A logic of probability with decidable model-checking

Danièle Beauquier[1 ♯]
University Paris 12, France
beauquier@univ-paris12.fr

Alexander Rabinovich[2 ♯]
Tel-Aviv University, Israel
rabino@math.tau.ac.il

Anatol Slissenko[1 ♯]
University Paris-12, France
slissenko@univ-paris12.fr

Abstract. A predicate logic of probability, close to logics of probability of Halpern and al., is introduced. Our main result concerns the following model-checking problem: deciding whether a given formula holds on the structure defined by a given Finite Probabilistic Process. We show that this model-checking problem is decidable for a rather large subclass of formulas of a second-order monadic logic of probability. We discuss also the decidability of satisfiability and compare our logic of probability with the probabilistic temporal logic $pCTL^*$.

1 Introduction

Logics with probabilities were considered in different contexts; on the one hand in artificial intelligence for reasoning about uncertainty in expert systems, and on the other hand for specification and verification of systems which exhibit some uncertainty such as fault-tolerant or randomized systems.

One can distinguish two families of logical approaches for reasoning about probabilities: (1) the first one extends the predicate logics (2) the second one extends temporal logics.

A fundamental contribution to the study of predicate logics of probability was done in [FHM90,Hal90], mainly motivated by the problems of artificial intelligence. The paper [Hal90] contains a good survey and analysis of previous works on predicate logics of probability. There has been very interesting works on extensions of predicate logic by probability quantifiers in model-theory community (see survey [Kei85]). The models considered there are different from thus considered in this paper and these works seems to be unrelated to our.

Most of the work related to the verification uses probabilistic extensions of temporal logics. The first applications of temporal logic to probabilistic systems were considered in studying which temporal properties are satisfied with probability 1 by systems modeled as finite Markov chains [LS82]. Later, papers [HJ94,ASB+95] introduced logics $pCTL$ and $pCTL^*$ that can express quantitative bounds on the probability of system evolutions. This approach is surveyed, for example in [Han94] and [CY95].

---

[1] *Address:* LACL, Dept. of Informatics, University Paris-12, 61 Av. du Gén. de Gaulle, 94010, Créteil, France.

[♯] Partially supported by French-Israeli Arc-en-ciel/Keshet project No 30.

[2] *Address:* Department of Computer Science, Tel-Aviv University, Israel.

In this paper we are interested in the verification of probabilistic systems. However, unlike previous works on verification we take as a specification formalism a probabilistic extension of predicate logic. Predicate logics offer some advantages (over modal and temporal logics) due to their expressiveness and convenience for formalization of complicated properties. We follow the general setting of [FHM90,Hal90,AH94] to introduce a rather expressive predicate logic of probability.

Our main result is a description of a fragment of a second-order monadic logic of probability with the following decidable model-checking problem.

Model-checking problem: decide whether a given formula holds on the structure defined by a given finite Markov chain.

The paper is organized as follows. In Section 2 we give a general description of the logic we use and emphasize how it deviates from [Hal90,FHM90]. In Section 3 we show that monadic predicate logic of probability is undecidable. Section 4 contains our main result about a fragment of probabilistic monadic logic of order with decidable model-checking. Section 5 compares our logic with probabilistic temporal logic $pCTL^*$. We prove that the property "there is a moment at which Q holds with probability one" cannot be expressed in $pCTL^*$, but is directly formalized in our logic of probability. Finally, section 6 presents some further results and open questions.

2 Logic of probability

We follow Halpern's presentation of logic of probability [Hal90]. There, arithmetic operations on probabilities are allowed, probabilities may be variables which are quantified. In our setting, we just compare probabilities with rational constants. However, we consider second order logics, when [Hal90] confines himself to first order ones.

We consider a language that consists of a collection Σ of predicate symbols of various arities. We also have a collection of predicate variables of various arities. Given formulas φ and ψ in the logic, we allow formulas of the form $\mathbf{Prob}_{>q}(\varphi)$ and $\mathbf{Prob}_{>q}(\varphi|\psi)$, where q is a rational number which can be read as "the probability of φ is greater than q" and "the probability of φ under the condition ψ is greater than q" respectively.

2.1 Syntax

More formally we define the syntax as follows. The vocabulary consists of a set of deterministic predicate symbols, a set of probabilistic predicate symbols, predicate variables and individual variables. We also assume that rational constants are in the vocabulary.

Formulas:

– *Atomic formulas* are of the form $R(x_1,\dots,x_k)$, where R is a (deterministic or probabilistic) predicate symbol of arity k and $x_1,\dots,x_k$ are individual variables; or of the form $Q(x_1,\dots,x_k)$, where Q is a deterministic predicate variable of arity k and $x_1,\dots,x_k$ are individual variables.

– If φ_1 and φ_2 are formulas then $(\varphi_1 \vee \varphi_2)$ and $\neg\varphi_1$ are formulas.

– If φ is a formula then $\exists\, x\, \varphi$ and $\exists\, Q\, \varphi$, where x is an individual variable and Q is a deterministic predicate variable, are formulas.

– If φ, ψ are formulas, and q is a rational number then $\mathbf{Prob}_{>q}(\varphi)$ and $\mathbf{Prob}_{>q}(\varphi \mid \psi)$ are formulas.

Conjunction $(\varphi_1 \wedge \varphi_2)$, implication $(\varphi_1 \rightarrow \varphi_2)$, universal quantification $\forall\alpha\, \varphi$ are defined as usual, using disjunction, negation and existential quantifier. Expressions like $\mathbf{Prob}_{<q}$, $\mathbf{Prob}_{\leq q}$, $\mathbf{Prob}_{\geq q}$, $\mathbf{Prob}_{=q}$ can be also defined in terms of $\mathbf{Prob}_{>p}$ using negation and modified bounds on probability in a syntactical manner. For example, we define $\mathbf{Prob}_{<p}(\varphi)$ as $\mathbf{Prob}_{>(1-p)}(\neg\varphi)$.

2.2 Semantics

First we recall some basic notions from probability theory.

A *measurable space* is a pair (Ω, Δ) consisting of a non empty set Ω and a σ-algebra Δ of its subsets that are called *measurable sets* and represent random events in probability context. A *σ-algebra* over Ω contains Ω and is closed under complementation and countable union. Adding to a measurable space a *probability measure* $\mu : \Delta \rightarrow [0,1]$ such that $\mu(\Omega) = 1$ and that is countably additive, we get a *probability space* (Ω, Δ, μ). Probabilistic predicates are interpreted as random predicates. Given a domain $\mathcal{U}$ and a probabilistic space (Ω, Δ, μ) a *random* (or *stochastic*) *predicate* P of arity k is a function from $\Omega \times \mathcal{U}^k$ to $Bool = \{\text{true, false}\}$ such that for any fixed $u_1, \ldots, u_k \in \mathcal{U}$ the set $\{\omega \in \Omega : P(\omega, u_1, \ldots, u_k)\}$ is measurable.

