Random Walks and Brownian Motion — Instructor: Ron Peled
Tel Aviv University Spring 2011

Lecture 3

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In this lecture we prove the law of iterated logarithm; First we prove the following lemma:

(i) (LCLT - local CLT) If $k = o\left(n^{3/4}\right)$ and $n+k$ is even, then $P\left(S_n = k\right) \underset{n\to\infty}{\sim} \sqrt{\frac{2}{\pi n}} e^{-\frac{k^2}{2n}}$.

(ii) For all $n,\, k \geq 0,\ P\left(S_n \geq k\right) \leq e^{-\frac{k^2}{2n}}$ (not tight, by a polynomial factor).

Then by using the Borel-Cantelli lemma we show that $\limsup\limits_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = 1$ a.s.,

and by symmetry, $\liminf\limits_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = -1$ a.s.

We discuss Higher dimensional RW and general 1D RW;

We prove the Hewitt-Savage 0-1 law: If $A \in \varepsilon$ (for a RW) then $P(A) \in \{0,\, 1\}$,

and its following application: For a RW in $\mathbb{R}$, exactly one of the following has probability 1:

(i) $S_n = 0$ for all n (trivial RW)

(ii) $S_n \to \infty$

(iii) $S_n \to -\infty$

(iv) $\limsup S_n = \infty$ and $\liminf S_n = -\infty$

1 Law of the iterated logarithm

For SRW, we know by CLT that $S_n \approx N(0,n)$.

Is it true that $P\left(S_n = k\right) \approx \frac{1}{\sqrt{2\pi n}} e^{-\frac{k^2}{2n}}$(density at k of $N(0,n)$)?

For parity reasons, this is false; $P\left(S_n = k\right) = 0$ if $k+n$ is odd.

Is it true that $P\left(S_n = k\right) \approx \frac{2}{\sqrt{2\pi n}} e^{-\frac{k^2}{2n}}$ when $k+n$ is even (and n is large)?

Yes, this is true.

Lemma. *(i) (LCLT - local CLT) If $k = o\left(n^{3/4}\right)$ and $n+k$ is even, then*

$P\left(S_n = k\right) \underset{n\to\infty}{\sim} \sqrt{\frac{2}{\pi n}} e^{-\frac{k^2}{2n}}$.

Remark: This estimate is uniform for $k = o\left(n^{3/4}\right)$.

(ii) For all n, $k \geq 0$, $P\left(S_n \geq k\right) \leq e^{-\frac{k^2}{2n}}$ (not tight, by a polynomial factor).

Proof. (i)

$$P\left(S_n = k\right) = \binom{n}{\frac{n+k}{2}} 2^{-n} = \frac{n!\, 2^{-n}}{\left(\frac{n+k}{2}\right)!\left(\frac{n-k}{2}\right)!} \underset{\substack{n\to\infty \\ k=o(n)}}{\overset{\textit{Stirling}}{\sim}} \frac{\sqrt{2\pi n}}{\sqrt{2\pi\frac{n+k}{2}}\sqrt{2\pi\frac{n-k}{2}}} \cdot \frac{\left(\frac{n}{e}\right)^n 2^{-n}}{\left(\frac{n+k}{2e}\right)^{\frac{n+k}{2}}\left(\frac{n-k}{2e}\right)^{\frac{n-k}{2}}} =$$

$$= \sqrt{\frac{2}{\pi n\left(1 - \frac{k^2}{n^2}\right)}} \cdot \frac{n^n}{(n+k)^{\frac{n+k}{2}}(n-k)^{\frac{n-k}{2}}}$$

$$\frac{n^n}{(n+k)^{\frac{n+k}{2}}(n-k)^{\frac{n-k}{2}}} = \exp\left(n\log n - \frac{n+k}{2}\log(n+k) - \frac{n-k}{2}\log(n-k)\right)$$

$$n\log n - \frac{n+k}{2}\log(n+k) - \frac{n-k}{2}\log(n-k) \triangleq (\star)$$

Since $\log(n+k) = \log(n) + \log\left(1 + \frac{k}{n}\right)$,

and since $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \underset{x\to 0}{O}\left(x^4\right)$,

$$(\star) = -\frac{n+k}{2}\log\left(1 + \frac{k}{n}\right) - \frac{n-k}{2}\log\left(1 - \frac{k}{n}\right) =$$

$$= -\frac{n+k}{2}\left(\frac{k}{n} - \frac{k^2}{2n^2} + \frac{k^3}{3n^3} + O\left(\frac{k^4}{n^4}\right)\right) + \frac{n-k}{2}\left(\frac{k}{n} + \frac{k^2}{2n^2} + \frac{k^3}{3n^3} + O\left(\frac{k^4}{n^4}\right)\right) =$$

$$= \frac{k^2}{4n} + \frac{k^2}{4n} - \frac{k^2}{2n} - \frac{k^2}{2n} + O\left(\frac{k^4}{n^3}\right) = -\frac{k^2}{2n} + O\left(\frac{k^4}{n^3}\right) \qquad \left(\begin{array}{c} n\to\infty \\ k = o\left(n^{3/4}\right)\end{array}\right)$$

