Abstract: Given two Delone sets \( Y, Z \) in \( \mathbb{R}^d \), we study the existence of a bounded-displacement (BD) map between them, namely a bijection \( f \) from \( Y \) to \( Z \) so that the quantity \( |y - f(y)| \) for \( y \in Y \) is bounded. This notion induces an equivalence relation on collections \( X \) of Delone sets and we study the cardinality of \( \text{BD}(X) \), a set of distinct BD-class representatives. In this talk we focus on set \( X \) of point sets that correspond to tilings in a substitution tiling space. We provide a sufficient condition under which \( |\text{BD}(X)| = 2^{2d} \). In particular, we show that, in the context of primitive substitution tiling, \( |\text{BD}(X)| \) can be greater than one.
Bound-displacement non-equivalence in substitution tilings

**Def.** A point set \( \Lambda \subset \mathbb{R}^d \) is called a Delone set if it is

(i) Uniformly discrete: \( \exists r > 0 \forall x, y \in \Lambda \quad \|x - y\| > r \)

(ii) Relatively dense: \( \forall R > 0 \quad \exists \Lambda \subset \mathbb{R}^d \) such that

Examples:

(I) Lattices / translated lattices.

(II) Perturbed lattices: move each point by at most \( M \) (keeping uniform discreteness).

(III) More constructions:
- Random point processes.
- Cut-and-project sets.
- Sets of 'return times' to a section in an \( \mathbb{R}^d \)-action on a manifold.
- \( \tau = \text{tiling} \mapsto \Lambda_{\tau} \) pick a point in each tile (keeping uniform discreteness).
Note that there is a correspondence:

\[ \text{Tilings} \leftrightarrow \text{Delone sets} \]

Finitely many tile types with bounded diameter and inradius.

\[ \text{\rightarrow} : \text{tiling } \mapsto \Lambda \tau \text{ point from each tile.} \]

\[ \text{\leftarrow} : \text{Using "approx. Voronoi cells";} \]

\[ \Lambda \text{ Delone set. } \forall x \in \Lambda \text{ define} \]

\[ T_y = \bigcup \{ \mathcal{C} \mid \forall x \in \Lambda \text{ dist}(y, \mathcal{C}) \leq \text{dist}(x, \mathcal{C}) \} \]

Cubes with vertices in \( \frac{r}{2} \mathbb{Z}^d \)
If \( T_\Lambda \) is a tiling obtained from \( \Lambda \) then \( \Lambda \) and \( \Lambda_{T_\Lambda} \) differ by moving each point a bounded distance.

**Def.** Two point sets \( \Lambda, \Gamma \subseteq \mathbb{R}^d \) are **bounded displacement (BD) equivalent** if there is a BD-map between them, that is a bijection \( \psi : \Lambda \to \Gamma \) with

\[
\sup_{x \in \Lambda} \| x - \psi(x) \| < \infty
\]

- similarly, tilings \( T_1, T_2 \) are BD-eqiv if \( \Lambda_{T_1} \) and \( \Lambda_{T_2} \) are.
Observations:

- The definition induces an equivalence relation on collections of Delone sets.

- Every two lattices of the same covolume are BD equivalent.

- Every tiling of \( \mathbb{R}^d \) by tiles of equal volume \( \alpha \) is BD to \( \alpha \mathbb{Z}^d \).

- Z Delone sets that are not BD to any lattice, e.g. \( \mathbb{Z} \otimes \mathbb{Z} \). 
Previous works

Thm [Laczkovich '81]: TFAE for $\alpha > 0$ and $\Lambda \subseteq \mathbb{R}^d$:

1. BD-map $\varphi: \Lambda \rightarrow \mathbb{R}^d$.
2. $\exists C > 0 \ A \in \mathbb{Q}_d \left[ \text{Finite unions of lattice cubes} \right]$
   \[ \left| \#(\Lambda \cap A) - \alpha \text{Vol}(A) \right| \leq C \cdot \text{Vol}_{d-1}(\partial A) \]

2. For tilings $\mathcal{T}$:
   $\exists C > 0 \ A \in \mathbb{Q}_d$
   \[ \left| \# \{ \text{tiles that intersects } A \} - \alpha \text{Vol}(A) \right| \leq \_ \_ \_ \_ \]