A *probabilistic structure* for the language described above is a tuple $(\langle\mathcal{U}, \delta\rangle, \langle\Omega, \Delta, \mu\rangle, \pi)$, where

– $\langle\mathcal{U}, \delta\rangle$ is a first-order structure with universe $\mathcal{U}$, and δ assigns a relation over $\mathcal{U}$ of the appropriate arity to each deterministic predicate symbol;

– $\langle\Omega, \Delta, \mu\rangle$ is a probabilistic space;

– π assigns to each probabilistic predicate symbol P of arity k a random predicate $\pi(P) : \Omega \times \mathcal{U}^k \rightarrow Bool$.

Define a *valuation* ν to be a function which assigns to each individual variable an element of $\mathcal{U}$, and to each deterministic predicate variable a *finite* relation over $\mathcal{U}$ of the appropriate arity ('finite' means that the set of tuples for which the deterministic predicate is true is finite).

Given a probabilistic structure $M = (\langle\mathcal{U}, \delta\rangle, \langle\Omega, \Delta, \mu\rangle, \pi)$, an element $\omega \in \Omega$ and a valuation ν, formally we define when a formula φ holds at ω in M under a valuation ν, written $M, \nu, \omega \models \varphi$, by the following inductive clauses:

(S1) $M, \nu, \omega \models R(x_1, \ldots, x_k)$ for a deterministic predicate symbol R of arity k and individual variables $x_1, \ldots, x_k$ iff $\delta(R)(\nu(x_1), \ldots, \nu(x_k))$ is true.

(S2) $M, \nu, \omega \models Q(x_1, \ldots, x_k)$ for a deterministic predicate variable Q of arity k iff $\nu(Q)(\nu(x_1), \ldots, \nu(x_k))$ is true.

(S3) $M, \nu, \omega \models P(x_1, \ldots, x_k)$ for a probabilistic predicate P of arity k iff $\pi(P)(\omega, \nu(x_1), \ldots, \nu(x_k))$ is true.

(S4) Quantifiers over individual variables and Boolean connectors are treated as usually.

(S5) Quantifiers over deterministic predicate variables are interpreted as quantifiers over deterministic predicate variables that range only over finite relations over $\mathcal{U}$.

(S6) $M, \nu, \omega \models \mathbf{Prob}_{>q}(\varphi)$ iff $\mu(\{\omega' \in \Omega : M, \nu, \omega' \models \varphi\}) > q$, that is iff the set of all ω' for which $M, \nu, \omega' \models \varphi$ holds has a measure greater than q.

(S7) $M, \nu, \omega \models \mathbf{Prob}_{>q}(\varphi|\psi)$ iff
$\mu\{\omega' \in \Omega : M, \nu, \omega' \models (\varphi \wedge \psi)\} > q \cdot \mu\{\omega' \in \Omega : M, \nu, \omega' \models \psi\}$,
i. e. the conditional probability of φ under ψ is $> q$.

Remark that (S6) is a particular case of (S7) when $\psi = true$.

The semantics is well defined only if the sets that appear in (S6) and (S7) are measurable. From now on we assume that

(**Countability Assumption**) *The domain* $\mathcal{U}$ *of probabilistic structures is countable.*

Proposition 1 *Under Countability Assumption the sets that appear in (S6) and (S7) are measurable, and the semantics is well defined.*

Proof. By induction on the structure of formulas. The only not quite straightforward step is quantification. For formula $\exists x\ \varphi$ we use that any σ-algebra is closed under countable union. For formula $\exists Q\ \varphi$ we use the fact that the set of finite predicates over a countable domain is countable and thus we can again use that any σ-algebra is closed under countable union. ∎

Proposition 2 *Suppose that two valuations* ν_1 *and* ν_2 *agree on the free variables of a formula* φ*. Then* $M, \nu_1, \omega \models \varphi$ *iff* $M, \nu_2, \omega \models \varphi$.

Proposition 3 *If all the occurrences of probabilistic predicates in a formula* φ *are in the scope of some operator* **Prob** *then* $M, \nu, \omega_1 \models \varphi$ *iff* $M, \nu, \omega_2 \models \varphi$ *for every* $\omega_1, \omega_2 \in \Omega$*. In particular, for any formula* ψ *we have*
$M, \nu, \omega_1 \models \mathbf{Prob}_{>q}(\psi)$ *iff* $M, \nu, \omega_2 \models \mathbf{Prob}_{>q}(\psi)$.

3 Undecidability of monadic logic of probability

The decidability of the probabilistic propositional logic follows from [FH94] where the decidability of a more general logic was proved. For first-order logic it is well-known that the satisfiability problem is decidable if the language has only unary predicates (Monadic Logic) and the satisfiability problem is undecidable even with one binary predicate [Hod93]. Many undecidability results for probabilistic logics can be found in [AH94] where this question was investigated in detail. It was shown in [AH94] that the satisfiability problem of their probabilistic logic even with one unary predicate is Σ_1^2 complete. However the logics considered there admit addition of probabilities or even multiplication of probabilities and quantifiers over reals and the methods of [AH94] are not applicable for our (much weaker) probabilistic logic.

In this section we prove (Theorem 1) that the satisfiability/validity problem for monadic logic of probability (that is a logic of probability where all predicates are monadic and the domain is $\mathbb{N}$) is undecidable. We reduce the satisfiability problem for first-order predicate logic with one binary predicate to the satisfiability problem for monadic logic of probability.

First, we define a translation from the first-order formulas over a binary predicate to formulas of probabilistic logic with two unary predicates. Let R be a binary predicate symbol and ϕ be a formula in the signature $\{R\}$. Replace in ϕ every occurrence of $R(x,y)$ by $\mathbf{Prob}_{>0}(P(x) \wedge Q(y))$, where P and Q are unary predicate symbols. The resulting formula $\psi(P,Q)$ is called the *translation* of ϕ.

Proposition 4 *The formula $\phi(R)$ is satisfiable iff its translation $\psi(P,Q)$ is satisfiable.*

Proof. It is clear that if the translation of ϕ is satisfiable in a probabilistic structure M then ϕ is satisfiable in the structure $\langle |M|,\ R^* \rangle$, where $|M|$ is the universe of M and $R^*(a,b)$ holds iff $M, a, b \models \mathbf{Prob}_{>0}(P(x) \wedge Q(y))$.

