Lecture 5

1 Denotational Semantics for typed λ-terms

Semantical domains - provides the meaning for type expressions.

1. For each primitive type α - a corresponding primitive domain D^{α}.
2. For derived types, by induction $D^{\alpha\to\beta}$ an <u>appropriate</u> set of functions from D^{α} into D^{β}.

What is an appropriate? For example $D^{real\to real}$ set of continuous functions; $D^{int\to int}$ - set of computable functions.

Meaning of terms: For a term of type α its meaning is an element of D^{α}. The meaning of terms depends on meaning of its free variables.

Terminology. (1) **Environment** - a function which assigns to variables (syntactical objects) corresponding semantical objects. An environment should <u>respect</u> types, i.e., $\rho(x^{\alpha}) \in D^{\alpha}$.

(2) **updated environment**: $\rho[d/x]$, here d is a semantical object and x is a syntactical object.

$$\rho[d/x] \text{ differs from } \rho \text{ only on } x;\ \rho(x) = d.$$

Inductive definition of semantics $[\![t^{\alpha}]\!]\rho \in D^{\alpha}$, here t is a term, ρ is an environment.

Definitions by Structural induction on terns:

Atoms
1. $[\![c]\!]\rho$ - meaning of constants is provided by interpretation and is independent of environment.
2. $[\![x]\!]\rho$ is defined as $\rho(x)$.

Application $[\![M^{\alpha\to\beta}N^{\alpha}]\!]$?

By induction $[\![M^{\alpha\to\beta}]\!]\rho$ is already defined and is a function from D^{α} into D^{β}. Also $[\![N^{\alpha}]\!]\rho$ is already defined and is an element of D^{α}. Hence $[\![MN]\!]\rho$ is defined as $[\![M]\!]\rho[\![N]\!]\rho$. (Note that in MN we have syntactical application vs semantical application in $[\![M]\!]\rho[\![N]\!]\rho$.)

Abstraction $[\![\lambda x^{\alpha}.\, M^{\beta}]\!]$ is the function f such that for every $d \in D^{\alpha}$

$$fd = [\![M]\!] \quad \underbrace{\rho[d/x]}_{\text{updated env}}$$

Note that we have to ensure that this is an element of $D^{\alpha\to\beta}$. Here, where it is important how $D^{\alpha\to\beta}$ was chosen.

2 Two Models

A Full Frame - the standard model of typed λ-calculus.

$D^{\alpha\to\beta}$ is chosen as the set of all functions from D^α to D^β. It is clear that this choice insure that the abstraction is well defined.

Monotonic Frame

1. For primitive α - D^α is chosen as a <u>partially ordered</u> set.
2. $D^{\alpha\to\beta}$ is defined as the set of all <u>monotonic</u> functions from D^α to D^β.
3. $D^{\alpha\to\beta}$ is ordered pointwise, i.e,

$$f \leq_{\alpha\to\beta} g \ \mathtt{if}\ \forall d \in D^\alpha.\, fd \leq_\beta gd$$

Lemma 1 $[\![t]\!]\rho$ *is a monotonic function and depends monotonically on* ρ.

Hence, abstraction can be well defined in a monotonic frame.

3 Main Properties of Denotational Semantics

1. <u>Sanity Test:</u> The meaning of a term t in an environment ρ is independent from the meaning assigned by ρ to the variables which do not occur free in t. More precisely, Given a term t and two environments ρ_1 and ρ_2 which coincide on all free variables of t. Then $[\![t]\!]\rho_1 = [\![t]\!]\rho_2$.
2. <u>Type correctness:</u> for term t of type σ,

$$[\![t]\!]\rho \in D^\sigma$$

3. <u>Modularity.</u> Assume that t_1 and t_2 are <u>semantically equivalent</u>, i.e.,

$$\forall\rho.\ [\![t_1]\!]\rho = [\![t_2]\!]\rho.$$

   Assume that t_1 is a subterm of t and t' is obtained from t by replacing an occurrence of t_1 by t_2. Then t and t' are semantically (denotationally) equivalent.

