BROWNIAN MOTION HOMEWORK ASSIGNMENT 7

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To get full credit for this exercise it suffices to solve correctly three of the problems. You are encouraged, however, to consider all problems in order to practice your understanding of martingale theory. If you solve more than three problems, your grade will be based on the three problems with the highest score.

(i) (Superharmonic functions on graphs) Let G be an infinite connected, locally finite graph. A simple random walk on G starting at a vertex v_0 is a Markov chain (X_n), $n \geqslant 0$, with $X_0 = v_0$ and $P(X_{n+1} = w | X_n = v) = 1/\text{degree}(v)$ for each pair of neighboring vertices v, w. We say that G is *recurrent* if a simple random walk starting from a fixed vertex of G returns to the starting vertex infinitely often. It is not difficult to check that this property does not depend on the starting vertex. We say G is *transient* otherwise. A function f on the vertices of G is called *superharmonic* if

$$f(v) \geqslant \frac{1}{\text{degree}(v)} \sum_{w\colon w \sim v} f(w) \quad \text{for all vertices } v$$

where we write $w \sim v$ to say that w is a neighbor of v.

(a) Show that if G is recurrent than any superharmonic function which is either bounded or positive is necessarily constant.

(b) Show that if G is transient there exists a positive, bounded superharmonic function which is not constant.

(ii) Consider the probability space $[0, 1]$ endowed with the Borel sigma algebra and uniform probability measure. Denote $I_{n,k} := [k/2^n, (k + 1)/2^n)$, $n \geqslant 1, 0 \leqslant k < 2^n$. For a point $x \in [0, 1)$ let $I_n(x) := I_{n,k}$ for the unique k such that $x \in I_{n,k}$.

(a) (Lebesgue density theorem) Let $A \subseteq [0, 1]$ be a Borel set. Prove that for almost every $x \in [0, 1]$ we have

$$\lim_{n\to\infty} 2^n |I_n(x) \cap A| \to \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases},$$

where $|B|$ denotes the Lebesgue measure of a set B.

Hint: Use the filtration $(\mathcal{F}_n)$ where $\mathcal{F}_n$ is the sigma algebra generated by $(I_{n,k})_k$.

(b) (Rademacher's theorem) Let $F : [0, 1] \to \mathbb{R}$ be a Lipschitz function. That is, a function for which there exists some $L < \infty$ such that

$$|F(x) - F(y)| \leqslant L|x - y| \quad \text{for all } x, y \in [0, 1].$$

Prove that there exists an integrable function $f : [0, 1] \to \mathbb{R}$ such that

$$F(x) - F(y) = \int_x^y f(t)dt \quad \text{for all } x, y \in [0, 1].$$

Hint: Consider the sequence of random variables

$$X_n(x) := 2^n \left(F\left(\frac{k+1}{2^n}\right) - F\left(\frac{k}{2^n}\right)\right)$$

where k is such that $I_n(x) = I_{n,k}$.

(iii) (Galton-Watson branching process) Let X be a random variable taking non-negative integer values and satisfying $\mathbb{E}X < \infty$. Assume that $\mathbb{P}(X = 1) < 1$. Define a Markov chain (Z_n), $n \geqslant 0$, by setting $Z_0 := 1$ and

$$\text{Conditioned on } Z_0, \ldots, Z_{n-1}, \quad Z_n := \sum_{k=1}^{Z_{n-1}} X_{n,k}$$

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where the $(X_{n,k})$, $n, k \geqslant 1$ are independent, identically distributed random variables with the distribution of X. One may think of Z_n as the size of a population at generation n with the rule that each generation is obtained from the previous one by replacing each individual with a random number of children distributed according to X, independently between the individuals and generations. Denote by E the event of extinction of the population, that is,

$$E := \{\text{there exists some } n \geqslant 1 \text{ for which } Z_n = 0\}.$$

(a) Let $m := \mathbb{E}X$. Prove that the process (M_n), $n \geqslant 0$, defined by $M_n := Z_n/m^n$ is a martingale. Deduce that $\mathbb{P}(E) = 1$ if $m \leqslant 1$.

(b) Define $f(s) := \mathbb{E}(s^X)$ for $0 \leqslant s \leqslant 1$ (where we use the convention that $f(0) = \mathbb{P}(X = 0)$ so that f is real analytic on $[0, 1]$). Suppose there exists some $0 \leqslant \rho < 1$ satisfying $f(\rho) = \rho$. Prove that the process (G_n), $n \geqslant 0$, defined by $G_n := \rho^{Z_n}$ is a martingale. Deduce that $\mathbb{P}(E) = \rho$ and $\mathbb{P}(Z_n \to \infty) = 1 - \rho$. Infer also that the equation $f(\rho) = \rho$ has at most one solution in $[0, 1)$.