Let $\mathcal{M}$ be a structure for a binary predicate name R where the interpretation of R is a relation R^* over a countable universe $\mathcal{U} = \{a_1, a_2, \ldots, a_n, \ldots\}$. Let us define a probabilistic structure M as follows. Take as a probabilistic space $\Omega = \mathcal{U}$ with a discrete distribution of probabilities $\mu(\{a_n\}) = 1/2^n$ for every n if Ω is infinite, and μ is uniform if Ω is finite.

For each $a_n \in \Omega$, set $\pi(P)(a_n, t) = \text{true}$ iff $t = a_n$ and set $\pi(Q)(a_n, t) = \text{true}$ iff $R^*(a_n, t)$.

Observe that for every $a, b \in \mathcal{U}$, $R^*(a,b)$ iff $M, a, b \models \mathbf{Prob}_{>0}(P(x) \wedge Q(y))$. Hence for every sentence ϕ in the signature $\{R\}$ and its translation ψ we have $\mathcal{M} \models \phi$ iff $M \models \psi$ In particular, if ϕ is satisfiable then its translation is satisfiable. ∎

From Proposition 4 we can deduce:

Theorem 1 *The satisfiability problem for monadic logic of probability is undecidable.*

We do not know the exact complexity for the satisfiability problem of monadic logic of probability, however we believe that it is much lower than Σ_1^2. We also have the following property:

Proposition 5 *There exists a satisfiable formula of monadic logic of probability with equality such that all its models have an infinite probabilistic space.*

Proof. There is a closed predicate formula $\phi(R)$ over a binary predicate R which is satisfiable only in structures where the universe is infinite. For example take for $\phi(R)$ the conjunction of the three properties, R is transitive, irreflexive and $\forall x\ \exists y\ R(x,y)$. Consider the formula $\psi(P,Q)$ obtained as above, replacing in $\phi(R)$ every occurrence of $R(x,y)$ by $\mathbf{Prob}_{>0}(P(x) \wedge Q(y))$. Consider the probabilistic monadic formula

$$\Psi(P,Q) = \psi(P,Q) \wedge \mathbf{Prob}_{=1}(\exists! x\ P(x)) \wedge \forall x\ \mathbf{Prob}_{>0}(P(x))$$

We claim that :

(1) $\Psi(P,Q)$ is satisfiable,

(2) Every model of $\Psi(P,Q)$ has an infinite probabilistic space.

In order to prove (1), consider the following model M. Take a countable infinite universe $\mathcal{U} = \{a_1, a_2, \ldots, a_n, \ldots\}$. Take as a probabilistic space $\Omega = \mathcal{U}$ with a discrete distribution of probabilities $\mu(\{a_n\}) = 1/2^n$ for every n.

For each $a_n \in \Omega$ set $\pi(P)(a_n, t) = true$ iff $t = \{a_n\}$ and $\pi(Q)(a_n, t) = true$ iff $t \in \{a_{n+1}, a_{n+2}, \ldots\}$. Then it is clear from the construction that M satisfies $\Psi(P, Q)$.

Here is the proof of (2). Suppose there is a structure M that is a model of $\Psi(P, Q)$ with a finite probabilistic space $\Omega = \{\omega_1, \ldots, \omega_k\}$. We can suppose that $\mu(\omega_i) > 0$ for $i = 1, \ldots, k$. Thus for $i = 1, \ldots, k$ there exists a unique $a_i \in \mathcal{U}$ such that $\pi(P)(\omega_i, a_i)$ because M satisfies $\mathbf{Prob}_{=1}(\exists! x\ P(x))$. Choose an element a in universe $\mathcal{U}$ different from all the a_i. Since M satisfies $\forall x\ \mathbf{Prob}_{>0}(P(x))$, there exists an $\omega \in \Omega$ such that $\pi(P)(\omega, a) = true$. A contradiction. ■

4 Model-checking for a fragment of logic of probabilities

In this section we consider a logic of probability where all predicates are monadic and the domain is $\mathbb{N}$ with order. This logic is denoted *PMLO*. The probabilistic structures used in this section are defined by Finite Probabilistic Processes. We study the following model-checking problem: decide whether a given *PMLO*-formula φ holds on the structure defined by a given Finite Probabilistic Process. We introduce a rather large subclass $\mathcal{C}$ of formulas for which the model-checking problem is 'almost always decidable'.
Subsection 4.1 explains how Finite Probabilistic Processes define probabilistic structures. Subsection 4.2 introduces a class $\mathcal{C}$ of formulas with decidable model-checking problem.

4.1 Probabilistic Structures defined by Finite Probabilistic Processes

Definition. A *Finite Probabilistic Process* is a finite labelled Markov chain [KS60] $M = (S, P, V, \mathcal{L})$, where S is a finite set of states, P is a transition probability matrix: $S^2 \to [0, 1]$ such that $P(i, j)$ is a rational number for all $(i, j) \in S^2$, $\sum_{j \in S} P(i, j) = 1$ for every $i \in S$, and $V : S \to 2^{\mathcal{L}}$ is a valuation function which assigns to each state a set of symbols from a finite set $\mathcal{L}$.

The pair (S, P) is called a *finite Markov chain.*

The following Lemma is a well known fact in the theory of matrices (see e. g. [Gan77],13.7.5, 13.7.1)

Lemma 1 *Let (S, P) be a finite Markov chain. There exists a positive natural number d* period of the Markov chain *such that the limits*

$$\lim_{m \to \infty} P^{r+dm} = P_r \quad (r = 0, 1, \ldots, d-1)$$

exist. Moreover if the elements of P are rational, then these limits are computable from P and the convergence to the limits is geometric, i. e. $|P^{r+dm}(i, j) - P_r(i, j)| < a \cdot b^m$ when $m \geq m_0$ for some positive rationals a, $b < 1$ and natural m_0 also computable from P.