Modularity follows from

Lemma 2 $[\![\ \underbrace{M[N/x]}_{\text{substitition}}\ ]\!]\rho = [\![M]\!]\ \underbrace{\rho[[\![N]\!]\rho/x]}_{\text{updated env}}$.

4 Relationship between Denotational and Operational semantics

Theorem 3 *$[\![\ ]\!]$ respect α, β and η reductions., i.e., $t_1 \rightarrow_\beta t_2$ implies $\forall \rho.\ [\![t_1]\!]\rho = [\![t_2]\!]\rho$. Similarly, for $\rightarrow_\alpha$ and $\rightarrow_\eta$ reductions.*

Hence, α, β and η reductions preserve denotational meaning, i.e. they are equivalence transformations.

Theorem 4 *(Soundness) $t_1 =_{\alpha\beta\eta} t_2$ implies $t_1 =_{den} t_2$.*

Note that soundness theorem is valid in every frame. What about completeness?

Completeness A frame M is complete if $M \models t_1 = t_2$ implies $\vdash t_1 =_{\alpha\beta\eta} t_2$.

The answer depends on a frame.

Example 5 *(1) M is a full frame with one element primitive domains - all terms of a given type a semantically equivalent.*

(2) M_k is a full frame with k element primitive domains.

(3) M_ω is a full frame with countable (non-finite) primitive domains.

Theorem 6 *Completeness for a full frame with an infinite base type.*

Theorem 7 *$t_1 =_{\alpha\beta\eta} t_2$ if $M_k \models t_1 = t_2$ for every k.*

Theorem 8 *Let Mon be the monotonic frame with primitive domain $\langle Nat, \leq \rangle$. Them $Mon \models t_1 = t_2$ iff $t_1 =_{\alpha\beta\eta} t_2$.*

Adequacy of denominational and operational semantics. Assume that bool is a primitive type and $D^{bool} = \{TRUE, FALSE\}$.

Adequacy property: $[\![M^{bool}]\!]$=TRUE iff $M \rightarrow^*_{red} True$

5 Proof of Consistency of typed λ calculus

Theorem 9 *The simple typed λ-calculus is consistent, i.e., for distinct variables x and y it is not the case that $x =_{\alpha\beta\eta} y$.*

Proof 1: Follows from Church Rosser property for β, η reduction. The proof of Church Rosser theorem is far from trivial.

Proof 2: Immediately follows from soundness theorem. If $x =_{\alpha\beta\eta} y$ then $\forall \rho.\ [\![x]\!]\rho = [\![y]\!]\rho$. Now choose ρ that violates the last condition.

6 About Normalization

Theorem 10 (Weak Normalization) *Every typed term M has β normal form.*

Proof 1. follows from Strong Normalization.

Proof 2. define the degree of a redex $(\lambda x : \sigma.M)N$ be the length of the type $\sigma \to \tau$ of $\lambda x : \sigma.M)$. The measure of a term M is the pair $\langle i, j \rangle$ where i the highest degree of the subterm of M and j is the number of subterms of degree i.

(a) Show that if $(\lambda x : \sigma.M)N$ has degree n and measure $\langle n, 1 \rangle$ then $M[N/x]$ has no redex of degree greater than $n - 1$.

(b) Show that for every term M not in normal form there is some reduction $M \to M'$ such that the measure of M' is less than the measure of M. Hence we obtain weak normalization.

(c) Give an example of a term M and a reduction $M \to M'$ such that the measure of M' is greater than the measure of M. Hence this assignment does not prove SN.

7 β Equivalence for typed terms

Theorem 11 *The β equivalence of typed terms is decidable.*

Theorem 12 *The complexity of the decision problem for β-equivalence is non-elementary*