PERCOLATION: HOMEWORK ASSIGNMENT 7

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This homework assignment needs to be submitted in class on May 21.

The first two exercises discuss questions from class, see there for more details.

(1) A labeled skeleton with k leaves is a tree all of whose vertices have degree 1 or 3, having exactly k vertices of degree 1, and with the vertices of degree 1 labeled by $\{1, \ldots, k\}$. Suppose that $x_1, \ldots, x_k$ are vertices in a connected graph G. Prove that there exists a labeled skeleton with k leaves S and a mapping $\psi : V(S) \to V(G)$ (not necessarily one-to-one) such that
  (a) ψ maps the vertex with label i to x_i.
  (b) For any adjacent $u, v \in S$ there exists a path $P_{u,v}$ in G between $\psi(u)$ and $\psi(v)$ such that all these paths are edge-disjoint.

(2) Let G be an infinite graph. A trifurcation point is a vertex of G lying in an infinite connected component, having exactly 3 incident edges, and such that deleting the vertex and its incident edges splits the connected component of the vertex into exactly 3 infinite components. Let $S \subseteq V(G)$ be a finite set and define $\partial^o S$, the outer boundary of S, as the set of vertices in $V(G) \setminus S$ which are adjacent to a vertex in S. Prove that the number of trifurcation points in S is at most $|\partial^o S| - 2$.

(3) Consider bond percolation on $\mathbb{Z}^d$ with $p > p_c$. Prove that

$$\inf_{x,y \in \mathbb{Z}^d} \mathbb{P}_p(x \leftrightarrow y) > 0.$$

(4) Consider bond percolation on $\mathbb{Z}^d$ and recall that $\theta(p) := \mathbb{P}_p(0 \leftrightarrow \infty)$. Prove that $\theta(p)$ is left-continuous on $(p_c, 1]$.
Hint: Recall the standard coupling of bond percolation for all p. To each edge e assign a random variable $U(e)$ uniform on $[0, 1]$, independently between the edges, and declare e to be open at parameter p if $U(e) < p$.
Remark: We proved in class the right continuity of θ on $[0, 1]$. Together with this exercise we obtain that θ is continuous on $(p_c, 1]$.

(5) (a) Let $A_1, \ldots, A_n$ be increasing events for bond percolation on a graph, all having the same probability. Prove that

$$\mathbb{P}(A_1) \geqslant 1 - (1 - \mathbb{P}(A_1 \cup \cdots \cup A_n))^{1/n}.$$

Remark: This is sometimes called the square-root trick. It shows that for increasing events of equal probability, if their union has probability close to 1 then each event has probability close to 1. This is not true for general events.

  (b) For the next two parts, consider bond percolation on $\mathbb{Z}^d$ with $p > p_c$. For an integer $n \geqslant 1$, let $F_1^n, \ldots, F_{2d}^n$ be the faces of the box $[-n, n]^d$ (these are subsets of $[-n, n]^d \setminus [-n+1, n-1]^d$). Prove that for every $\varepsilon > 0$ there exists an integer $m \geqslant 0$ such that for any $n > m$ and any $1 \leqslant i \leqslant 2d$,

$$\mathbb{P}_p([-m, m]^d \leftrightarrow F_i \text{ in } [-n, n]^d) \geqslant 1 - \varepsilon.$$

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The above event is the event that there exists an open path, fully contained in $[-n,n]^d$, connecting a vertex in $[-m,m]^d$ with a vertex of F_i.

(c) Write F_L^n and F_R^n for the left and right faces of $[-n,n]^d$, respectively (the faces with x coordinate $-n$ and n, respectively). Prove that

$$\mathbb{P}_p(F_L^n \leftrightarrow F_R^n \text{ in } [-n,n]^d) \to 1 \quad \text{as } n \to \infty.$$

The above event is the event of left-right crossing of $[-n,n]^d$, that there exists an open path, fully contained in $[-n,n]^d$, connecting a vertex of F_L^n with a vertex of F_R^n.