Given a Finite Probabilistic Process $M = (S, P, V, \mathcal{L})$ and a state s, we define a probabilistic structure M_s as follows:
Signature: a deterministic binary predicate $<$, and monadic probabilistic predicates Q for every label $Q \in \mathcal{L}$.
Interpretation:

- the universe of the structure M_s is the set $\mathbb{N}$ of natural numbers;
- $<$ is interpreted as the standard less relation over $\mathbb{N}$;
- probabilistic space (Ω, Δ, μ)(see [KSK66]) : $\Omega = sS^\omega$ is the set of all infinite sequences of states starting from s, Δ is the σ-algebra generated by the basic cylindric sets $D_u = uS^\omega$, for every $u \in sS^*$, and the probability measure μ is defined by $\mu(D_u) = \prod_{i=0,\ldots,n-1} P(s_i, s_{i+1})$ where $u = s_0 s_1 \ldots s_n$;
- interpretation of monadic probabilistic predicates:

for each $\omega = s_0 s_1 \ldots s_n \ldots \in \Omega$, for each $n \in \mathbb{N}$ we have $\pi(Q)(\omega, n)$ iff $Q \in V(s_n)$ (i.e. Q belongs to the label of state s_n). At this point, notice that for every integer n, the set $\{\omega \in \Omega : \pi(Q)(\omega, n)\}$ is μ-measurable since it is a finite union of basic cylinders.

Example. Let us consider a Call Establishment procedure in a simple telephone network where the capacity of simultaneous outgoing calls is less than the number of users. An abstraction of this procedure represents the behavior of a user where time is assumed to be discrete (Figure 1).

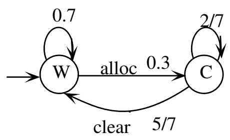

Figure 1.

To simplify it is assumed that a user which is not connected is continuously attempting to get a connection (state $Wait$) and at each time moment he succeeds to be connected with probability 3/10. Moreover when the calling is established the duration of the call (state $Call$) follows a geometric distribution: at each time moment, the probability to finish the call has probability 5/7.

One can write a liveness property such that:

$$\varphi =_{df} \forall t\ Prob_{=1}(\exists\, t' > t\ Call(t') \mid Wait(t)) \tag{1}$$

which expresses that at every time, if the user is waiting for a connection, the probability that he will be served later is equal to one.

One can also express some probabilistic property concerning the time the user has to wait before being served:

$$\psi =_{df} \forall t\ Prob_{\geq 0.9}(\exists t'\ (t < t' \ \wedge\ t' < t + 3\ \wedge\ Call(t')) \mid Wait(t)) \tag{2}$$

The set of labels here is equal to the set of states, and the label of a state is the state itself. One can prove that $M_{Wait} \models \varphi$ and $M_{Wait} \not\models \psi$.

4.2 A fragment of logic of probability with decidable model-checking

Recall that MLO denotes monadic second order logic of order over natural numbers and $WMLO$ denotes monadic second order logic of order over natural numbers where second-order quantification is over finite sets instead of arbitrary sets.

Below, when speaking about $WMLO$-formulas, we consider only $WMLO$-formulas without free second order variables. The predicate symbols of these formulas are interpreted as arbitrary sets. When we apply a **Prob** operator to such a formula we interpret all its predicate symbols as probabilistic ones.

Definition. A $PMLO$-formula φ belongs to the class $\mathcal{C}$ iff operators $\mathbf{Prob}_{>q}$ are not nested and are applied only to $WMLO$-formulas with at most one free individual variable. For example

$$\exists t \exists t' (t < t' \wedge \mathbf{Prob}_{>1/3}(P(t) \wedge \exists Q \forall t'' > t\ Q(t'')) \wedge \mathbf{Prob}_{>1/2}(\neg P(t'))) \tag{3}$$

where P is a probabilistic predicate and Q a deterministic one, belongs to $\mathcal{C}$.

The properties expressed in (1) and (2) are also in the class $\mathcal{C}$.

As one more example that needs a weak second order quantification we can mention the following property: the probability that a given probabilistic predicate has an even number of elements is greater than 0.9.

The main result of this subsection is Theorem 2 which, roughly speaking, says that it is decidable whether a given formula $\varphi \in \mathcal{C}$ holds in the structure defined by a given Finite Probabilistic Process M.

In order to express our decidability result about model checking, we need to introduce the notion of *parametrized* formula of logic of probability.

The set of *parametrized formulas* is defined similarly to the set of formulas except that operators $\mathbf{Prob}_{>q}$ with $q \in \mathbb{Q}$ are replaced by $\mathbf{Prob}_{>p}$, where p is a parameter name.

For example

$$\exists t \exists t' (t < t' \wedge \mathbf{Prob}_{>p_1}(P(t) \wedge\ \exists Q \forall t'' > t\ Q(t'')) \wedge \mathbf{Prob}_{>p_2}(\neg P(t')))$$

is a parametrized formula.

A formula φ is said to be *completely closed* if it is closed, and no probabilistic predicate is out of scope of an operator **Prob**. If φ is a completely closed formula, $M \models \varphi$ stands for $M, \omega \models \varphi$, that is well-defined and is independent from ω due to Proposition 3.

Let φ be a parametrized formula with parameters $p_1, \ldots, p_m$ and $\alpha_1, \ldots, \alpha_m$ be a sequence of rational values. We denote by $\varphi_{\alpha_1,\ldots,\alpha_m}$ the formula obtained by replacing in φ each parameter p_i by the value α_i. The set of parametrized completely closed formulas is defined exactly like the set of completely closed formulas. By abuse of terminology, we say that a parametrized formula φ belongs to $\mathcal{C}$ if all (or, equivalently, any of) its instances $\varphi_{\alpha_1,\ldots,\alpha_m}$ are in $\mathcal{C}$.

Theorem 2 *Given a Finite Probabilistic Process M, a state s_0 of M and a parametrized completely closed formula φ in the class $\mathcal{C}$ with m parameters, one can compute for each parameter p_i in φ a finite set P_i of rational values ($i = 1, \ldots, m$), such that for each tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ where $\alpha_i \in \mathbb{Q} \setminus P_i$, $i = 1, \ldots, m$, one can decide whether (M, s_0) satisfies φ_α.*

Remarks. 1. The complexity of our decision procedure is mainly determined by the complexity of decision procedure for MLO-formulas (that is non-elementary in the worst case).

2. In the definition of class $\mathcal{C}$ we allow to apply probabilistic operators only to formulas with one free individual variable. This is not essential restriction. The decidability result can be extended to the case when **Prob** is applied to formulas with many free individual variables. However the proof of the decidability of this extended fragment is more subtle and will be given in the full version of the paper.

3. The fact that we cannot treat some finite number of exceptional values seems to be essential from mathematical point of view. One cannot exclude that the model checking problem is undecidable for these exceptional values. However, for practical properties the values of probabilities can always be slightly changed without loss of its essential significance, and this permits to eliminate these exceptional values of probabilities.

4.3 Proof of Theorem 2

In the rest of this section the proof of Theorem 2 is given.

We introduce a notation: $\mathbb{N}_{\geq a} = \{n \in \mathbb{N} | n \geq a\}$ and we recall what are future and past $(W)MLO$-formulas.

