Quantitative Temporal Logic

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Abstract. We define a quantitative Temporal Logic that is based on a simple modality within the framework of Monadic Predicate Logic. Its canonical model is the real line (and not an ω-sequence of some type). We prove its decidability using general theorems from Logic (and not Automata theory). We show that it is as expressive as any alternative suggested in the literature.

1 Introduction

1.1 Summary of the Results

Temporal Logic (TL) is a convenient framework for reasoning about the evolving of a system in time. This made TL a popular subject in the Computer Science community and it enjoyed extensive research during the last 20 years. In temporal logic the relevant properties of the system are described by *Atomic Propositions* that hold at some points in time and not at others. More complex properties are described by formulas built from the atoms using Boolean connectives and *Modalities* (temporal connectives): a k-place modality C transforms statements $\varphi_1, \ldots, \varphi_k$ on points possibly other than the given point t_0 to a statement $C(\varphi_1, \ldots, \varphi_k)$ on the point t_0. The rule that specifies when is the statement $C(\varphi_1, \ldots, \varphi_k)$ true for the given point is called *Truth Table* in [GHR94]. The choice of the particular modalities with their truth table determines the different temporal logics. The most basic modality is the one place "diamond" modality $\diamondsuit X$ saying "X holds some time in the future". Its truth table is usually formalized by $\varphi_\diamondsuit(t_0, X) \equiv (\exists t > t_0)X(t)$ [GHR94].

The truth table of $\diamondsuit$ is a formula of the Monadic Logic of Order (MLO). MLO is a fundamental formalism in Mathematical Logic, part of the general framework of Predicate Logic. Its formulas are built using atomic propositions $X(t)$ (similar to the atoms X of TL), atomic relations between elements $t_1 = t_2$, $t_1 < t_2$ and using Boolean connectives and (first order) quantifiers $\exists t$ and $\forall t$ (occasionally we shall be interested in second order MLO that has also quantifiers $\exists X$ and $\forall X$). Practically all the modalities used in the literature have their truth table defined in MLO and as a result every formula of the temporal logic translates directly into an equivalent formula of MLO. Therefore, the different temporal logics may be considered a convenient way to use fragments of MLO. There is a lot to be gained from adopting this point of view: the rich theory concerning MLO and in particular the decidability results concerning MLO apply to TL. MLO can also serve as a yardstick by which to check the strength of the temporal logic chosen: a temporal logic is *expressively complete* if every formula of MLO with single variable t_0 is equivalent to a temporal formula. An expressively complete temporal logic is as strong as can be expected.

Actually the notion of expressive completeness refers to a temporal logic and to a model (or a class of models) since the question if two formulas are equivalent depends on the domain over which they are evaluated. Any ordered set with monadic predicates is a model for TL and MLO, but the main, *canonical*, intended models are the non-negative integers $\langle N, <, 0\rangle$ for discrete time and the non-negative reals $\langle R^+, <, 0\rangle$ for continuous time. There may be reasons to use other models but they should be spelled out explicitly if the need arises to ignore the natural model. A major result concerning TL is Kamp's proof [Kamp68] (reproved in [GPSS80]) that the pair of modalities X *until* Y and X *since* Y is expressively complete for the two canonical models (but not for less natural models like the rationals). *Note* that since this paper is not concerned with discrete time we mean from now on only the model of non-negative reals R^+ when we speak of the canonical model.

Sometimes, in particular in Computer Science it is natural to restrict the attention to a subclass of unary predicates over R^+, the class of predicates with "finite variability"; i.e - predicates that change from *true* to *false* only finitely often in any finite interval of time. We shall call the standard model with these finite variability predicates the "canonical finite variability model".

It is clear that a predicate P is a finite variability predicate if and only if there is an unbounded increasing sequence t_i such that P is constant on any interval (t_i, t_{i+1}).

MLO and its derived temporal logics are not suitable to deal with *quantitative* properties. The most basic quantitative property is "X will happen within one unit of time". This is analogue to the discrete case modality "X will happen at the next step". A natural way to deal with quantitative modalities is by extending MLO to MLO_1-monadic logic of order with the $+1$ function: adding to the language the function $S(t) = t + 1$. The basic modality described above is then described by the formula:

$$\varphi(t_0, X) \equiv \exists t(t_0 < t < t_0 + 1 \wedge X(t)) \tag{1}$$

The canonical model R^+ is also the canonical model for MLO_1 but there may be other models: R^+ with a different time scale (e.g. $S(t) = 2^t$) or any ordered set with a function $+1$ that satisfies some obvious axioms (see Section 3). Unfortunately, MLO_1 is too expressive and it is undecidable over the canonical model (this follows also from [AFH96]). Quite a few formalisms have been suggested in the literature in order to supply TL with the capability to deal with quantitative properties in a decidable way. Many of them are surveyed in [AH92]. We add one that seems to be the simplest and the right one:

