BROWNIAN MOTION HOMEWORK ASSIGNMENT 8

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(i) Solve exercise 2.15 from the Brownian motion book.

(ii) Solve exercise 2.18 from the Brownian motion book.

(iii) The law of the iterated logarithm for Brownian motion states that, almost surely,

$$\limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t\log(\log(t))}} = 1. \tag{1}$$

Let (X_i), $i \geqslant 1$, be a sequence of independent and identically distributed random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$. Define the random walk (S_n), $n \geqslant 0$, by $S_n := \sum_{i=1}^n X_i$. Use (1) and the Skorohod embedding theorem to prove that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log(\log(n))}} \leqslant 1.$$

Remark: In fact equality holds but this is somewhat more difficult to prove.

(iv) This problem is not part of the homework but since we discussed it in class I list it here for your consideration. It is a little tricky but not difficult and you are encouraged to try it. Let X be a random variable with $\mathbb{E}|X| < \infty$. Consider the collection of all random variables of the form $\mathbb{E}(X \mid \mathcal{F})$ as $\mathcal{F}$ ranges over all sub-sigma algebras of the probability space. Prove that this collection of random variables is uniformly integrable.

The Brownian motion book is available at: http://research.microsoft.com/en-us/um/people/peres/brbook.pdf

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