Definition. A $(W)MLO$-formula $\varphi(x_0, X_1, X_2, ..., X_m)$ with only one free first-order variable x_0 is a *future formula* if for every $a \in \mathbb{N}$ and every m subsets $S_1, S_2, .., S_m$ of $\mathbb{N}$, the following holds:

$(\mathbb{N}, a, S_1, S_2, .., S_m) \models \varphi(x_0, X_1, X_2, ..., X_m)$
iff $(\mathbb{N}_{\geq a}, a, S'_1, S'_2, .., S'_m) \models \varphi(x_0, X_1, X_2, ..., X_m)$

where $S'_i = S_i \cap \mathbb{N}_{\geq a}$ for $i = 1, 2, ..., m$.

Past $(W)MLO$-formulas are defined in a symmetric way. Note that this is a semantic notion.

Theorem 4.1.7 [CY95] gives the following corollary that we will use:

Theorem 3 *Let $\varphi(t)$ be a future $(W)MLO$-formula with only one free variable and M be a Finite Probabilistic Process. One can compute for each state s of M, the probability f_s of the set of $\omega \in \Omega = sS^\omega$ that satisfy $\varphi(0)$.*

Recall that a set $S \subseteq \mathbb{N}$ is *ultimately periodic* if there are $h, d \in \mathbb{N}$ such that for all $n > h$, $n \in S$ iff $n + d \in S$.

Below, for simplicity, we will write $\mathbf{Prob}_{M_s}(\varphi(n))$ instead of $\mu\{\omega : M_s, n, \omega \models \varphi(t)\}$ for a Finite Probabilistic Process M, state s of M and $n \in \mathbb{N}$.

Lemma 2 *Let $M_1, \ldots, M_k$ be Finite Probabilistic Processes, s_i be a state of M_i ($1 \leq i \leq k$), $\varphi_1(t), \ldots, \varphi_k(t)$ be future $WMLO$-formulas with only one free variable t and $c_1, \ldots, c_k \in \mathbb{Q}$. For all (rational) values of p except a finite number of computable values, the set*

$$\{n \in \mathbb{N} : \textstyle\sum_{1 \leq i \leq k} c_i \cdot \mathbf{Prob}_{M_{i,s_i}}(\varphi_i(n)) > p\}$$

is finite or ultimately periodic, and is computable.

Proof. We give a proof for $k = 1$. The general case is treated similarly. Let $\varphi(t)$ be a future $WMLO$-formula with only one free variable t. Using Theorem 3, one can compute for each state s of M, the probability f_s of the set of $\omega \in \Omega = sS^\omega$ that satisfy $\varphi(0)$. Let F be the column vector $(f_s)_{s\in S}$. Let P be the transition probability matrix of M. Let I be the row vector with elements all equal to zero except the element in place s_0 which is equal to 1. Vector I represents the initial probability distribution over states of M.

For a given n, the probability that (M_{s_0}, n) satisfies $\varphi(t)$ is equal to $I \cdot P^n \cdot F$.

So, we have to compute the set $\mathbb{N}_{\varphi,p}$ of integers n such that $I \cdot P^n \cdot F > p$. In the general case, P^n does not converge when $n \to \infty$.

Let d be the period of the Markov chain from Lemma 1. For each $r \in D = \{0, \ldots, d-1\}$ consider the set $\mathbb{N}_r = r + d\mathbb{N}$. For $n \in \mathbb{N}_r$, the product $I \cdot P^n \cdot F$ has a limit p_r when $n \to \infty$ (Lemma 1). Define $\mathcal{P} = \{p_0, p_1, \ldots, p_{d-1}\}$. Fix a value $p \in \mathbb{Q} \setminus \mathcal{P}$. Let D^+ be the set of integers r such that $p_r > p$, and D^- be the set of integers r such that $p_r < p$.

For $r \in D^-$, let $K_{r,p}$ be the set $\{n \in \mathbb{N}_r : I \cdot P^n \cdot F > p\}$. Note that $K_{r,p}$ is finite and computable from p. For $r \in D^+$, let $K'_{r,p}$ be the set $\{n \in \mathbb{N}_r : I \cdot P^n \cdot F \leq p\}$. Note that $K'_{r,p}$ is finite and computable from p. Thus for $p \in \mathbb{Q} \setminus \mathcal{P}$, the set $\mathbb{N}_{\varphi,p}$ is equal to the union $\bigcup_{r\in D^-} K_{r,p} \cup \bigcup_{r\in D^+} \mathbb{N}_r \setminus K'_{r,p}$, and $\mathbb{N}_{\varphi,p}$ is finite or ultimately periodic and is computable. ∎

Lemma 3 *Let $M_1, \ldots, M_k$ be Finite Probabilistic Processes, s_i be a state of M_i $(1 \leq i \leq k)$, $\varphi_1(t), \ldots, \varphi_k(t)$ be past $WMLO$-formulas with only one free variable t and $c_1, \ldots, c_k \in \mathbb{Q}$. For all (rational) values of p except a finite number of computable values, the set*

$$\{n \in \mathbb{N} : \textstyle\sum_{1\leq i\leq k} c_i \cdot \mathbf{Prob}_{M_i,s_i}(\varphi_i(n)) > p\}$$

is finite or ultimately periodic, and is computable.

Proof. We prove this lemma for $k = 1$. Let $\varphi(t)$ be a past $WMLO$-formula with only one free variable t. A structure S for such a formula φ is defined as an infinite word on the alphabet $\Sigma = 2^{\mathcal{L}}$ where $\mathcal{L}$ is the set of monadic symbols of $\varphi(t)$. The property defined by $\varphi(t)$ depends only on the prefix of size $t+1$ of a model. Thus [Büc60], there exists a finite complete deterministic automaton $\mathcal{A}$ on the alphabet Σ accepting a language of finite words $L(\mathcal{A})$ such that $S, n \models \varphi(t)$ iff the prefix of S of size $n+1$ belongs to $L(\mathcal{A})$.

Therefore, given the automaton $\mathcal{A}$ and the Finite Probabilistic Process M, we build a new Finite Probabilistic Process M', "product" of M and $\mathcal{A}$ in the following way:

States of M' are pairs (q, s) where q is a state of $\mathcal{A}$ and s is a state of M. There is a transition from (q, s) to (q', s') iff (q, σ, q') is a transition in $\mathcal{A}$, where σ is the valuation of s in M and the probability of this transition is the same as the probability of (s, s') in M.

