Electronic Notes in Theoretical Computer Science 68 No. 2 (2002)
URL: http://www.elsevier.nl/locate/entcs/volume68.html LastPage pages

The probability nesting game

Danièle Beauquier [1,2,a] Yoram Hirshfeld [1,3,b]
Alexander Rabinovich [1,3,c] Anatol Slissenko [1,2,a]

[a] *Laboratory for Algorithmics, Complexity and Logic*
University Paris 12, France

[b] *School of Mathematical Sciences*
Tel-Aviv University, Israel

[c] *School of Computer Science*
Tel-Aviv University, Israel

Abstract

We investigate the first order predicate logic of probability and establish the game theoretic tools to prove that this language does not support unnesting of the probability operator. This is contrasted with the probabilistic propositional logic where an unnesting algorithm exists.

1 Introduction

Classical predicate and propositional logics and their relatives (modal and temporal logics) were long recognized as the main formal languages to express properties of systems. It was also obvious that some important properties cannot be stated in these languages. In particular statements concerning probability are entirely outside the scope of classical logic, even though they are needed when dealing with uncertainty, fault tolerance and randomized systems. Much effort was invested trying to incorporate probability into formal logic. Major contributions are [6,8] and other papers on the subject are [12,9,3] ([6] contains a good survey and analysis of previous work on predicate logic of probability). These efforts did not converge to a commonly accepted probability logic. The logic suggested in [6] is natural, but too strong; it is wildly undecidable (it is Σ_1^2 complete by [1] even for a single unary predicate). In [2] a very natural weaker logic is considered, called Logic of Probability. It allows to state that a statement φ has probability larger than some *fixed*

[1] Partially supported by French-Israeli Arc-en-ciel/Keshet project No 30.
[2] {beauquier,slissenko}@univ-paris12.fr
[3] {joram,rabino}@math.tau.ac.il

(not variable) rational number. The statement φ itself may again speak about probability. It seems obvious (though it is not proved) that the satisfiability problem for monadic logic of probability is far simpler than Σ_1^2 complete. Yet in spite of its naturalness and simplicity the monadic logic of probability is undecidable [2].

Here we continue to research the expressive power of the logic of probability. In particular our aim is to show that the nested use of the probability operator can yield statements that are not expressible without the nesting. This turns out to be a non trivial question and the answer involves a new variant of the Ehrenfeucht-Fraïssé game technic [5,7]. Variants of Ehrenfeucht-Fraïssé games were extensively used for proof of inexpressibility both in predicate and modal logics [4,11,10,13].

The rest of the paper is arranged as follows: In section 2 we define the syntax and the semantics of the logic of probability. In section 3 we show that for propositional logic the power of the probability statement is restricted: there is an unnesting procedure that replaces every formula of propositional probability logic by an equivalent formula without nested probability operators (theorem 1). In section 4 we identify a fragment of monadic predicate logic of probability that we call simple formulas. These are unnested formulas which in addition have no quantifiers inside the range of a probability operator. We define a Ehrenfeucht-Fraïssé game and prove in theorem 2 that this game characterizes equivalence of structures with respect to corresponding classes of simple closed formulas. We use this game characterization in section 5 where we prove theorem 3: there is a simple closed formula which is not equivalent to any unnested formula.

2 Logic of probability

We follow Halpern's presentation of logic of probability [8]. There, arithmetic operations on probabilities are allowed and probabilities may be variables which are quantified. In our setting, we just compare probabilities with rational constants.

We consider a vocabulary that consists of a collection Σ of predicate symbols of various arities and constant symbols. Given formulas φ and ψ in the logic, we allow formulas of the form $\mathbf{Prob}_{>q}(\varphi)$ and $\mathbf{Prob}_{>q}(\varphi|\psi)$, where q is a rational number which can be read as "the probability of φ is greater than q" and "the probability of φ under the condition ψ is greater than q" respectively.

2.1 Syntax

More formally we define the syntax as follows. The vocabulary consists of a set of deterministic predicate symbols, a set of probabilistic predicate symbols, individual deterministic constants and individual variables. We also assume that rational constants are in the vocabulary.