2. $\exists C > 0 \ \forall \text{patch } P \ in \ \mathcal{T}$
   \[ \left| \# P - \alpha \cdot \text{Vol} (\text{supp}(P)) \right| \leq C \cdot \text{Vol}_{d-1}(\partial P) \]
\( \mathbb{R}^n, n > d \geq 2 \)
\( \mathbb{V} \subseteq \mathbb{R}^d \)
\( S \) = Poincaré section
\( n \)-d dimensional.

\[ \Lambda_{S, x_0} = \{ v \in \mathbb{V} \mid v \cdot x_0 \in S \} \]

- For a.e. \( \mathbb{V} \), \( \forall x_0 \), and \( S \) "nice enough",
  \( \Lambda_{S, x_0} \) is BD to a lattice.
- For a residual set of \( \mathbb{V} 's \), \( \forall x_0 \),
  \( S \) "nice enough" , \( \Lambda_{S, x_0} \) is not BD to a lattice.

Remark: For linear sections \( S \), \( \Lambda_{S, x_0} \) is a cut-and-project set.

\( S = \mathbb{T}(k), \quad k \leq \mathbb{U} \) an \( n \)-d-dim subspace, transverse to \( \mathbb{V} \), with non-empty interior in \( \mathbb{U} \).
Substitution tilings:

- $F = \{ T_1, \ldots, T_n \}$ finite set of tiles in $\mathbb{R}^d$, $\beta > 0$ is a fixed expansion constant, and each $T_i$ has a tiling by elements of $\beta F = \{ \beta T_1, \ldots, \beta T_n \}$.

- The substitution rule $\sigma$ is the operation of expanding by $\beta$ and substitute by the given rule.

- By applying $\sigma$ repeatedly and take limits one obtains tilings of $\mathbb{R}^d$ - substitution tilings

- $X_0$, the tiling space, is the collection of all these tilings, is usually minimal and uniquely ergodic w.r.t. the $\mathbb{R}^d$-action by translations.
Examples

1. Penrose tiles

- Substitution matrix: \( M_\omega = (a_{ij}) \)
  
  \( a_{ij} : \) \# tiles of type \( i \) in \( \sigma(T_j) \).

- \( \sigma \) is called primitive if \( M_\omega \) is primitive.
For primitive substitution systems:

- Eigenvalues: \( \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_d| \)

\[
\text{Thm [S. Hal]} \quad \text{If } |\lambda_2| < 1 \text{ then every } \tau \in \mathcal{X}_0 \text{ is BD to a lattice.}
\]

\[
\text{Thm [Aliste-Prieto, Coronel, Gambando '13]} \quad \text{If } |\lambda_2| < \frac{\lambda_1}{d} \Rightarrow \forall \tau \in \mathcal{X}_0 \text{ is BD to a lattice.}
\]

\[
\text{Thm [S. Hal] (for polygonal tiles)} \quad \text{Let } \tau_{\text{min}} \text{ be minimal s.t. } \forall \tau \neq \tau_{\text{min}} \text{ such that } |	au| = 1.
\]

- If \( |\lambda_2| < \frac{\lambda_1}{d} \Rightarrow \forall \tau \in \mathcal{X}_0 \text{ is BD to a lattice.} \)
- If \( |\lambda_2| > \frac{\lambda_1}{d} \Rightarrow \forall \tau \in \mathcal{X}_0 \text{ is not BD to a lattice.} \)
Thom [Smolinsky 1997] Every incommensurable multiscale substitution tiling is not BD to a lattice.
**BD for non-lattices \( \Lambda_1, \Lambda_2 \)**

Thm [Frettlöh, Smilansky, ‘19] (non-BD condition)

Let \( \Lambda_1, \Lambda_2 \) be two Delone sets in \( \mathbb{R}^d \) and assume \( \mathcal{I} \) a sequence of sets \((\Lambda_m)_{m=1}^\infty\)

in \( \mathbb{Q}^d \) s.t.

\[
\left\lvert \#(\Lambda_m \cap \Lambda_1) - \#(\Lambda_m \cap \Lambda_2) \right\rvert \quad m \to \infty
\]

\[
\frac{\#(\Lambda_m \cap \Lambda_1) - \#(\Lambda_m \cap \Lambda_2)}{\text{Vol}_{d-1}(\partial \Lambda_m)} \to \infty
\]

Then \( \Lambda_1 \) is not BD to \( \Lambda_2 \).