At last, the set of labels $\mathcal{L}'$ of M' is reduced to one symbol F and the valuation of (q, s) is $\{F\}$ if q is a final state in $\mathcal{A}$, and $\emptyset$ otherwise.

It is clear that :
$M_{s_0}, n \models \mathbf{Prob}_{>p}(\varphi(t))$ iff $M'_{(q_0,s_0)}, n \models \mathbf{Prob}_{>p}(F(t))$
where q_0 is the initial state of $\mathcal{A}$ and F is the monadic probabilistic symbol defined by $\mathcal{L}'$.

Since $F(t)$ is a future $WMLO$-formula, using Lemma 2 we get the result. ■

Lemma 4 *Let M be a Finite Probabilistic Process, s_0 be a state of M, $\varphi(t)$ and $\psi(t)$ be $WMLO$-formulas with only one free variable t. For all rational values of p, except a finite computable set $\mathcal{P}$, the sets*

(1) $\mathbb{N}_{\varphi,p} =_{df} \{n \in \mathbb{N} | M_{s_0}, n \models \mathbf{Prob}_{>p}(\varphi(t))\}$,

(2) $\mathbb{N}_{\varphi,\psi,p} =_{df} \{n \in \mathbb{N} | M_{s_0}, n \models \mathbf{Prob}_{>p}(\varphi(t)|\psi(t))\}$

are finite or ultimately periodic, and are computable.

Proof. (1) Let $\varphi(t)$ be a $WMLO$-formula with only one free variable t. Such a formula $\varphi(t)$ is equivalent (Lemma 9.3.2 in [GHR94]) to a finite disjunction of mutually exclusive formulas $\varphi_i(t)$ of the form $(\alpha_i(t) \wedge \beta_i(t))$, where $\alpha_i(t)$ are past formulas and $\beta_i(t)$ are future formulas. Moreover the $\alpha_i(t)$ and $\beta_i(t)$ are computable from formula $\varphi(t)$.

For each state s_j of M we introduce a new probabilistic predicate S_j, and add S_j in the valuation of s_j. Let M' be the new Finite Probabilistic Process obtained in this way.

The following equalities hold:

$\mathbf{Prob}_{M_s}(\varphi(n)) = \mathbf{Prob}_{M_s}(\bigvee_i \varphi_i(n)) = \sum_{i \in I} \mathbf{Prob}_{M_s}(\varphi_i(n))$
$= \sum_{i \in I} \mathbf{Prob}_{M_s}(\alpha_i(n) \wedge \beta_i(n))$
$= \sum_{i \in I} \mathbf{Prob}_{M'_s}(\bigvee_{j \in J}((\alpha_i(n) \wedge S_j(n)) \wedge (\beta_i(n) \wedge S_j(n)))$
$= \sum_{i \in I} \sum_{j \in J} \mathbf{Prob}_{M'_s}((\alpha_i(n) \wedge S_j(n)) \wedge (\beta_i(n) \wedge S_j(n)))$
$= \sum_{i \in I} \sum_{j \in J} \mathbf{Prob}_{M'_s}\big(\ \alpha_i(n) \wedge S_j(n)\ \big) \cdot \mathbf{Prob}_{M'_s}\big(\ \beta_i(n) \wedge S_j(n) \mid \alpha_i(n) \wedge S_j(n)\ \big)$
$= \sum_{i \in I} \sum_{j \in J} \big(\ \mathbf{Prob}_{M'_s}(\alpha_i(n) \wedge S_j(n)) \cdot \mathbf{Prob}_{M_{s_j}} \beta_i(0)\ \big)$.

We can compute the rational constants $\mathbf{Prob}_{M_{s_j}} \beta_i(0)$ using Theorem 3 and then we apply Lemma 3 to finish the proof.

The proof of (2) can be reduced to the proof of (1). ■

Proof.(*of Theorem 2*) For each $i = 1, \ldots, m$, let ψ_i be the subformula of ϕ of the form $\mathbf{Prob}_{>p_i}\varphi_i(t_i)$. One can compute using Lemma 4 a finite set of probabilities P_i such that for each value $\alpha_i \in \mathbb{Q} \setminus P_i$, the set $R_{\alpha_i} = \{n : M_{s_0}, n \models \mathbf{Prob}_{>\alpha_i}\ \varphi_i(t_i)\}$, is computable and is finite or ultimately periodic.

For each $i = 1, \ldots, m$, each value $\alpha_i \in \mathbb{Q} \setminus P_i$ each subformula ψ_i of ϕ, of the form $\mathbf{Prob}_{>p_i}\varphi_i(t_i)$, one can compute, using Lemma 4 the set $R_{\alpha_i} = \{n : M_{s_0}, n \models \mathbf{Prob}_{>\alpha_i}\ \varphi_i(t_i)\}$, and this set is an ultimately periodic set. There exists a first-order MLO-formula $\theta_{\alpha_i}(X)$ which characterizes R_{α_i}, i. e. R_{α_i} is the unique predicate that satisfies $\theta_{\alpha_i}(X)$. For example if R_{α_i} is the set of even integers, then $\theta_{\alpha_i}(X)$ will be "$X(0) \wedge \forall t(X(t) \leftrightarrow X(t+2))$".

Introduce new monadic predicate names N_{α_i}. Let Ψ_α be the formula obtained from φ_α by replacing $\mathbf{Prob}_{>\alpha_i}\ \varphi_i(t_i)$ by $N_{\alpha_i}(t_i)$. Consider now the MLO-formula $\Psi'_\alpha = \big(\ (\bigwedge_{0 \le i \le m} \theta_{\alpha_i}(N_{\alpha_i})) \rightarrow \Psi_\alpha\ \big)$. Clearly, (M, s) satisfies φ_α iff the

MLO-formula Ψ'_α is valid. Since the validity problem for MLO is decidable, it follows that the problem whether (M_{s_0}) satisfies φ_α is decidable. ■

5 Comparison with probabilistic temporal logic $pCTL^*$

The logic $pCTL^*$ is one of the most widespread among probabilistic temporal logics [ASB+95]. The relationship between our logic and $pCTL^*$ is rather complex. The semantics for logic of probability is defined over arbitrary probabilistic structures, however $pCTL^*$ is defined only for Finite Probabilistic Processes. Moreover, unlike logic of probability, the truth value of $pCTL^*$ formula depends not only on the probabilistic structure defined by a Finite Probabilistic Process but also on the 'branching structure' of this process. Hence, there is no meaning preserving translation from $pCTL^*$ to *monadic* logic of probability. We also show below that even on the class of models restricted to Finite Probabilistic Processes no $pCTL^*$ formula is equivalent to the probabilistic formula $\exists t\ \mathbf{Prob}_{\geq 1}Q(t)$, where Q is a probabilistic predicate symbol.