Plugging back into $(\star)$:

$$P\left(S_n=k\right)\sim\sqrt{\frac{2}{\pi n\left(1-\frac{k^2}{n^2}\right)}}\,e^{-\frac{k^2}{2n}+O\left(\frac{k^4}{n^3}\right)}\sim\sqrt{\frac{2}{\pi n}}\,e^{-\frac{k^2}{2n}}\qquad\begin{pmatrix} n\to\infty \\ k=o\left(n^{3/4}\right)\end{pmatrix}$$

(ii) We will use that for any X,

$$P\left(X\geq t\right)\underset{\text{assuming }\theta>0}{=}P\left(e^{\theta X}\geq e^{\theta t}\right)\underset{\text{Markov}}{\leq}\frac{E\,e^{\theta X}}{e^{\theta t}}$$

Taking $X=S_n,\ t=k$ and by comparing Taylor coefficients,

$$E\,e^{\theta X_1}=\left(\frac{1}{2}e^{\theta}+\frac{1}{2}e^{-\theta}\right)\leq e^{\theta^2/2}$$

$$P\left(S_n\geq k\right)\underset{\theta>0}{\leq}\frac{E\,e^{\theta S_n}}{e^{\theta k}}=\frac{\left(\frac{1}{2}e^{\theta}+\frac{1}{2}e^{-\theta}\right)^n}{e^{\theta k}}\leq\frac{e^{\theta^2\frac{n}{2}}}{e^{\theta k}}\underset{\text{take }\theta=\frac{k}{n}}{=}e^{-\frac{k^2}{2n}}\qquad\square$$

Lemma. *For $k>\sqrt{n},\ k=o\left(n^{3/4}\right)$ we have $P\left(S_n\geq k\right)\geq c\frac{\sqrt{n}}{k}e^{-k^2/2n}$ for some $c>0$.*

Remark: If $\mid k\mid\leq\sqrt{n}$, then $P\left(S_n\geq k\right)\geq c$ for some $c>0$ by the CLT.

Proof.

$$P\left(S_n\geq k\right)\geq P\left(k\leq S_n\leq k+\frac{n}{k}\right)\underset{(i)}{\geq}\sqrt{\frac{c}{n}}\sum_{k\leq j\leq\left\lfloor k+\frac{n}{k}\right\rfloor}e^{-j^2/2n}$$

For these j we have

$$e^{-j^2/2n}\geq e^{-\left(k+\frac{n}{k}\right)^2/2n}\geq e^{-\frac{k^2}{2n}-1-\frac{n}{2k^2}}\underset{k\geq\sqrt{n}}{\geq}c'e^{-k^2/2n}$$

Thus,

$$P\left(S_n\geq k\right)\geq\frac{c''}{\sqrt{n}}e^{-k^2/2n}\cdot\frac{n}{k}\qquad\square$$

Reminder: Define $M_n=\max_{0\leq j\leq n}S_j$. Then $P\left(M_n\geq k\right)=P\left(S_n=k\right)+2P\left(S_n>k\right)$.

S_n is typically of order $\sqrt{n}$. By (ii) and Borel-Cantelli, a.s. $S_n\leq\sqrt{2n\log n}\,(1+\epsilon)$ eventually.

Lemma. *(Borel-Cantelli)*

If $\{A_n\}$is a sequence of events, then:

1) If $\sum P(A_n) < \infty$, then a.s. only finitely many occur.

2) If $\sum P(A_n) = \infty$ and A_n are independent, then a.s. infinitely many of them occur.

Application: Define $A_n = \left\{S_n \geq (1+\epsilon)\sqrt{2n\log n}\right\}$. By (ii), $P(A_n) \leq \frac{1}{n^{1+\epsilon}}$.

So $\sum P(A_n) < \infty$, and therefore a.s. $S_n \leq (1+\epsilon)\sqrt{2n\log n}$ eventually.

