PERCOLATION: HOMEWORK ASSIGNMENT 1

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This homework assignment needs to be submitted in class on March 5.

(1) Let $G = (V, E)$ be an infinite connected graph. Perform a percolation on G, retaining each edge with probability $0 \leqslant p \leqslant 1$ independently. Let G_p be the subgraph of G obtained after the percolation. Let E be the event that there exists an infinite connected component in G_p.
  (a) Prove that E is an event, i.e., that it is measurable. Here, we identify a subgraph G_p as an element of $\{0,1\}^E$, and take our probability space to be $\{0,1\}^E$ with the Borel sigma-algebra induced by the product topology. By saying that G is a graph we implicitly mean that V and E are countable (or finite) sets.
  (b) Prove that $\mathbb{P}_p(E) \in \{0,1\}$ for every p (hint: the Kolmogorov 0-1 law).
  (c) For a vertex $v \in V$, let $\mathcal{C}_v$ be the connected component of v in G_p. Prove that the following are equivalent: $\mathbb{P}_p(E) = 1$, $\mathbb{P}_p(|\mathcal{C}_v| = \infty) > 0$ for every $v \in V$, $\mathbb{P}_p(|\mathcal{C}_v| = \infty) > 0$ for some $v \in V$.

(2) Complete the proof of the theorem that a Galton-Watson tree has positive probability to be infinite if and only $\mathbb{E}(X) > 1$ (or $\mathbb{P}(X = 1) = 1$), where X is a random variable with the offspring distribution of the tree. As in class, assume (without loss of generality) that $m := \mathbb{E}(X) < \infty$, $\mathbb{P}(X = 1) < 1$ and $0 < \mathbb{P}(X = 0) < 1$. Let (Z_n) be the generation sizes of the Galton-Watson tree (Z_n is the size of the nth generation), with $Z_0 := 1$. Use the following steps:
  (a) Define the moment-generating function $f(s) := \mathbb{E}(s^X)$. Prove that
    (i) f is continuous and non-decreasing on $[0,1]$.
    (ii) f is strictly convex on $[0,1]$ if $\mathbb{P}(X \geqslant 2) > 0$.
    (iii) $f(1) = 1$ and $0 < f(0) < 1$.
    (iv) $f'(s)$ exists on $[0,1]$ and satisfies $f'(s) = \mathbb{E}(X s^{X-1})$.
  (b) Define $f_n(s) := \mathbb{E}(s^{Z_n})$. Recall from class (no need to prove again) that $f_{n+1}(s) = f(f_n(s))$. Let $q_n := \mathbb{P}(Z_n = 0) = f_n(0)$. Prove that $q_n \to q = \mathbb{P}(\text{tree is finite})$ and q satisfies $f(q) = q$.
  (c) Use the previous parts to conclude that $q < 1$ if and only if $m > 1$, proving the theorem (hint: by the first part, $m = f'(1)$).

Date: February 27, 2013.