Let us recall the syntax and the semantics of the logic $pCTL^*$ as defined in [ASB+95]. Formulas are evaluated on a probabilistic structure associated to a Finite Probabilistic Process $(S, P, V, \mathcal{L})$.

There are two types of formulas in $pCTL^*$: state formulas (which are true or false in a specific state) and path formulas (which are true or false along a specific path).

Syntax. State formulas are defined by the following syntax:

1. each a in $\mathcal{L}$ is a state formula
2. If f_1 and f_2 are state formulas, then so are $\neg f_1$, $f_1 \vee f_2$
3. If g is a path formula, then $Pr_{<q}(g)$, $Pr_{>q}(g)$ are state formulas for every rational number q.

Path formulas are defined by the following syntax:

1. A state formula is a path formula
2. If g_1 and g_2 are path formulas, then so are $\neg g_1$, $g_1 \vee g_2$
3. If g_1 and g_2 are path formulas, then so are Xg_1, $g_1 U g_2$.

(X and U are respectively the $Next$ and $Until$ temporal operators).

Semantics. Given a Finite Probabilistic Process $M = (S, P, V, \mathcal{L})$ state formulas and path formulas are interpreted as defined below. Formulas f_1 and f_2 are state formulas and g_1 and g_2 are path formulas. Let s be a state, and Π be an arbitrary infinite path in M. Satisfaction of a state formula is defined with respect to s and satisfaction of a path formula with respect to Π. For each integer $k \geq 0$, we denote by Π^k the path obtained from Π when removing the first k states (thus $\Pi^0 = \Pi$) and by $[\Pi]_k$ the kth state of Π.

- $M, s \models Q$ iff $a \in V(Q)$,
- $M, s \models \neg f_1$ iff $M, s \not\models f_1$, $M, s \models f_1 \vee f_2$ iff $M, s \models f_1$ or $M, s \models f_2$,
- $M, s \models \mathbf{Prob}_{>q}(g_1)$ iff $\mu\{\sigma \in sS^\omega | M, \sigma \models g_1\} > q$,

$M, s \models \mathbf{Prob}_{<q}(g_1)$ is defined in a similar way,

- $M, \Pi \models f_1$ iff $[\Pi]_0 \models f_1$,
- $M, \Pi \models \neg g_1$ iff $M, \Pi \not\models g_1$, $M, \Pi \models g_1 \vee g_2$ iff $M, \Pi \models g_1$ or $M, \Pi \models g_2$,

• $M, \Pi \models X g_1$ iff $M, \Pi^1 \models g_1$,

• $M, \Pi \models g_1 U g_2$ iff there exists $k \geq 0$ such that $M, \Pi^k \models g_2$ and for all $0 \leq j < k$, $M, \Pi^j \models g_1$.

Below we give an example that illustrate differences between the logic of probabilities and $pCTL^*$. Consider the Finite Probabilistic Processes K and L shown on Figure 2 below. Let φ be the following pCTL* formula

$\mathbf{Prob}_{=1} \big(X(\mathbf{Prob}_{=\frac{1}{2}}(X\ P) \wedge \mathbf{Prob}_{=\frac{1}{2}}(X\ Q)) \big)$.

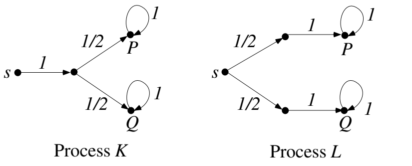

Figure 2

Note that $K, s \models \varphi$ but $L, s \not\models \varphi$. However, the probabilistic structures K_s and L_s are the same. Hence, unlike the truth value of logic of probability, the truth value of $pCTL^*$ formula depends not only on the probabilistic structure defined by the Finite Probabilistic Process but also on the 'branching structure' of this process. Therefore there is no direct, meaning preserving translation from $pCTL^*$ to *monadic* logic of probability.

In the rest of this section we show that even on the class of models restricted to Finite Probabilistic Processes no $pCTL^*$ formula is equivalent to the probabilistic formula $\exists t\ \mathbf{Prob}_{\geq 1} Q(t)$ where Q is a probabilistic predicate symbol. More precisely,

Theorem 4 *Let $\varphi = \exists t\ \mathbf{Prob}_{\geq 1} Q(t)$ where Q is a probabilistic predicate symbol. There is no $pCTL^*$ formula ψ such that for every Finite Probabilistic Process M and every state s of M one has $M_s \models \varphi$ iff $M, s \models \psi$.*

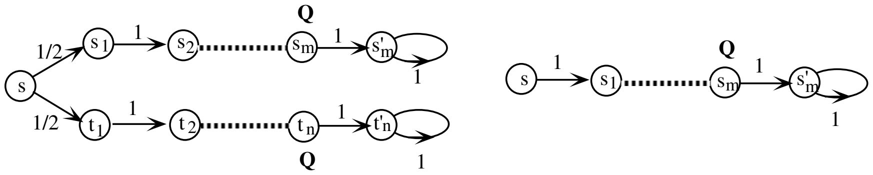

Figure 3: $\mathcal{K}_{m,n}$ and $\mathcal{K}_m$.

Consider the Finite Probabilistic Processes $\mathcal{K}_{m,n}$ and $\mathcal{K}_m$ for $m \geq 1$ and $n \geq 1$ as shown in Figure 3. Edges (i, j) are labeled by probabilities $P(i, j)$. Process $\mathcal{K}_m$ contains only one state (state s_m) labeled by the probabilistic predicate Q, other states have empty labels and process $\mathcal{K}_{m,n}$ contains only two states (states s_m and t_n) labeled by the probabilistic predicate Q. Let us call Π_m the unique infinite path starting in s in $\mathcal{K}_m$.

Lemma 5 *(1) For every $pCTL^*$ path formula g, there exists an integer $r \geq 1$ such that for every $m \geq r$, $\mathcal{K}_m, \Pi_m \models g$ iff $\mathcal{K}_r, \Pi_r \models g$.*

(2) For every $pCTL^*$ state formula f, there exists an integer $r \geq 1$ such that for every $m, n \geq r$, $\mathcal{K}_{m,n}, s \models f$ iff $\mathcal{K}_{r,r}, s \models f$.

Proof. The proof is by induction on the complexity of g and f. ■

Finally we are ready to prove Theorem 4.

