Small perturbations of non-Hermitian matrices

Ofer Zeitouni
Joint with Anirban Basak, Elliot Paquette and Martin Vogel

April 2020

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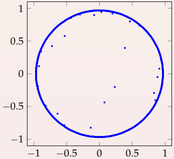

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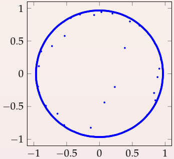

Goes back to Trefethen et als - pseudo-spectrum.

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For the empirical measure $L_N = N^{-1} \sum_{i=1}^N \delta_{\lambda_i^A}$ of eigenvalues of a matrix A, we have

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Thus, spectrum computations involves the determinant of a family of Hermitian matrices built from A!

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Main ingredient of proof compares the singular values $\Sigma_A(t) = (\sigma_1^A, \ldots, \sigma_N^A)$ of $A_N + tN^{-1/2} G_N$ to the singular values $\Sigma_0(t) = (\sigma_1, \ldots, \sigma_N)$ of $tN^{-1/2} G_N$;

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How can we take $t = t_N \to 0$?

Regularization by noise

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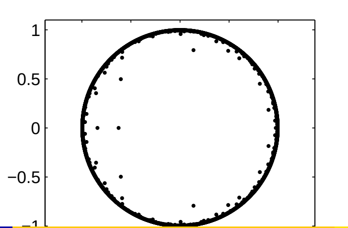

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General criterion - Guionnet, Wood, Z.

Noise Stability-Maximal Nilpotent

$a \in \mathcal{A}$ is regular if for ψ smooth, compactly supported,

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The proof uses the regularity (of the limit) to truncate the singularity of the $\log$... and depends crucially on convergence to ν_a. But it is not useful in maximally nilpotent example, since $L_N^A = \delta_0 \not\to \nu_a = \delta_{S^1}$!.

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Assume: $A_N \xrightarrow{*} a$, *regular,* $\|E_N\| \to 0$ *polynomially.* $L_N^{A_N+E_N} \to \nu_a$ *weakly.*

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So it is enough to find a perturbation with correct limiting behavior! Nilpotent example uses a- unitary element (which is regular), E_N is $(N, 1)$ element.

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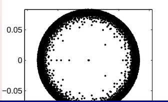
Theorem (Guionnet-Wood-Z '14)

If $b = a \log N$ and γ is large enough, then the spectral radius of $J_{b,N} + N^{-\gamma} G_N$ is uniformly strictly smaller than 1. In particular,

$$L_N^{J_{a \log N, N} + N^{-\gamma} G_N} \not\to \delta_{S^1}$$

even though $J_{a \log N, N}$ converges in $*$ moments to random unitary!



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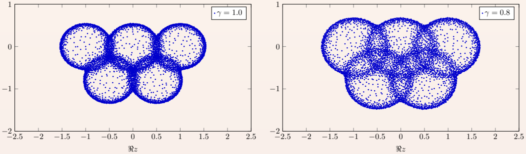
$$A_N = \begin{bmatrix} B^1 & & & \\ & B^2 & & \\ & & \ddots & \\ & & & B^{\ell(N)} \end{bmatrix}.$$

Noise Stability-Block Nilpotent IV

Simulations inconclusive!



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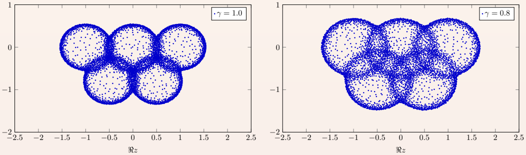

Analyzed by Feldheim-Paquette-Z. (2015).

More general models?

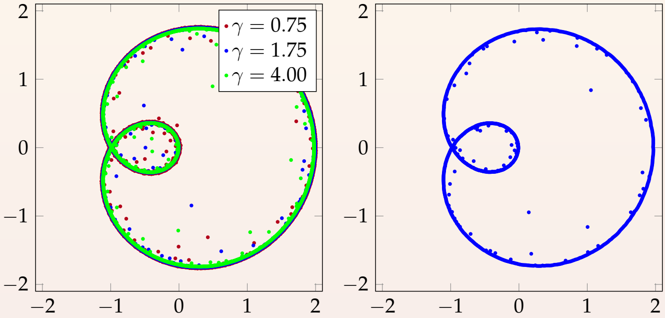

Figure: The eigenvalues of $J_N + J_N^2 + N^{-\gamma} G_N$, with $N = 4000$ and various γ. On left, actual matrix. On the right, $U_N(J_N + J_N^2)U_N^*$.