Theorem. *(LIL)*

$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1$ *a.s.*

Remark: By symmetry, $\liminf_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = -1$ a.s.

Notation: $u(n) = \sqrt{2n\log\log n}$

Proof. *Remark:* For ease of readability, we will not worry about integrality of indices in the proof. One may round the indices appropriately everywhere and the argument will still go through.

For $\gamma > 0,\ a > 1,\ k \in \mathbb{N}$

$$P\left(\max_{0\leq j\leq a^k} S_j \geq (1+\gamma)\, u\left(a^k\right)\right) \leq 2P\left(S_{a^k} \geq (1+\gamma)\, u\left(a^k\right)\right) \leq$$

$$\underset{(ii)}{\leq} 2e^{-\frac{(1+\gamma)^2 u\left(a^k\right)^2}{2a^k}} = 2e^{-(1+\gamma)^2 \log\log\left(a^k\right)} = \frac{2}{(k\log a)^{(1+\gamma)^2}}$$

These probability estimates are summable in k. Thus by B-C,

$$P\left(\max_{0\leq j\leq a^k} S_j \leq (1+\gamma)\, u\left(a^k\right) \text{ from some } k \text{ on}\right) = 1$$

Now, for large n, write $a^{k-1} \leq n \leq a^k$. Then

$$\frac{S_n}{u(n)} = \frac{S_n}{u(a^k)} \cdot \frac{u(a^k)}{a^k} \cdot \frac{a^k}{n} \cdot \frac{n}{u(n)} \leq a(1+\gamma)$$

since $\frac{S_n}{u(a^k)} \leq 1+\gamma$, $\frac{a^k}{n} \leq a$, and $\frac{u(a^k)}{a^k} \cdot \frac{n}{u(n)} \leq 1$ because $\frac{u(t)}{t}$ is eventually decreasing.

We deduce that

$$P\left(\max_{0\leq j\leq a^k} S_j \leq (1+\gamma)\, u\left(a^k\right) \text{ from some } k \text{ on}\right) = 1$$

$$P\left(\limsup_n \frac{S_n}{u(n)} \leq a(1+\gamma)\right) = 1$$

Since $\gamma > 0$ and $a > 1$ are arbitrary, we have

$$P\left(\limsup_n \frac{S_n}{u(n)} \leq 1\right)$$

For the lower bound, fix $0 < \gamma < 1$, $a > 1$.

Denote $A_k = \left\{S_{a^k} - S_{a^{k-1}} \geq (1-\gamma)\, u\left(a^k - a^{k-1}\right)\right\}$.

We need to show that $\sum P(A_k) = \infty$.

Denote: $n = a^k - a^{k-1}$.

For large k,

$$P(A_k) \underset{\text{lemma}}{\geq} c\frac{\sqrt{n}}{(1-\gamma)\, u(n)} e^{-\frac{(1-\gamma)^2 u^2(n)}{2n}} = \frac{c}{1-\gamma} \cdot \frac{1}{\sqrt{2\log\log n}} \cdot \frac{1}{(\log n)^{(1-\gamma)^2}}$$

Noting $\log n \approx k$. So this expression is not summable in k.

Therefore, by B-C,

$$P(\text{infinitely many of } A_k \text{ occur}) = 1$$

By the upper bound,

$$P\left(\liminf_n \frac{S_n}{u(n)} \geq -1\right) = 1$$

So we have, for large k,

\textrm{is\:\ permutable}

$$\frac{S_{a^k}}{u\left(a^k\right)} \geq (1-\gamma)\,\frac{u\left(a^k-a^{k-1}\right)}{u\left(a^k\right)} + \frac{S_{a^k}}{u\left(a^k\right)} \geq$$

$$\geq (1-\gamma)\,\frac{u\left(a^k-a^{k-1}\right)}{u\left(a^k\right)} - \frac{(1+\epsilon)\,u\left(a^{k-1}\right)}{u\left(a^k\right)} \underset{k\to\infty}{\to} (1-\gamma)\sqrt{1-\frac{1}{a}} - \frac{1+\epsilon}{\sqrt{a}}$$

Thus

$$P\left(\limsup_{n}\frac{S_n}{u\left(n\right)} \geq (1-\gamma)\sqrt{1-\frac{1}{a}} - \frac{1+\epsilon}{\sqrt{a}}\right) = 1$$

and we take $\gamma\to 0$ and $a\to\infty$ to obtain the lower bound. $\square$

2 Higher dimensional RW and general 1D RW

In 1D and in 2D SRW is recurrent. In 3D (and more D) it is transient.