Proof of Theorem 4. Let us suppose that such a $pCTL^*$ formula ψ exists. Using Lemma 5, there exists an integer $r \geq 1$ such that for every $m, n \geq r$, $\mathcal{K}_{m,n}, s \models \psi$ iff $\mathcal{K}_{r,r}, s \models \psi$. That contradicts the fact that $\mathcal{K}_{m,n}, s \models \varphi$ iff $m = n$. ■

6 Conclusion and further results

Our main result is a description of a fragment of a second-order monadic logic of probability with decidable model-checking. An important and difficult open question is whether one can prove the decidability of model-checking for all values of probabilities, without exceptions. Another open question is to consider other domains such as the real domain or the tree domain instead of the set of integers. It would be of great interest in specification of real-time uncertain systems. Below some extensions of our results are described.

A. Probabilities 0 and 1. Probabilities 0 and 1 play an important role in many questions related to specification and verification. Some probability logics, e. g. [LS82], consider only probabilistic operators $\mathbf{Prob}_{=0}$ and $\mathbf{Prob}_{=1}$. Theorem 2 can be strengthened as follows

Theorem 5 *Given a Finite Probabilistic Process M, a state s_0 of M and a parametrized completely closed formula φ in the class $\mathcal{C}$ with m parameters, one can compute for each parameter p_i in φ a finite set P_i of rational values, such that $0, 1 \notin P_i$ and for each tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ where $\alpha_i \in \mathbb{Q} \setminus P_i$ for $i = 1, \ldots, m$ one can decide whether (M_s) satisfies φ_α.*

In particular, we obtain the following corollary

Corollary 1 *Given a Finite Probabilistic Process M, a state s_0 of M and a completely closed formula φ in the class $\mathcal{C}$ with all probability operators only of the form $\mathbf{Prob}_{=0}$ and $\mathbf{Prob}_{=1}$. It is decidable whether (M_s) satisfies φ.*

B. Many variables inside Prob. In the definition of class $\mathcal{C}$ we allow to apply probabilistic operators only to formulas with one free individual variable. This is not essential restriction. The results of section 4 can be extended to the case when **Prob** is applied to formulas with many free individual variables. However the proof of the decidability of this extended fragment is more subtle and will be given in the full version of the paper.

C. On nesting. In class $\mathcal{C}$ we disallow nesting of **Prob** operators. Below we sketch how the decidability result can be extended to formulas with nested **Prob**. The main step in the proof of Theorem 2 shows that over a probabilistic structure M_s described by a Finite Probabilistic Process, the formula $\mathbf{Prob}_{>q}(\varphi(t))$ defines the set $S_q = \{n : M_s, n \models \mathbf{Prob}_{>q}(\varphi(t))\}$ which is ultimately periodic for all but finitely many q; the latter we call *exceptional* values and their complement *good* values. Now consider a nested formula of the form $\mathbf{Prob}_{>p_1}(\ldots \mathbf{Prob}_{>p_2}(\varphi) \ldots)$ with parameters p_1 and p_2. We can find a finite set of exceptional values for the innermost **Prob**. For each good value q_2 we can compute an ultimately periodic set which is definable also by a $WMLO$-formula and replace $\mathbf{Prob}_{>q_2}(\varphi)$ by this $WMLO$-formula. After the replacement we obtain an unnested formula ψ of the form $\mathbf{Prob}_{>p_1}(\dots)$. Now we can proceed and find for ψ a finite set of exceptional values of p_1 (for fixed q_2). If q_2 is a good value for $\mathbf{Prob}_{>p_2}(\varphi)$ and q_1 is a good value for the corresponding ψ then we can compute the truth value of the formula $\mathbf{Prob}_{>q_1}(\dots\mathbf{Prob}_{>q_2}(\varphi)\dots)$. Thus the whole set of exceptional values of (p_1, p_2) may be infinite but it is 'very sparse', in particular it is nowhere dense.

References

[AH94] M. Abadi and J. Halpern. Decidability and expressiveness for first-order logic of probability. *Information and Computation*, 112(1):1–36, 1994.

[ASB+95] A. Aziz, V. Singhal, F. Balarin, R. K. Brayton, and A. L. Sangiovanni-Vincentelli. It usually works: the temporal logic of stochastic systems. In *Computer Aided Verification. Proceeding of CAV'95*, pages 155–165. Springer Verlag, 1995. Lect. Notes in Comput. Sci., vol. 939.

[Büc60] J.R. Büchi. Weak second-order arithmetic and finite automata. *Z. Math. Logik u. Grundlag. Math.*, (6):66–92, 1960.

[CY95] C. Courcoubetis and M. Yannakakis. The complexity of probabilistic verification. *Journal of the ACM*, 42:857–907, 1995.

[FH94] R. Fagin and J. Halpern. Reasoning about knowledge and probability. *J. of the Assoc. Comput. Mach.*, 41(2):340–367, 1994.

[FHM90] R. Fagin, J.Y. Halpern, and N. Megiddo. A logic for reasoning about probabilities. *Information and Computation*, 87:1,2:78–128, 1990.

[Gan77] F. R. (Feliks Ruvimovich) Gantmakher. *The Theory of Matrices.* Chelsea Pub. Co., New York, 1977.

[GHR94] D. Gabbay, I. Hodkinson, and M. Reynolds. *Temporal Logic.* Clarendon Press, Oxford, 1994.

[Hal90] J. Halpern. An analysis of first-order logics of probability. *Artificial Intelligence*, 46:311–350, 1990.

[Han94] H. A. Hansson. *Time and Probability in Formal Design of Distributed Systems.* Elsevier, 1994. Series: "Real Time Safety Critical System", vol. 1.

[HJ94] H. A. Hansson and B. Jonsson. A logic for reasoning about time and probability. *Formal Aspects of Computing*, 6(5):512–535, 1994.

[Hod93] W. A. Hodges . *Model Theory.* Cambridge University Press, Cambridge, 1993.

[KS60] J. G. Kemeny and J. L. Snell. *Finite Markov Chains.* D Van Nostad Co., Inc., Princeton, N.J., 1960.

[KSK66] J. G. Kemeny, J. L. Snell, and A.W. Knapp. *Denumerable Markov Chains.* D Van Nostad Co., Inc., Princeton, N.J., 1966.

[Kei85] H. J. Keisler. Probability quantifiers. In J. Barwise and S. S.Feferman, editors, *Model Theoretic Logics*, pages 509–556. Springer, 1985.

[LS82] D. Lehmann and S. Shelah. Reasoning about time and chance. *Information and Control*, 53(3):165–198, 1982.

[Tho90] W. Thomas. Automata on infinite objects. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, pages 131–191. North-Holland, 1990.