Formulas:

– *Atomic formulas* are of the form $R(t_1, \dots, t_k)$, where R is a (deterministic or probabilistic) predicate symbol of arity k and $t_1, \dots, t_k$ are individuals (variables or constants).

- If φ_1 and φ_2 are formulas then $(\varphi_1 \vee \varphi_2)$ and $\neg\varphi_1$ are formulas.

- If φ is a formula then $\exists x\, \varphi$, where x is an individual variable, is a formula.

- If φ, ψ are formulas, and q is a rational number then $\mathbf{Prob}_{>q}(\varphi)$ and $\mathbf{Prob}_{>q}(\varphi \mid \psi)$ are formulas.

Conjunction $(\varphi_1 \wedge \varphi_2)$, implication $(\varphi_1 \to \varphi_2)$, universal quantification $\forall x$ are defined as usual, using disjunction, negation and existential quantifier. Expressions like $\mathbf{Prob}_{<q}$, $\mathbf{Prob}_{\leq q}$, $\mathbf{Prob}_{\geq q}$, $\mathbf{Prob}_{=q}$ can be syntactically defined in terms of $\mathbf{Prob}_{>p}$, using negation and modified bounds on probability. For example, we define $\mathbf{Prob}_{<p}(\varphi)$ as $\mathbf{Prob}_{>(1-p)}(\neg\varphi)$.

2.2 Semantics

First we recall some basic notions from probability theory.

A *probability space* is a triple (Ω, Δ, μ) consisting of a non empty set Ω, a σ-algebra Δ of its subsets called the *measurable sets* (representing random events in probability context), and a *probability measure* $\mu : \Delta \to [0, 1]$ (the probability of each event). The *σ-algebra* Δ includes Ω itself and is closed under complementation and countable union. The *probability measure* $\mu : \Delta \to [0, 1]$ satisfies $\mu(\Omega) = 1$ and is countably additive.

Probabilistic predicates are interpreted as random predicates. Given a domain $\mathcal{U}$ and a probability space (Ω, Δ, μ) a *random* (or *stochastic*) *predicate* P of arity k is a function from $\Omega \times \mathcal{U}^k$ to $Bool = \{\text{true}, \text{ false}\}$ such that for any fixed $u_1, \dots, u_k \in \mathcal{U}$ the set $\{\omega \in \Omega : P(\omega, u_1, \dots, u_k)\}$ is measurable.

A *probabilistic structure* for the vocabulary described above is a tuple $(\langle \mathcal{U}, \delta\rangle, \langle \Omega, \Delta, \mu\rangle, \pi)$, where

– $\langle \mathcal{U}, \delta\rangle$ is a first-order structure with universe $\mathcal{U}$, and δ assigns a relation over $\mathcal{U}$ of the appropriate arity to each deterministic[4] predicate symbol and assigns an element of $\mathcal{U}$ to each individual constant;

– $\langle \Omega, \Delta, \mu\rangle$ is a probabilistic space;

– π assigns to each probabilistic predicate symbol P of arity k a random predicate $\pi(P) : \Omega \times \mathcal{U}^k \to Bool$.

Define a *valuation* ν to be a function which assigns to each individual variable an element of $\mathcal{U}$.

Given a probabilistic structure $M = (\langle \mathcal{U}, \delta\rangle, \langle \Omega, \Delta, \mu\rangle, \pi)$, an element $\omega \in$

[4] Sometimes deterministic predicates are called *rigid* predicates

Ω and a valuation ν we proceed by induction to associate with every formula φ a truth value, writing $M, \nu, \omega \models \varphi$ if the value true is associated with φ by M, ν, ω:

(S1) $M, \nu, \omega \models R(t_1, \ldots, t_k)$ for a deterministic predicate symbol R of arity k and individual variables or constants $t_1, \ldots, t_k$ iff $\delta(R)(a_1, \ldots, a_k)$ is true, where $a_i = \nu(t_i)$ if t_i is a variable and $a_i = \delta(t_i)$ if t_i is a constant name.

