Exercise 1

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The exercise needs to be handed in by May 2'nd in class.

In all of the following, unless otherwise indicated, we assume $(S_n)_{n=0}^{\infty}$ is a random walk in $\mathbb{R}^d$ with $S_n := \sum_{i=0}^{n} X_i$ (the walk may or may not be on $\mathbb{Z}^d$).

1. Prove the second Wald identity (directly, without use of martingale theory)
2. Biased 1D RWs: The following exercise discusses the 1D random walk with $\mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) = p$ for $p \neq \frac{1}{2}$. Let $T := \min\{n \mid S_n \in \{0, M\}\}$ be the hitting time of 0 or M.
   (a) Calculate $\mathbb{P}^x(S_T = 0)$ for $0 < x < M$ and use it to find $\mathbb{P}^x$ (no return to 0) for $x > 0$. Compare the result with the Kesten-Spitzer-Whitman theorem.
   (b) Calculate $\mathbb{E}^x T$ for $0 < x < M$.
3. Do Exercises 5 and 12 from the notes of Steve Lalley: http://galton.uchicago.edu/~lalley/Courses/312/RW.pdf.
4. Prove that the minimum of two stopping times is also a stopping time. That is, if T and S are stopping times with respect to a filtration $\{\mathcal{F}_n\}_{n\geq 0}$ then $\min(T, S)$ is also such a stopping time.
5. Suppose X_1 is uniform on the interval (0,1) and let $T := \min\{n \mid S_n > 1\}$. Show that $\mathbb{P}(T > n) = 1/n!$, so $\mathbb{E}T = e$ and $\mathbb{E}S_T = e/2$.
6. Prove that each of the following conditions is sufficent to deduce that $\liminf_{n\to\infty} S_n = -\infty$ and $\limsup_{n\to\infty} S_n = \infty$.
   (a) X_1 has a symmetric distribution and $(\mathbb{P}(X_1 = 0) < 1$.
   (b) $\mathbb{E}(X_1) = 0$ and $0 < \mathbb{E}(X_1^2) < \infty$ (prove this without using the Chung-Fuchs theorem).
7. Let $P := \{x \mid \exists n\ \mathbb{P}(S_n = x) > 0\}$ be the set of possible values for S_n. Prove that if S_n is point-recurrent then $\mathbb{P}(\forall x \in P,\ S_n = x$ infinitely often$)$=1 (note that the walk is not necessarily on $\mathbb{Z}^d$).

8. Prove that if $\mathbb{P}(|S_n| < 1 \text{ infinitely often}) = 1$ then for every $\varepsilon > 0$, $\mathbb{P}(|S_n| < \varepsilon \text{ infinitely often}) = 1$. This justifies defining neighbourhood-recurrence using one particular value of ε.

9. Let $P := \{x \mid \forall \varepsilon > 0 \ \exists n \ \mathbb{P}(|S_n - x| < \varepsilon) > 0\}$ be the set of neighbourhood possible values for S_n. Prove that if S_n is neighbourhood-recurrent then P is a group under addition in $\mathbb{R}^d$.

10. Give an example of a point-recurrent 1D RW whose set of possible values $P := \{x \mid \exists n \ \mathbb{P}(S_n = x) > 0\}$ is dense in $\mathbb{R}$.

11. Give an example of a 1D RW which is neighbourhood-recurrent but not point-recurrent.

12. Prove that if S_n is recurrent on $\mathbb{Z}^d$ then so is its symmetrized version $\tilde{S}_n$, the walk whose increments are distributed as $X_1 - X_1'$ where X_1, X_1' are independent copies of X_1.
Hint: Use the Fourier-analytic criterion for recurrence and compare S_{2n} with $\tilde{S}_n$.
Remark: The same result is true also for RW in $\mathbb{R}^d$ and neighbourhood-recurrence.

13. Prove that if S_n is a RW on $\mathbb{Z}$ satisfying the weak law of large numbers, i.e., for every $\varepsilon > 0$, $\mathbb{P}(\frac{S_n}{n} > \varepsilon) \to 0$ as $n \to \infty$, then S_n is recurrent.
Hint: Similar to the 2D recurrence theorem.

14. Let p be the transition kernel of an irreducible Markov chain on a countable state space S. That is, for every $x \in S$, $\sum_{y \in S} p(x, y) = 1$ ($p(x, y)$ is the probability to go from x to y in one step) and for every $x, y \in S$, $\exists n$ such that the probability to go from x to y in n steps is non-zero. A function $h : S \to \mathbb{R}$ is called superharmonic with respect to the Markov chain if $h(x) \geq \sum_{y \in S} p(x, y)h(y)$. Show that the Markov chain is recurrent (that is, every $x \in$S is recurrent) if and only if all non-negative superharmonic functions with respect to the Markov chain are constants.

15. (* Optional exercise) Construct arbitrary heavy tail recurrent 1D distributions. More precisely, show that for any $\varepsilon(x) \downarrow 0$ as $x \to \infty$ there exists a recurrent 1D RW on $\mathbb{Z}$ such that $\mathbb{P}(|X_1| \geq x) \geq \varepsilon(x)$ for all large x.