For a general RW in $\mathbb{R}^d$ ($S_n = X_1+\cdots+X_n$, I.I.D., $X_1\in\mathbb{R}^d$), what can we say about $P\left(S_n = 0 \text{ infinitely often}\right)$? It turns out that this probability is 0 or 1.

Say that an event A is *permutable* if the occurrence of A is unaffected by applying a *finite permutation* (a function $\pi:\ \mathbb{N}\to\mathbb{N}\ \begin{matrix}1-1\\ \text{onto}\end{matrix}$ s.t. $\pi\left(n\right)=n$ for all $n\geq n_0$) to the increments of S_n.

More formally, if the increments take values in a state space S, let our probability space Ω be $S^{\mathbb{N}}$, with the product probability over increments.

An event $A\in\mathcal{F}$ (σ-field) is *permutable* if $A=\left\{\omega\in\Omega|\,\pi\left(\omega\right)\in A\right\}$ for all finite permutations π.

The collection of all permutable events forms σ-field ε, the *exchangable σ-field.*

Theorem. *(Hewitt-Savage 0-1 law)*

If $A\in\varepsilon$ (for a RW) then $P\left(A\right)\in\{0,\,1\}$.

More Examples:

1. For any B, $\{S_n\in B \text{ infinitely often}\}\in\varepsilon$.

2. For any C_n, $\left\{\limsup_{n\to\infty} \frac{S_n}{C_n} \geq 1\right\} \in \varepsilon$.

3. Any tail event is permutable (tail σ-field $\subseteq \varepsilon$).

A *tail event* is an event which, for any n, is a function only of X_n, X_{n+1}, $\cdots$.

Proof. Fix $A \in \varepsilon$. We will show that $P(A) = P(A)^2$ (in other words, A, is independent of itself).

Take a sequence of events A_n s.t. A_n is a function only of $X_1, \cdots, X_n$ and:

(1)
$$P(A_n \triangle A) \to 0$$

(*Xor:* $B \triangle C = (B \cup C) \setminus (B \cap C)$)

Let $\pi = \pi_n$ be the permutation which exchanges $1, \cdots, n$ with $n+1, \cdots, 2n$.

Write $A_n^{'} = \pi(A_n)$. Notice that A_n and $A_n^{'}$ are independent.

Since π preserves probability (since $X_1, \cdots$ are I.I.D.),

(2)
$$P(A_n \triangle A) = P(\pi(A_n \triangle A)) \underset{A \text{ is permutable}}{=} P\left(A_n^{'} \triangle A\right)$$

Noticing that $\mid P(B) - P(C) \mid \leq P(B \triangle C)$, then (1) and (2) imply that $P(A_n) \to P(A)$ and

(3)
$$P\left(A_n^{'}\right) \to P(A)$$

However, we also obtain that

$$P\left(A_n \triangle A_n^{'}\right) \leq P(A_n \triangle A) + P\left(A_n^{'} \triangle A\right) \to 0$$

from which

$$0 \leq P(A_n) - P\left(A_n \cap A_n^{'}\right) \leq P\left(A_n \cup A_n^{'}\right) - P\left(A_n \cap A_n^{'}\right) = P\left(A_n \triangle A_n^{'}\right) \to 0$$

But $P(A_n) \to P(A)$, and by independence $P(A_n) P\left(A_n^{'}\right) \underset{(3)}{\to} P(A)^2$. Thus,

$$P(A) = P(A)^2$$

as we wanted. $\square$

Application: *For a RW in* $\mathbb{R}$*, exactly one of the following has probability 1:*

(i) $S_n = 0$ *for all* n *(trivial RW)*

(ii) $S_n \to \infty$

(iii) $S_n \to -\infty$

(iv) $\limsup S_n = \infty$ *and* $\liminf S_n = -\infty$

Proof. By the 0-1 law, $\limsup_{n\to\infty} S_n$ is a constant $c \in [-\infty, \infty]$ (since it is a permutable RV).

Notice that by considering the first increment X_1, $c = c - X_1$.

Thus, either $X_1 \equiv 0$ (this is case (i)), or $c \in \{-\infty, \infty\}$.

Similarly, $\liminf S_n \in \{-\infty, \infty\}$. $\square$

Exercise: If $X_1 \in \mathbb{R}$ is non-degenerate (not always 0: $P(X_1 = 0) < 1$), then we are in case (iv) if either:

1. X_1 is symmetric.
2. $E(X_1) = 0$ and $E\left(X_1^2\right) < \infty$. (It is actually true that $E(X_1) = 0$ suffices, as we will see next time.)