(S2) $M, \nu, \omega \models P(t_1, \ldots, t_k)$ for a probabilistic predicate P of arity k iff $\pi(P)(\omega, a_1, \ldots, a_k)$ is true, where $a_i = \nu(t_i)$ if t_i is a variable and $a_i = \delta(t_i)$ if t_i is a constant name.

(S3) Quantifiers over individual variables and Boolean connectors are treated as usually.

(S4) $M, \nu, \omega \models \mathbf{Prob}_{>q}(\varphi)$ iff $\mu(\{\omega' \in \Omega : M, \nu, \omega' \models \varphi\}) > q$, that is iff the set of all ω' for which $M, \nu, \omega' \models \varphi$ holds has a measure greater than q.

(S5) $M, \nu, \omega \models \mathbf{Prob}_{>q}(\varphi|\psi)$ iff $\mu\{\omega' \in \Omega : M, \nu, \omega' \models (\varphi \wedge \psi)\} > q \cdot \mu\{\omega' \in \Omega : M, \nu, \omega' \models \psi\}$, i. e. the conditional probability of φ under ψ is greater than q.

Remark that (S4) is a particular case of (S5) when $\psi = true$.

The semantics is well defined only if the sets that appear in (S4) and (S5) are measurable. From now on we assume that

(**Countability Assumption**) *The domain $\mathcal{U}$ of probabilistic structures is countable.*

The following propositions are easily proved by induction on the structure of formulas [2] :

Proposition 2.1 *Under Countability Assumption the sets that appear in (S4) and (S5) are measurable, and the semantics is well defined.*

Proposition 2.2 *Suppose that two valuations ν_1 and ν_2 agree on the free variables of a formula φ. Then $M, \nu_1, \omega \models \varphi$ iff $M, \nu_2, \omega \models \varphi$.*

Proposition 2.3 *If all the occurrences of probabilistic predicates in a formula φ are in the scope of some operator* **Prob** *then $M, \nu, \omega_1 \models \varphi$ iff $M, \nu, \omega_2 \models \varphi$ for every $\omega_1, \omega_2 \in \Omega$. In particular, for any formula ψ we have $M, \nu, \omega_1 \models$ $\mathbf{Prob}_{>q}(\psi)$ iff $M, \nu, \omega_2 \models \mathbf{Prob}_{>q}(\psi)$.*

3 On nesting of probability operators for propositional logic

Our main interest lies in predicate logic. But we start by showing that for propositional logic the use of the probability operator has only restricted power; nesting one occurrence in the scope of another one does not add extra power to the language. We outline the analogue (yet simpler) syntax and semantics of the *Propositional Probability Logic* (henceforth *PPL*).

Formulas are:

- Elementary formulas are *true*, *false* or a symbol from a set V_d of *deterministic* Boolean variables or a symbol from a set V_p of *probabilistic* Boolean variables.
- If φ_1 and φ_2 are formulas then $(\varphi_1 \vee \varphi_2)$ and $\neg\varphi_1$ are formulas.
- If φ, ψ are formulas, and q is a rational number then $\mathbf{Prob}_{>q}(\varphi)$ and $\mathbf{Prob}_{>q}(\varphi \mid \psi)$ are formulas.

Given a probability space (Ω, Δ, μ), a structure M is defined through a pair of functions $M_p : \Omega \times V_p \longrightarrow Bool$ and $M_d : V_d \longrightarrow Bool$. The truth value of any formula is defined inductively similarly to (S3)-(S5).

