Homework 5

(return by Dec. 26)

Exercise 1 *Calculate the least fixed point of the following continuous function* $f : (Bool_\perp \to Bool_\perp) \to Bool_\perp$

$$f = \lambda g : \; Bool_\perp \to Bool_\perp . \begin{cases} \perp & \longmapsto \; \sqcap(g(true), g(false)) \\ true & \longmapsto \; true \\ false & \longmapsto \; false \end{cases}$$

where $\sqcap(g(true), g(false))$ *is the greatest lower bound of* $g(true)$ *and* $g(false)$.

Exercise 2 *Consider the function* $F : (nat \to nat) \to (nat \to nat)$ *defined by*

$$F =_{def} \lambda f : (nat \to nat).\lambda x. \; if \; x = 1 \; then \; 1 \; else f(x - 2).$$

where $x - y$ *yields 0 if* $y \geq x$ *and it is the standard value of the substraction for* $y < x$*; The equality* $=$ *is strict and if-then-else has the standard interpretation.*

1. *Compute the least fixed point of* F.
2. *Show that* F *has more than one fixed point.*

Exercise 3 *Let* D *be a CPO such that every strictly increasing sequence in* D *is finite.*

1. *Show that every monotone function from* D *to* D *is continuous.*
2. *Show that for every continuous function* $F : D \to D$ *there is* n *such that the least fixed point of* F *is equal to* $F^n(\perp)$.
3. *Is it true that there is* n *such that for every continuous function* $F : D \to D$ *the leat fixed point of* F *is equal to* $F^n(\perp)$.
4. *Give an example of a monotone function* $H : C \to C$ *such that the least fixed point of* H *is different from* $\lim H^n(\perp)$

Exercise 4 *Show that the following programs are denotationally equivalent in the continuous frame.*

1. *let* x *be* N *in* $M_1 M_2$ *and (let* x *be* N *in* M_1*)(let* x *be* N *in* M_2*).*

2. *let y be N in $\lambda x.\ M$ and $\lambda x.$ let y be N in M, when there is no free occurrence of x in N.*

Exercise 5 *Let A, B and C be (directed) CPOs. Let $F : A \times B \to C$. Show that if F is continuous in each of its arguments (i,e, $\lambda x.F(a, x)$ is ω-continuous (for each $a \in A$) and $\lambda y.F(y, b)$ is ω-continuous (for each $b \in B$)), then F is continuous.*

Exercise 6 *Let A, B and C be (directed) CPOs. Assume that $g : A \to B$ and $f : B \to A$ are continuous. Show that $Fix(f \circ g) = f(Fix(g \circ f))$, where $f_1 \circ f_2$ is the composition of f_1 and f_2, and Fix is the least fixed point operator.*