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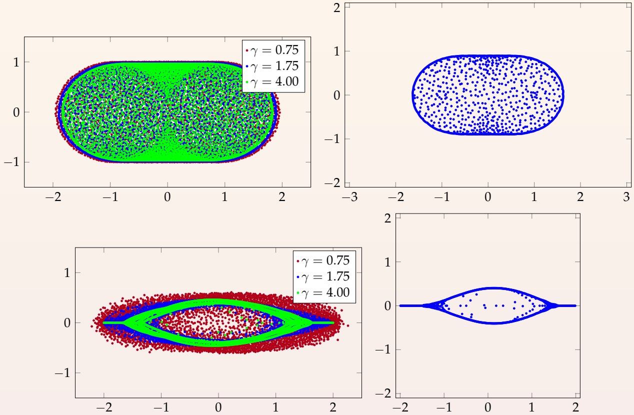

Figure: The eigenvalues of $D_N + J_N + N^{-\gamma} G_N$, with $N = 4000$ and various γ. Top: $D_N(i,i) = -1 + 2i/N$. Bottom: D_N i.i.d. uniform on $[-2, 2]$. On left, actual matrix. On the right, $U_N(D_N + J_N)U_N^*$.

More general models

Theorem (Basak, Paquette, Z. '17)

$T_N = D_N + J_N$, $M_N = T_N + N^{-\gamma} G_N$, $\gamma > 1/2$.
d_i iid uniform on $[-1, 1]$.

More general models

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Then $L_N \to \mu$, μ explicit: log-potential of μ at z is $(E \log |z - d_1|) \vee 0)$.

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Confirms simulations and predictions (based on pseudo-spectrum) of Trefethen et als. Some two-diagonal Toeplitz cases studied by Sjöstrand and Vogel (2016)

BPZ

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So only need to understand small singular values of M_N.

Non triangular Toeplitz, non Gaussian noise

- $E \sum G_N(i,j)^2 = O(N^2)$
- There is $\beta = \beta(\alpha, \gamma)$ so that for any M_N deterministic with $\|M_N\| = O(N^{-\alpha})$, $P(s_{\min}(M_N + N^{-\gamma} G_N) < N^{-\beta}) = o(1)$

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Related (different methods, Gaussian noise - Grushin problem) - Sjöstrand and Vogel '19.

Proof ingredients

Theorem (Replacement principle - after GWZ)

A_N - deterministic, bounded operator norm. Δ_N and G_N - independent random matrices. Assume

(a) *G_N and Δ_N are independent. $\|\Delta_N\| < N^{-\gamma_0}$ whp and G_N noise matrix as before.*

(b) *For Lebesgue a.e. $z \in B_{\mathbb{C}}(0, R_0)$, the empirical distribution of the singular values of $A_N - zI_N$ converges weakly to the law induced by $|X - z|$, where $X \sim \mu$ and $\mathrm{supp}\mu \subset B_{\mathbb{C}}(0, R_0/2)$.*

(c) *For Lebesgue a.e. every $z \in B_{\mathbb{C}}(0, R_0)$,*

$$\mathcal{L}_{L_N^{A+\Delta}}(z) \to \mathcal{L}_\mu(z), \qquad \text{as } N \to \infty, \text{ in probability.} \tag{1}$$

Then, for any $\gamma > \frac{1}{2}$, for Lebesgue a.e. every $z \in B_{\mathbb{C}}(0, R_0)$,

$$\mathcal{L}_{L_N^{A+N^{-\gamma}G}}(z) \to \mathcal{L}_\mu(z), \qquad \text{as } N \to \infty, \text{ in probability.} \tag{2}$$

Proof ingredient II

Theorem

Let T_N be any $N \times N$ banded Toeplitz matrix with a symbol **a**. *Then, there exists a random matrix Δ_N with*

$$P(\|\Delta_N\| \geq N^{-\gamma_0}) = o(1), \tag{3}$$

for some $\gamma_0 > 0$, so that $L_N^{T+\Delta}$ converges weakly, in probability, to $\mu_{\mathbf{a}}$.

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This works for Toeplitz with banded symbol, but not for twisted Toeplitz! Main issue - Toeplitz determinant of un-perturbed matrix requires work, e.g. Widom's theorem.