Theorem 3.1 *Every PPL-formula is equivalent to an unnested formula.*

Proof. We show first that if Ψ is a boolean combination of formulas $\mathcal{A}_1, \dots, \mathcal{A}_n$ of *PPL* where $\mathcal{A}_n$ has the form $\mathbf{Prob}_{>q}(\dots)$, and if α_0, α_1 are obtained from Ψ replacing $\mathcal{A}_n$ by false and true respectively, then for every formula φ,

(1). $\mathbf{Prob}_{>p}(\varphi|\Psi)$ is equivalent to
$(\mathbf{Prob}_{>p}(\varphi|\alpha_1) \wedge \mathcal{A}_n) \vee (\mathbf{Prob}_{>p}(\varphi|\alpha_0) \wedge \neg\mathcal{A}_n)$,

(2). $\mathbf{Prob}_{>q}(\Psi|\varphi)$ is equivalent to
$(\mathbf{Prob}_{>q}(\alpha_1|\varphi) \wedge \mathcal{A}_n) \vee (\mathbf{Prob}_{>q}(\alpha_0|\varphi) \wedge \neg\mathcal{A}_n)$.

This is true because for every structure M and $\omega \in \Omega$
If $M, \omega \models \mathcal{A}_n$ then $M, \omega \models \mathbf{Prob}_{>p}(\varphi|\Psi)$ iff $M, \omega \models \mathbf{Prob}_{>p}(\varphi|\alpha_1)$ and
$M, \omega \models (\mathbf{Prob}_{>p}(\varphi|\alpha_1) \wedge \mathcal{A}_n) \vee (\mathbf{Prob}_{>p}(\varphi|\alpha_0) \wedge \neg\mathcal{A}_n)$ iff
$M, \omega \models \mathbf{Prob}_{>p}(\varphi|\alpha_1)$
(the second claim is proved in a similar way).

The theorem is now proved by induction on the nesting depth of the operator **Prob**. We use (1) and (2) to pull out the innermost occurrences of **Prob** reducing the rank to one for which the induction assumption holds. □

4 Unnested formulas and their Ehrenfeucht-Fraïssé game

We turn next to nested and unnested closed formulas in first order predicate logic of probability. For our method to work we must be careful to require that the formulas have another property beside unnestedness. Later we will show that this extra property does not restrict us:

Definition 1 *A formula is* pseudoatomic *if it is either atomic or of the form* $\mathbf{Prob}_{>q}(\psi|\varphi)$ *where* ψ *and* φ *are quantifier-free and* **Prob** *free. A closed formula (i.e., a formula without free variables) is called* simple *if it is constructed from pseudoatomic formulas by Boolean connectives and quantifiers.*

Thus in a simple formula there is no nesting of the operator **Prob** and no quantifier is in the scope of the operator **Prob**.

Next we define two kinds of equivalence relation on the class of structures. In the first case two structures will be equivalent to each other if they cannot be distinguished by a simple formula with quantifier depth n. In the second case they will be equivalent if the spoiler (first player) has no winning strategy in an appropriate game.

Definition 2 *Let M_0 and M_1 be two structures over a given vocabulary and let ω_0 (respectively ω_1) be an element of the probabilistic space of M_0 (respectively of M_1). We write $(M_0, \omega_0) \equiv_n (M_1, \omega_1)$ if for any simple sentence φ with quantifier depth $\leq n$ we have $M_0, \omega_0 \models \varphi$ iff $M_1, \omega_1 \models \varphi$.*

An Ehrenfeucht-Fraïssé *k-round game* on the structures (M_0, ω_0) and (M_1, ω_1) is played by two players *Spoiler* and *Duplicator*, and it involves k moves for each player; *Spoiler* plays first. He chooses one of the structures, say M_i and an element in that structure. *Duplicator* responds by choosing an element in the other model, i. e. in M_{1-i}, which he believes 'matches' the element chosen by *Spoiler*. This pair of moves completes one round of the game. The game is repeated for k rounds. *Spoiler* may or may not alternate the model where he chooses the next element from, and *Duplicator* always reacts by choosing an element in the other model. When the game is over there are two sequences of elements: $a_1, \cdots, a_k$ in M_0 and $b_1, \cdots, b_k$ in M_1.