(c) Observe that $f'(s) = \mathbb{E}(Xs^X)$ and in particular $f'(1) = m$. Deduce that $P(E) < 1$ if $m > 1$.

(d) Suppose that $m > 1$. Write $M_\infty = \lim_{n\to\infty} M_n$ (why does it exist?). Since (M_n) is a martingale one may speculate that Z_n grows as m^n on the event of non-extinction, i.e., that $M_\infty > 0$ on E^c. The Kesten-Stigum theorem shows that the sharp condition for this to occur is $\mathbb{E}X \log(X + 1) < \infty$. We will instead prove it here under the stronger condition that

$$\mathbb{E}(X^2) < \infty. \tag{1}$$

Prove that $\mathbb{E}M_\infty = 1$ under the assumption (1).
Hint: Bound $\mathbb{E}(M_n^2)$.

(e) Still under the assumptions $m > 1$ and (1), observe that $\theta := \mathbb{P}(M_\infty = 0)$ satisfies $f(\theta) = \theta$ and deduce that $P(M_\infty = 0) = \mathbb{P}(E)$.
Hint: Condition on Z_1.

(iv) (Kolmogorov's three series theorem) Let (X_n), $n \geqslant 1$, be a sequence of independent (not necessarily identically distributed) random variables. Define their truncated versions,

$$Y_n := \begin{cases} X_n & |X_n| \leqslant 1 \\ 0 & |X_n| > 1 \end{cases}.$$

In this exercise we prove Kolmogorov's theorem, that the conditions

$$(i) \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > 1) < \infty \quad (ii) \sum_{n=1}^{\infty} \mathbb{E}(Y_n) \text{ converges} \quad (iii) \sum_{n=1}^{\infty} \mathrm{Var}(Y_n) < \infty$$

are necessary and sufficient for the almost sure convergence of $\sum_{n=1}^{\infty} X_n$.

(a) Use the Borel-Cantelli lemma and the L^2 martingale convergence theorem to deduce that the above conditions suffice for the convergence.

(b) Prove that condition (i) is necessary for the convergence.

(c) To prove that (ii) and (iii) are necessary we require the following lemma. Suppose that (ξ_n), $n \geqslant 1$, are independent and satisfy $\mathbb{E}(\xi_n) = 0$ and $|\xi_n| \leqslant K$ for some $K < \infty$ and all n. Let $S_n := \sum_{k=1}^{n} \xi_k$ and $s_n^2 := \sum_{k=1}^{n} \mathbb{E}(\xi_k^2)$. Prove that $(S_n^2 - s_n^2)$, $n \geqslant 1$, is a martingale and use this together with optional stopping to show that

$$\mathbb{P}\left(\max_{1\leqslant k\leqslant n} |S_k| \leqslant t\right) \leqslant \frac{(t + K)^2}{\mathrm{Var}(S_n)}.$$

(d) Assume that $\sum_n Y_n$ converges almost surely. Let (Y_n') be a sequence of independent random variables, independent also from (Y_n), with $Y_n' \overset{d}{=} Y_n$. Define $Z_n := Y_n - Y_n'$ and apply the previous part to deduce that $\sum_n \mathrm{Var}(Z_n) < \infty$.

(e) Deduce that conditions (ii) and (iii) are necessary for the convergence.

(v) (Biased random walk) Let $\frac{1}{2} < p < 1$ and let (X_n), $n \geqslant 1$, be a sequence of independent identically distributed random variables with $\mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) = p$. Let $S_n := \sum_{k=1}^{n} X_k$. Fix integers $a, b > 0$ and let $T := \min(n \colon S_n \in \{-a, b\})$.

(a) Find a martingale of the form α^{S_n} with $0 < \alpha < 1$.

(b) Prove that T is almost surely finite and use the above martingale to calculate $\mathbb{P}(S_T = -a)$.
(c) Deduce that $\mathbb{P}(\exists n,\, S_n = b) = 1$ and calculate $\mathbb{P}(\exists n,\, S_n = -a)$.
(d) Let $T_b = \min(n\colon S_n = b)$. Find a martingale of the form $S_n - \alpha n$ and use it to calculate $\mathbb{E}T_b$.
Hint: To apply optional stopping consider first the truncated hitting times $T_b \wedge n$. Use the previous part.

The Brownian motion book is available at: http://research.microsoft.com/en-us/um/people/peres/brbook.pdf