PERCOLATION: HOMEWORK ASSIGNMENT 8

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This homework assignment needs to be submitted in class on May 28.

We proved the Russo-Seymour-Welsh theorem for site percolation on the triangular lattice but you may assume its correctness also for bond percolation on the square lattice (the proof is similar).

(1) Consider bond percolation on $\mathbb{Z}^2$ with $p > 1/2$.
  (a) Prove that there exist $C, c > 0$ such that for all n,
  $$\mathbb{P}_p(\{0 \leftrightarrow \partial[-n,n]^2\} \setminus \{0 \leftrightarrow \infty\}) \leqslant C\exp(-cn).$$
  (b) Prove that there exist $C, c > 0$ such that for all $x, y \in \mathbb{Z}^2$,
  $$\theta(p)^2 \leqslant \mathbb{P}_p(\{x \leftrightarrow \infty\} \cap \{y \leftrightarrow \infty\}) \leqslant \theta(p)^2 + C\exp(-c|x-y|).$$
  (c) Let C_∞ be the infinite connected component, $\Lambda_n := [-n,n]^2$ and
  $$M_n := \frac{|\Lambda_n \cap C_\infty|}{|\Lambda_n|}.$$
  Estimate $\mathrm{Var}(M_n)$ and use the Borel-Cantelli lemma to conclude that $M_n \to \theta(p)$ almost surely.
  Hint: By Chebyshev's inequality, $\mathbb{P}_p(|M_n - \mathbb{E}\,M_n| > \varepsilon) \leqslant \frac{1}{\varepsilon^2}\mathrm{Var}(M_n)$.
  Remark: The result also follows from the ergodic theorem.

(2) Consider bond percolation on $\mathbb{Z}^2$. Let $A = ((n,m) \in \mathbb{Z}^2 \,:\, n, m \geqslant 0)$, the upper-right quadrant.
  (a) Define $p_c(A)$ by
  $$p_c(A) := \sup(p \,:\, \mathbb{P}_p(\text{there exists an infinite open path in } A) = 0).$$
  Prove that $p_c(A) = 1/2$.
  Hint: First prove that if $p > 1/2$ then the probability that there is no long-way crossing of a $2n \times n$ rectangle decays exponentially in n.
  (b) Prove that there exist $C, c, \alpha, \beta > 0$ such that
  $$cn^{-\alpha} \leqslant \mathbb{P}_{\frac{1}{2}}(\text{there is an open path in } A \text{ from } 0 \text{ to } \partial[-n,n]^2) \leqslant Cn^{-\beta}.$$

*(3) Consider bond percolation on a general graph G with possibly different percolation probabilities assigned to each edge. Suppose the graph contains the triangle configuration shown on the left of figure 1. Prove that the connectivity properties of the graph remain unchanged by changing the triangle to the star configuration on the right of figure 1, so long as
$$p_0 + p_1 + p_2 = p_0 p_1 p_2 + 1.$$
Remark: Since the triangular lattice may be transformed into the hexagonal lattice by such transformations this suggests the (correct) fact that the critical point p for bond percolation on the triangular lattice satisfies $3p = p^3 + 1$.

Date: May 26, 2013.

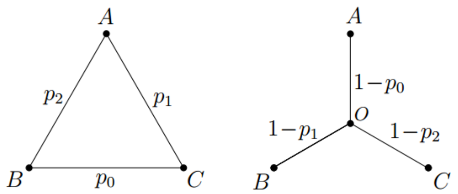

FIGURE 1. The star-triangle transformation.