**Thm [FSS ‘19]**

I mixed substitution system s.t. the tiling space contains continuously many distinct BD-classes.

**Question:** Can this happen in systems of Delone sets which are minimal (w.r.t. translations)?
**Thm [Frettlöh, Garber '16]**

A cut-and-project set $\Lambda$ whose hull contains a set $\Gamma$ not BD to $\Lambda$.

For primitive substitution $\sigma$.

- Recall, if $|x| < \tau^{d-1/d}$ $\implies \# BD(x_0) = 1$

- For a patch $p$ let $v(p) \in \mathbb{R}^n$ be s.t.
  \[ v(p)_i = \# \{ \text{tiles of type } i \text{ in } p \} \]

**Thm [S. '20]**

Suppose $|x| > \tau^{d-1/d}$ and let $v_+ \in \mathbb{R}^n$ be an eigenvector of $\Lambda_t$. Assume that 3 patches $P, Q$ s.t.

1. $\text{supp}(P), \text{supp}(Q)$ differ by translation
2. $v(P) - v(Q) \in V_+$

Then $|BD(x_0)| = 2^{x_0}$. 
Example

\[ M_0 = \begin{pmatrix} \frac{7}{8} & \frac{2}{8} \\ \frac{1}{8} & \frac{6}{8} \end{pmatrix} \Rightarrow r_1 = 9 \ (r_t) \ r_2 = 6 \]

\[ P = \boxed{\phantom{1}} , \ Q = \boxed{\phantom{1}} \Rightarrow V_p = \{0\} \quad V_Q = \{2\} \]

\[ V_p - V_Q = \left( \begin{array}{c} -2 \\ 1 \end{array} \right) = V_t \]
Idea of the proof:
- Assumptions 1+2 \( \Rightarrow |BD(X_o)| \geq 2 \).

**Recall:** if \( I \) a sequence of sets \((A_m)_{m \geq 1}\)
in \( \mathbb{R}^d \) s.t.
\[
\left| \frac{\#(A_m \cap \Lambda_1) - \#(A_m \cap \Lambda_2)}{\text{Vol}_{d-1}(\partial A_m)} \right| \xrightarrow{m \to \infty} \infty
\]
then \( \Lambda_1 \) is not BD to \( \Lambda_2 \).

- In the example: \( A_m = \text{supp}(\sigma_m^m(p)) \) or \( \mathbb{R} \)

\[
\# \{\text{tiles in } \sigma_m^m(p) \} = \left< M_m^m(1), \langle 1 \rangle \right>
\]
\[
\# \{\text{tiles in } \sigma_m^m(\sigma) \} = \left< M_m^m(2), \langle 1 \rangle \right>
\]
\[
\Rightarrow \left| \# \{\text{tiles in } \sigma_m^m(p) \} - \# \{\text{tiles in } \sigma_m^m(\sigma) \} \right|
\]
\[
= \left| \left< M_m^m(1^2), \langle 1 \rangle \right> - \left< M_m^m(2), \langle 1 \rangle \right> \right|
\]

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Step 2: \( |BD(x_0)| = 2 \Rightarrow |BD(x_0)| = 2^{x_0} \):
- Construct inductively a sequence of sets \( H_i \) for the condition \((\star)\), \( H_i = A_i \). If \( (k_i) \) grows "fast enough", the discrepancy grows faster than the boundary.
- Suppose \( \mathcal{T} \) and \( \mathcal{L} \) are two non-BD tilings in \( X_0 \).
Three observations:

(i) It is possible to find the above combination both in $A_{k_i} \cap \tau$ and in $A_{k_i} \cap \eta$.

(ii) We can position the patch in $A_{k_i}$ by fixing the position of $A_{k_i} \cap \tau$ or $A_{k_i} \cap \eta$ (the one we are interested in).

(iii) If $(k_i)$ grows fast enough then the size of the translation vector that maps $\text{supp} (A_{k_i} \cap \tau)$ to $\text{supp} (A_{k_i} \cap \eta)$ is negligible.