BROWNIAN MOTION HOMEWORK ASSIGNMENT 9

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(i) Let (X_i), $i \geqslant 1$, be a sequence of independent, identically distributed random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$. Define (S_n), $n \geqslant 0$, by $S_0 := 0$ and

$$S_n := \sum_{i=1}^{n} X_i, \quad n \geqslant 1.$$

Denote $M_n := \max(S_k\colon 0 \leqslant k \leqslant n)$ and $T_n := \min(0 \leqslant k \leqslant n\colon S_k = M_n)$ (the maximum until time n and the first time at which it is attained). Similarly, for a standard Brownian motion B define $M := \max(B(t)\colon 0 \leqslant t \leqslant 1)$ and $T := \min(0 \leqslant t \leqslant 1\colon B(t) = M)$. Prove that $\frac{T_n}{n}$ converges in distribution to T as $n \to \infty$.

(ii) (a) Let S be a metric space with metric d. Suppose that (X_n), (Y_n) are two sequences of random variables on the same probability space such that X_n converges in distribution to a random variable X and $d(X_n, Y_n)$ converges to zero in probability. Prove that Y_n converges in distribution to X.

(b) Let (X_i), $i \geqslant 1$, be a sequence of independent, identically distributed random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$. In a primitive model for a stock price one is given a volatility parameter $\sigma > 0$ and a number n of epochs and defines the stock price process as the random $P_{n,\sigma} : [0, 1] \to [0, \infty)$ given by $P_{n,\sigma}(0) := 1$,

$$P_{n,\sigma}\left(\frac{k}{n}\right) := \prod_{i=1}^{k} \left(1 + \frac{\sigma}{\sqrt{n}} X_i\right), \quad 1 \leqslant k \leqslant n$$

and $P_{n,\sigma}(t)$ defined as the linear interpolation of these values. It is assumed that $n > \sigma^2$ so that the process is indeed non-negative.

Let B be a standard Brownian motion. Prove that $(P_{n,\sigma}(t))$, $0 \leqslant t \leqslant 1$, converges in distribution (as a random function in the space $C[0, 1]$) to $\left(\exp\left(\sigma B(t) - \frac{\sigma^2}{2} t\right)\right)$, $0 \leqslant t \leqslant 1$, as $n \to \infty$ (with σ fixed).

Hint: Show first that it suffices to prove convergence in distribution for the logarithms of the processes. Now use a Taylor expansion, Donsker's invariance principle and the first part.

(iii) Solve exercise 5.2 from the Brownian motion book.

The Brownian motion book is available at: http://research.microsoft.com/en-us/um/people/peres/brbook.pdf

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