Duplicator is the winner if for any unnested quantifier free formula $\varphi(x_1, \ldots, x_k)$ with at most k free variables the truth value of $\varphi(a_1, \ldots, a_k)$ evaluated in M_0 at ω_0 equals to the value of $\varphi(b_1, \ldots, b_k)$ evaluated in M_1 at ω_1. Otherwise *Spoiler wins.*

In particular if *Duplicator* is the winner then for any quantifier-free formulas $\varphi(x_1, \ldots, x_k)$ and $\psi(x_1, \ldots, x_k)$ with at most k free variables the truth value of $\mathbf{Prob}_{<q}\big(\varphi(a_1, \ldots, a_k) \,\big|\, \psi(a_1, \ldots, a_k)\big)$ evaluated in M_0 at ω_0 equals to the value of $\mathbf{Prob}_{<q}\big(\varphi(b_1, \ldots, b_k) \,\big|\, \psi(b_1, \ldots, b_k)\big)$ evaluated in M_1 at ω_1.

A *winning strategy*, for either player, is a strategy which when followed, guarantees winning, no matter what the other player plays.

Definition 3 *Let M_0 and M_1 be two structures over a given vocabulary and let ω_0 (respectively ω_1) be an element of the probabilistic space of M_0 (respectively of M_1). We write $(M_0, \omega_0) \equiv_n^G (M_1, \omega_1)$ if Duplicator has a winning strategy in the n-round Ehrenfeucht-Fraïssé game.*

The following theorem enables us to use Ehrenfeucht-Fraïssé games to prove that two structures cannot be distinguished by simple (closed) formulas with a predetermined quantifier depth n:

Theorem 4.1 (Soundness) *Let M_0 and M_1 be structures over some vocabulary. If $(M_0, \omega_0) \equiv_n^G (M_1, \omega_1)$ then $(M_0, \omega_0) \equiv_n (M_1, \omega_1)$.*

Proof. We shall prove the claim by induction on n for all vocabularies simultaneously.

For $n = 0$ this follows from the definitions.

Assume next that the claim holds for n and that Duplicator has a winning strategy in the $n+1$ game. For contradiction we assume that there is a shortest simple formula of quantifier depth $n+1$ that is true in one structure and not in the other. The shortest counterexample cannot be a disjunction or a negation which necessarily involves a subformula that is a counterexample. Hence there is a counterexample $\exists x\varphi(x)$ which holds (say) in M_0 at ω_0 while $\forall x\neg\varphi(x)$ holds in M_1 at ω_1. We let now Spoiler choose an element a_0 in M_0 for which $M_0, \omega_0 \models \varphi(a_0)$. Duplicator follows by choosing an element b_0 in M_1. We now enrich the vocabulary by an individual constant c and interpret this constant as a_0 in M_0 and as b_0 in M_1. Since Duplicator had a strategy for $n+1$ rounds, starting with the response of b_0 to the move of a_0 it is clear that he still has a winning strategy in the n-round game in the enriched vocabulary and models with c interpreted by a_0 and b_0. Now $M_0, \omega_0 \models \varphi[c/x]$ and hence by induction assumption $M_1, \omega_1 \models \varphi[c/x]$ which contradicts the choice of φ. $\square$

Remark 4.2 (Incompleteness) Unfortunately, the converse direction of Theorem 4.1 fails. In the full paper, we will provide an example of two structures which are $\equiv_n$-equivalent but not $\equiv_n^G$-equivalent.

5 The expressive power of nesting

We are almost ready to prove that there is no unnesting procedure in predicate probability logic. First a lemma:

Lemma 5.1 (i) *Let L be the first order predicate language (without probability) with the single unary predicate P and with no equality symbol. Let Σ be the class of all structures that have at least one element satisfying P and at least one element satisfying $\neg P$. Then every L-formula is equivalent over Σ to a formula without quantifiers.*

(ii) *Let L be the first order predicate language of probability with the single unary predicate P and with no equality symbol. Let Σ be the class of all probabilistic structures that have at least one element satisfying P and at least one element satisfying $\neg P$* for each $\omega \in \Omega$. *Then over Σ every unnested formula is equivalent to a simple formula.*

Proof. Let φ be a *basic* formula, i.e, *True*, *False* or a conjunction of atoms and negation of atoms in which every variable occurs at most once. Then in Σ the formula $\exists x\varphi$ is equivalent to the formula φ with the atom $P(x)$ (or $\neg P(x)$) removed, let us denote the resulting formula φ^{-x}.

