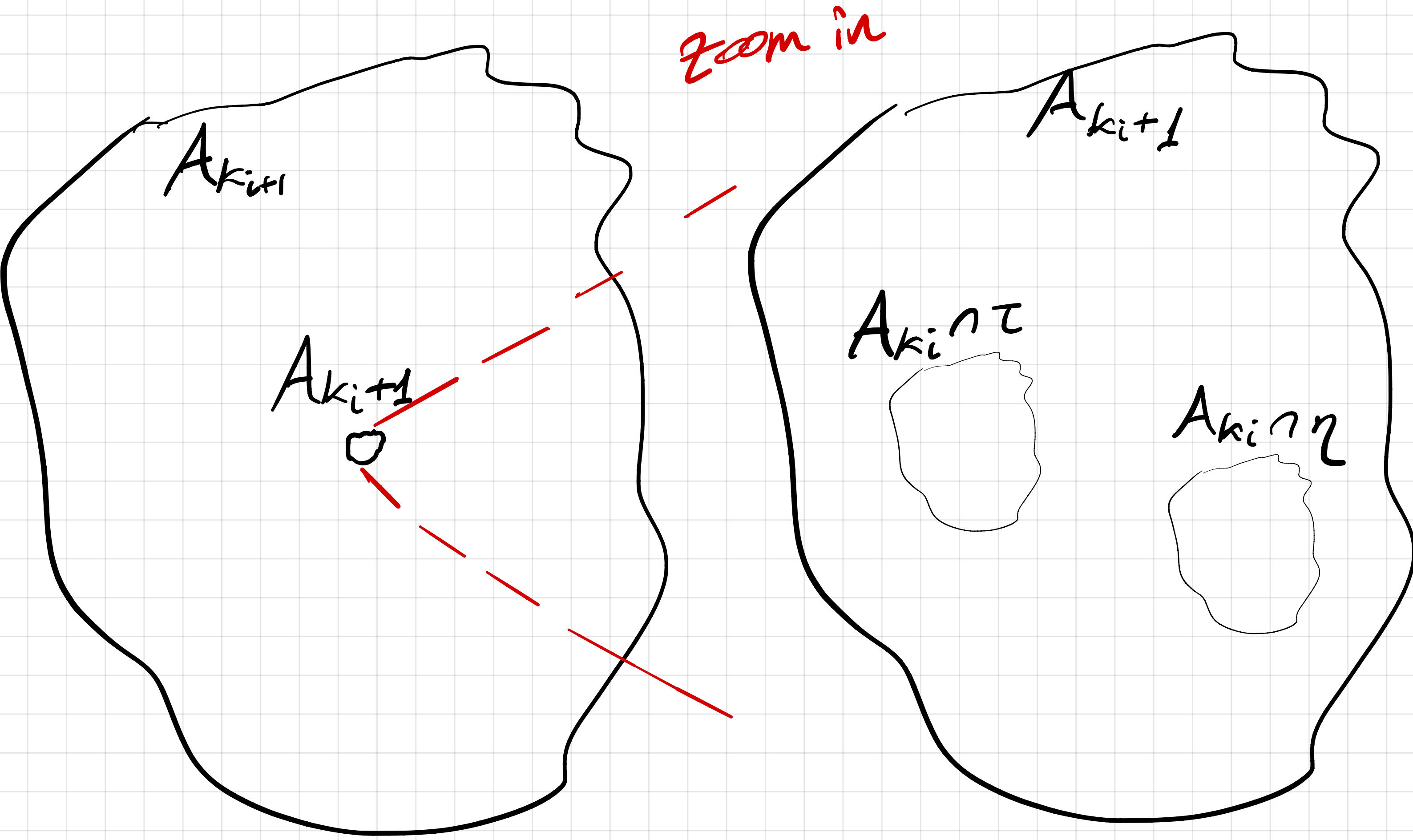
$$\Rightarrow \left| \#\{\text{tiles in } \sigma^m(P)\} - \#\{\text{tiles in } \sigma^m(Q)\} \right|$$

$$= \left| \langle M_\sigma^m \begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \rangle \right| = |\lambda_t|^m \left| \langle v_t, \begin{pmatrix}1\\1\end{pmatrix} \rangle \right|$$

Step 2: $|BD(X_\sigma)| \geq 2 \Rightarrow |BD(X_\sigma)| = 2^{\aleph_0}$:

- Construct inductively a sequence of sets H_i for the condition $(*)$, $H_i = A_{k_i}$. If (k_i) grows "fast enough", the discrepancy grows faster than the boundary.

- Suppose τ & η are two non-BD tilings in X_σ



Three observations:

(i) It is possible to find the above combination both in $A_{k_{i+1}} \cap \tau$ and in $A_{k_{i+1}} \cap \eta$.

(ii) We can position the patch in $A_{k_{i+1}}$ by fixing the position of $A_{k_i} \cap \tau$ or $A_{k_i} \cap \eta$ (the one we are interested in).

(iii) If (k_i) grows fast enough then the size of the translation vector that maps $\mathrm{supp}(A_{k_i} \cap \tau)$ to $\mathrm{supp}(A_{k_i} \cap \eta)$ is negligible.