BROWNIAN MOTION HOMEWORK ASSIGNMENT 11

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We detail some definitions used below:

(i) For a closed $A \subseteq \mathbb{R}^d$ let $T_A := \min\{t \colon B_t \in A\}$ be its *hitting time* by Brownian motion (defined to be infinite if $B_t \notin A$ for all t).

(ii) For $x \in \mathbb{R}^d$ and closed $A \subseteq \mathbb{R}^d$ define the *harmonic measure* $\mu_A(x, \cdot)$ as the (possibly sub-probability) measure satisfying $\mu_A(x, S) := \mathbb{P}_x(B_{T_A} \in S, T_A < \infty)$ for Borel $S \subseteq A$. In other words, $\mu_A(x, \cdot)$ is the distribution of the hitting position of A of a Brownian motion started at x, on the event that it hits A.

(iii) A closed set $A \subseteq \mathbb{R}^d$ is called *polar for the point* $x \in \mathbb{R}^d$ if a Brownian motion started from x will miss A with probability one, i.e., if $\mathbb{P}_x(T_A < \infty) = 0$.

We will sometimes identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. We define $D := \{z \in \mathbb{C} \colon |z| < 1\}$ and $H := \{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\}$.

In the following questions you may use the conformal invariance of Brownian motion as needed.

(i) (a) Prove that if a closed set $A \subseteq \mathbb{R}^d$ is polar for some point $x \in \mathbb{R}^d$ then it is polar for all points $x \in A^c$.
Remark: Consequently we will say that A is polar without mentioning the point x.

(b) Let $U, V \subset \mathbb{R}^2$ be domains and $f : \bar{U} \to \bar{V}$ be continuous and such that f maps U conformally onto V. Prove that for any $x \in U$, $\mu_{\partial U}(x, \cdot) \circ f^{-1} = \mu_{\partial V}(f(x), \cdot)$.

(c) Let $K \subset \mathbb{R}^2$ be a compact non-polar set. Prove the existence of the *harmonic measure from infinity* for K. That is, show that for each Borel $S \subseteq K$ the limit

$$\lim_{x \to \infty} \mu_K(x, S)$$

exists. Prove also that if K is a ball then this limit measure is uniformly distributed on its boundary.
Remark: Extensions of this fact exist to higher dimensions as well, where we first condition that the Brownian motion hits K.

(ii) Use conformal invariance to prove the following results:

(a) Prove that for any $z \in D$, the harmonic measure $\mu_{\partial D}(z, \cdot)$ is absolutely continuous with respect to the uniform measure on ∂D and has density

$$\frac{|1 - |z|^2|}{|z - x|^2}, \quad x \in \partial D.$$

Remark: This density is called the *Poisson kernel.* In higher dimensions a similar formula holds with $|z - x|^2$ replaced by $|z - x|^d$.

(b) Prove that for any $z \in H$, the harmonic measure $\mu_{\partial H}(z, \cdot)$ is absolutely continuous with respect to Lebesgue measure on ∂H and has density

$$\frac{\operatorname{Im}(z)}{\pi(\operatorname{Im}(z)^2 + (\operatorname{Re}(z) - x)^2)}, \quad x \in \partial H.$$

Remark: This distribution is called the Cauchy distribution.

(iii) (Bañuelos and Carroll, Markowsky) Let $f : D \to V$ be a conformal map onto some domain $V \subset \mathbb{C}$, with the Taylor series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n.$$

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(a) Prove that for each $0 < r < 1$,

$$\mathbb{E}_{f(0)}(T_{f(rD)^c}) = \frac{1}{2}\sum_{n=1}^{\infty} |a_n|^2 r^{2n}, \tag{1}$$

where $\mathbb{E}_{f(0)}$ stands for expectation with respect to a Brownian motion started at $f(0)$ and $rD := \{z \in \mathbb{C} \colon |z| < r\}$.
Hint: Use the martingale $|B_t|^2 - 2t$. Use also the fact that the L^2 norm of a function on the circle equals the L^2 norm of its Fourier coefficients.

(b) Prove that the formula (1) continues to hold for $r = 1$.

(c) Let $V = \{z \in \mathbb{C} \colon |\operatorname{Re}(z)| < \frac{\pi}{4}\}$. Prove that the principal branch of $\arctan(z)$ maps D conformally onto V.

(d) Use the previous parts to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hint: You may use that the principal branch of $\arctan(z)$ admits the Taylor expansion

$$\arctan(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{2n-1}}{2n-1}.$$

The Brownian motion book is available at: http://research.microsoft.com/en-us/um/people/peres/brbook.pdf