It is easy to see that every quantifier free formula is equivalent to a disjunction of basic formulas and that for such a disjunction $\psi = \varphi_1 \vee \cdots \vee \varphi_n$ the formula $\exists x \psi$ is equivalent in Σ to $\varphi_1^{-x} \vee \cdots \vee \varphi_n^{-x}$.

It follows that if θ is equivalent to a quantifier free formula (and therefore also $\neg\theta$) then $\exists x\theta$ (and also $\forall x\theta$) is equivalent to a quantifier free formula. This proves part (i). Part (ii) follows trivially from part 1. $\square$

In contrast to the propositional case, nesting of probabilistic operators increases the expressive power for first-order logic. We are now ready our main result.

Theorem 5.2 *No unnested formula is equivalent to*

$$\mathbf{Prob}_{=3/4}\left[\,\exists t(\mathbf{Prob}_{=1/2}[P(t)] \wedge P(t))\,\right]. \qquad (1)$$

Proof. We show a slightly more general statement. Let Ψ be the formula

$$\mathbf{Prob}_{=\eta}\left[\,\exists t(\mathbf{Prob}_{=\theta}[P(t)] \wedge P(t))\,\right],$$

where $0 < \theta < \eta < 1$ and $2\eta = \theta + 1$. Then there is no unnested formula which is equivalent to Ψ.
Assume that Ψ is suspected of being equivalent to some unnested formula Θ over the class Σ of all probabilistic structures that have at least one element satisfying P and at least one element satisfying $\neg P$ *for each* $\omega \in \Omega$. By the lemma we may assume that Θ is simple. Assume now that the quantifier depth of Θ is k. We build two structures M_0 and M_1 with the same finite probability space $\Omega = \{\omega_0, \omega_1, \cdots \omega_{2k+1}\}$ such that

(i) both M_1 and M_2 are in the class Σ, and M_1 satisfies Ψ while M_2 does not.

(ii) $(M_0, \omega_0) \equiv_k^G (M_1, \omega_0)$, so that by theorem 2 either both satisfy Θ or none of the two satisfies Θ.

Thus Ψ is not equivalent to Θ.

Figure 1 and Figure 2 represent the two structures.

We assume that $k \geq 1$. Both domains have $2k+1$ elements (denoted here by $0, 1, \cdots, 2k$). Ω has $2k+2$ elements with probability as specified on the left column. Rationals θ,η and ε are chosen so that $(k+1)\varepsilon = \eta - \theta + \varepsilon$ and $k\varepsilon = 1 - \eta$. Thus $\eta - \theta = 1 - \eta$ or equivalently $2\eta = 1 + \theta$, and if $\eta = \frac{3}{4}$ then $\theta = \frac{1}{2}$.

Each horizontal line represents the domain corresponding to a particular ω_i. The marking $*$ means that the element i corresponding to this column satisfies P in the domain corresponding to this row. There are two types of elements: elements of the first type satisfy P in two of the domains and they satisfy P with probability θ. We call them θ-elements. The other elements are ε-elements which satisfy P in one domain only and their probability of