Grushin's problem

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$$R_+ = \sum_{i=1}^{M} \delta_i \circ e_i^*, \quad R_- = \sum_{i=1}^{M} f_i \circ \delta_i^*,$$

$$\mathcal{P} = \begin{pmatrix} A & R_- \\ R_+ & 0 \end{pmatrix} : \mathbb{C}^N \times \mathbb{C}^M \longrightarrow \mathbb{C}^N \times \mathbb{C}^M \quad \text{bijection!}$$

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We have

$$\mathcal{P}^{-1} = \mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}$$

with

$$E = \sum_{M+1}^{N} \frac{1}{t_i} e_i \circ f_i, \quad E_+ = \sum_{1}^{M} e_i \circ \delta_i^*,$$

$$E_- = \sum_{1}^{M} \delta_i \circ f_i^*, \quad E_{-+} = -\sum_{1}^{M} t_j \delta_j \circ \delta_j^*,$$

and the norm estimates

$$\|E(z)\| \leq \frac{1}{\alpha}, \quad \|E_\pm\| = 1, \quad \|E_{-+}\| \leq \alpha, \quad |\det \mathcal{P}|^2 = \prod_{M+1}^{N} t_i^2.$$

Noisy Grushin problem

$$A^\delta = A + \delta G, \quad 0 \le \delta \ll 1.$$

$$\mathcal{P}^\delta = \begin{pmatrix} A^\delta & R_- \\ R_+ & 0 \end{pmatrix} : \mathbb{C}^N \times \mathbb{C}^M \longrightarrow \mathbb{C}^N \times \mathbb{C}^M$$

Applying $\mathcal{E} = \mathcal{P}^{-1}$ from the right:

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Suppose that $\delta \|G\| \alpha^{-1} \le 1/2$, then

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$$\|E^\delta\| = \|E(1+\delta GE)^{-1}\| \le 2\alpha^{-1}, \|E_+^\delta\| \le 2, \|E_-^\delta\| \le 2, \|E_{-+}^\delta - E_{-+}\| \le \alpha.$$

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The Schur complement formula applied to $\mathcal{P}^\delta$ and $\mathcal{E}^\delta$ shows that

$$\log|\det A^\delta| = \log|\det \mathcal{P}^\delta| + \log|\det E_{-+}^\delta|.$$

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Complementary lower bound requires just a bit more work.

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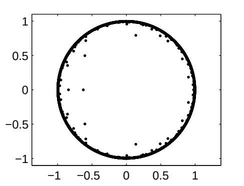
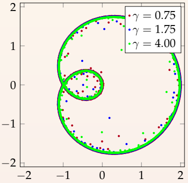
Since $\det \mathcal{P}$ is like erasing the small singular values of A, this gives a version of the deterministic equivalence lemma for general noise (Vogel-Z. '20)

Outliers

$J_N + N^{-\gamma} G_N$ $\qquad$ $J_N + J_N^2 + N^{-\gamma} G_N$





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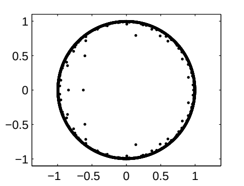

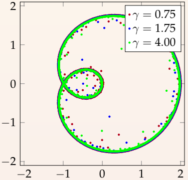

Outliers are random. What is structure of outliers?

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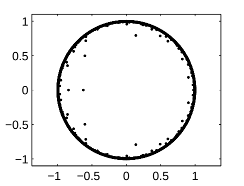

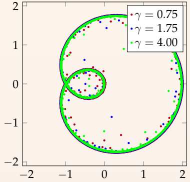

Outliers are random. What is structure of outliers?

- $J_N + N^{-\gamma} G_N$: outliers are zeros of a limiting Gaussian field, all inside disc.

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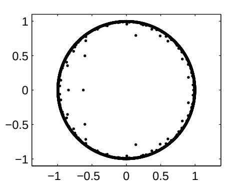

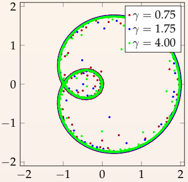

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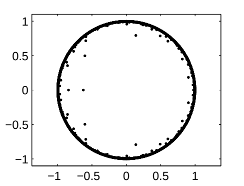

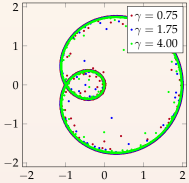

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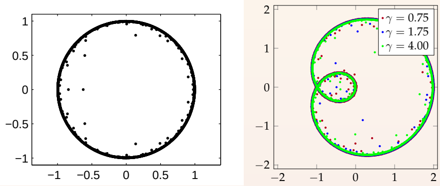

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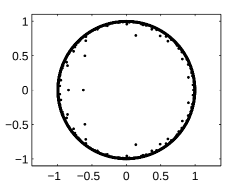

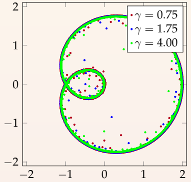

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- Toeplitz, finite symbol $a(\lambda) = \sum_{i=-k_1}^{k_2} a_i \lambda^i$, set

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This does not mean there are no outliers, as $a(S^1) \subsetneq D_\infty(a)$.

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Improves on counting estimates of Sjöstrand and Vogel ('19).

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Computation of intensity first performed by Sjostrand and Vogel (2018).

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