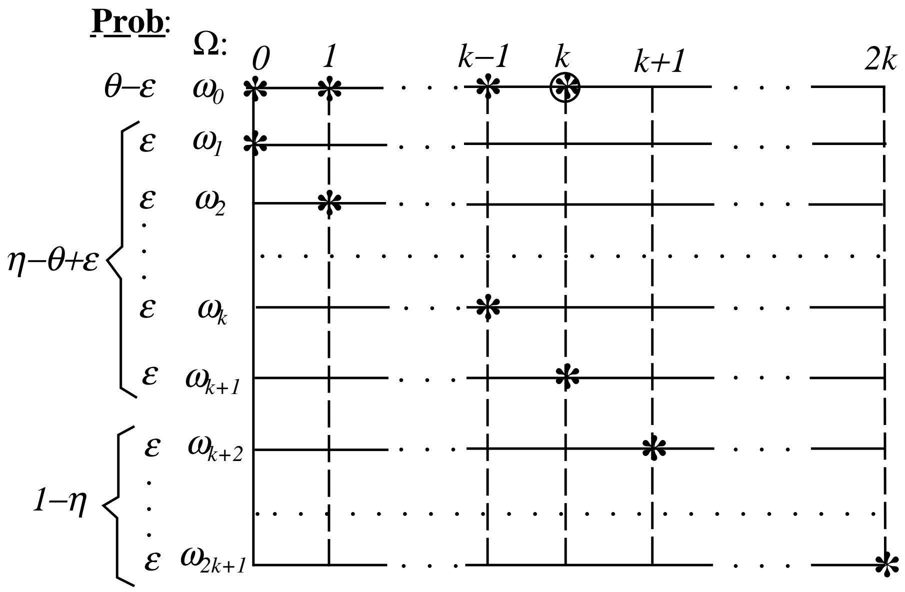

Fig. 1. Structure M_0.

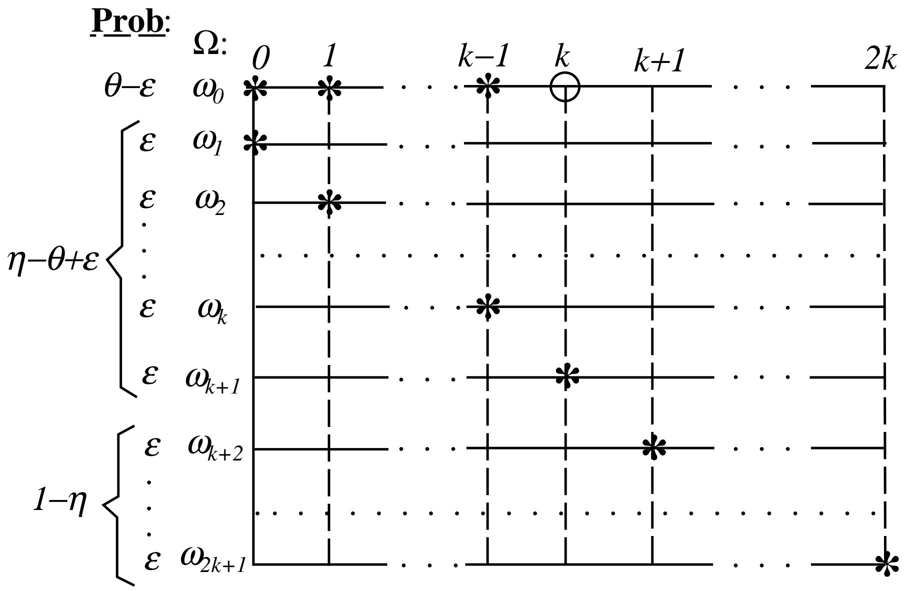

Fig. 2. Structure M_1.

satisfying P is ε. M_0 has $k+1$ θ-elements and k ε-elements. M_1 has k θ-elements and $k+1$ ε-elements. Note that the two structures satisfy the condition of lemma 1.

The formula $\mathbf{Prob}_{=\theta}[P(t)]$ holds for $t = 0, ..., k$ in M_0 and for $t = 0, ..., k-1$ in M_1. It follows that $\exists t\,(\mathbf{Prob}_{=\theta}[P(t)] \wedge P(t))$ holds in $k+1$ lines of M_0 and only in k lines in M_1. We conclude that the formula (1) is true on M_0 (Figure 1) and false on M_1 (Figure 2).

It remains to show that *Duplicator* has a winning strategy for the k-round Ehrenfeucht–Fraïssé game between (M_0, ω_0) and (M_1, ω_0). Assume that $0 \leq i < k$ and that $a_1, \cdots, a_i$ in M_0 and $b_1, \cdots, b_i$ in M_1 were already chosen according to the rules of the game. Assume that *Spoiler* chooses a_{i+1} in M_0 (the dual case is treated similarly). If for some $j \leq i$ $a_{i+1} = a_j$ then *Duplicator* chooses $b_{i+1} = b_j$. Else if a_{i+1} is an element of type θ (satisfying P in two domains), *Duplicator* chooses an element not yet chosen in M_1 of type θ. This is always possible, as there are at least k elements of type θ in either structure. Finally, if a_{i+1} is a new ε element then b_{i+1} is chosen as a new ε-element in M_1. This is possible since there are at least k ε-elements in each structure. It is clear that with this strategy:

(i) For every $i \leq k$

$$M_0, \omega_0 \models P(a_i) \text{ iff } M_0, \omega_0 \models P(b_i)$$

(ii) After a simple rearrangement of $\omega_1, \omega_2 \cdots \omega_{2k+1}$ for M_1, a_i and b_i satisfy P in the same lines. I.e, for every $i \leq k$ and for every $j \leq 2k+1$

$$M_0, \omega_j \models P(a_i) \text{ iff } M_0, \omega_j \models P(b_i)$$

.

Thus the sequences $a_1, \cdots, a_k$ and $b_1, \cdots, b_k$ are isomorphic modulo the order rearrangement. Since the rearrangement does not change the truth value of pseudoatomic statements we conclude that (M_0, ω_0) and (M_1, ω_0) satisfy the same pseudoatomic statements. $\square$

6 Conclusion

Ehrenfeucht-Fraïssé games were used to prove that nested probability statements are more expressive than unnested statement. This is a key observation when studying a language and examining its expressive power. We would have liked to sharpen the results and claim a hierarchy of languages with increasing expressive power, according to the depth of the **Prob** nesting. Unfortunately the particular form of Ehrenfeucht-Fraïssé games that we used does not generalize easily to non simple formulas. This leaves open the questions:

- Is there a proper hierarchy of languages according to the depth of nested **Prob** operators allowed?
- What is the proper Ehrenfeucht-Fraïssé game to use in the analysis of probability logic?

References

[1] M. Abadi and J. Halpern. Decidability and expressiveness for first-order logic of probability. *Information and Computation*, 112(1):1–36, 1994.

[2] D. Beauquier, A. Rabinovich, and A. Slissenko. A logic of probability with decidable model-checking. In *Proc. of CSL02, Lect. Notes in Comput. Sci., vol. 2471*, 2002.

[3] C. Courcoubetis and M. Yannakakis. The complexity of probabilistic verification. *Journal of the ACM*, 42:857–907, 1995.

[4] H.-D. Ebbinghaus, J. Flum, and W. Thomas. *Mathematical Logic.* Springer, 2nd edition,, 1994.

[5] A. Ehrenfeucht. An application of games to the completeness problem for formalized theories. *Fund. Math.*, 49:129–141, 1961.

[6] R. Fagin, J. Halpern, and N. Megiddo. A logic for reasoning about probabilities. *Information and Computation*, 87(1–2):78–128, 1990.

[7] R. Fraïssé. Sur quelques classifications des systèmes de relations. *Publ. Sci. Univ. Alger., Sér. A*, 1:35–182, 1954.

[8] J. Halpern. An analysis of first-order logics of probability. *Artificial Intelligence*, 46:311–350, 1990.

[9] H. A. Hansson. *Time and Probability in Formal Design of Distributed Systems.* Elsevier, 1994. Series: "Real Time Safety Critical System", vol. 1. Series Editor: H. Zedan.

[10] W. Hodges. *A Shorter Model Theory.* Cambridge University Press, 1997.

[11] N. Immerman. *Descriptive Complexity.* Springer, 1998.

[12] H. J. Keisler. Probability quantifiers. In J. Barwise and S. Feferman, editors, *Model-Theoretic Logics*, pages 509–556. Springer-Verlag, Berlin/New York, 1985.

[13] C. Stirling. Games and modal mu-calculus. In *TACAS*, volume 1055 of *Lecture Notes in Computer Science*, pages 217–232. Springer, 1996.