a) We take **(1)** as a table for the modality $\Diamond_1 X$ and its companion (2) as the table for the modality $\overleftarrow{\Diamond}_1 X$ - "X happened during the previous unit of time":

   $$\psi(t_0, X) \equiv \exists t(t_0 - 1 < t < t_0 \wedge X(t)) \ . \tag{2}$$

   The $t - 1$ is interpreted as 0 for $t < 0 + 1$ and as the standard $t - 1$ function for $t \geq 0 + 1$.
b) We add those two simple modalities to TL obtaining *quantitative temporal logic* QTL. We also identify the corresponding fragment QMLO of MLO_1.
c) We show that these logics are as expressive as any of the previously suggested constructs by showing how these constructs are defined in QMLO.
d) We show that the validity (and satisfiability) problems for this logic is decidable over the canonical model by reducing the problem to decidability of pure MLO and using the decidability results for MLO in [BG85].

e) We show that the satisfiability problem for QTL in the canonical finite variability model is PSPACE complete. In view of (c) the last two items reprove the decidability and the complexity results for logics like MITL without resorting in the proof to automata theory. It may seem surprising that automata theory does not yield any tighter bounds than straightforward translations and general logical considerations.

f) The whole discussion except for the complexity results applies uniformly to the canonical finite variability model and to the canonical general model. No other approach was even able to represent the most natural canonical model.

While QTL is as expressive as any of its rivals in the literature, it can be enriched to stronger yet still decidable logics. For example A. Pnueli's modality in X and Y: $P(t_0, X, Y) \equiv \exists t_1 t_2 [t_0 < t_1 < t_2 < t_0 + 1 \wedge X(t_1) \wedge Y(t_2)]$ and similar constructs can be added retaining decidability. However, none of these modalities seem general enough to be officially added to QTL. The search for a natural stronger logic still continues [HR99].

We believe that our formalism is *exactly the right one*: We use the most natural model, we use the most simple pair of quantitative modalities, everything is done within plain mathematical logic with no new or ad-hoc constructs, we prove decidability using mainstream logic and our formalism works just as well for general systems that may vary infinitely often in a finite length of time.

1.2 Comparison with Previous Works

Many formalisms were suggested in the last 15 years to extend temporal logic to deal with quantitative properties, as surveyed in [AH92] and [Wilke94].

> "The number of formalisms that purportedly facilitate the modeling, specifying and proving of timing properties for reactive systems has exploded over the past few years. The authors, who confess to have added to the confusion by advancing a variety of different syntactic and semantic proposals, feel that it would be beneficial to pause for a second - to pause and look back to sort out what has been accomplished and what needs to be done." [AH92].

> "Recent research in reactive and hybrid systems is dominated by a plethora of Concepts, Terminology, Notations. Unfortunately, this background is not free of ad-hoc and ambiguous decision which are liable to misjudgments and to infliction of myths into the area" [Trak99].

So it is not easy to compare our work with the related literature. We shall try to compare our work to the previous results regarding four aspects: the model used, the temporal connectives introduced, the decidability proof technique and the complexity of the method.

Models In [AH92] (section 2.2) there is a classification of all the models leading to "sixteen possible formal semantics of real time sequences". All sixteen models are ω-sequences of some kind. None of them is the real line itself. None of them reflects the real line faithfully and none avails itself to be adjusted to systems without finite variability. In section 7 we shell compare the canonical model with two of the most popular models.

Formal Language Some of the formalisms are obtained from logics or temporal logics by adding programming language constructs which lack the universality of logical notions. Notions such as freeze quantification, clock variables, half-order logics, explicit-clock notations, reset quantifications, position variables and position quantifications, nonstandard reals (which have nothing to do with the non-standard analysis) etc were invented [AH92,BL95,HRS98].

Others use temporal connectives which are definable in a fragment of first or second order Monadic Logic. We will discuss only the latter. Notably among logics with first-order defined connectives is MITL of [AFH96]. All such decidable languages are equal in expressive power to the most basic logic QTL on the finite variability canonical model. On the different ω-sequences models the logics differ in expressive power, causing a proliferation of logics. Thus "ϕ will happen within two units of time" is expressible in the canonical models through the connective "ϕ will happen within one time unit" but not so in the popular model of state sequences (see section 7).

Some modalities suggested in the literature were motivated by pragmatic considerations. Some other were obtained annotating the authors favorite modalities for TL by some time constrains. For example, the operator $\Diamond_{[1,3]}$ is interpreted as "eventually within one to three time units." There is no yardstick by which to measure the appropriateness of a temporal logic.

The most important characteristic of these formalisms are (1) expressiveness, which refers to correctness properties which the formalism can specify and (2) Complexity and decidability - the complexity of the model checking and the satisfiability problems for the formalism. One of the main goals was to formulate the most expressive logic which is still decidable.

Decidability Alur and Dill introduced time automata and proved that the emptiness problem is decidable for the class of these automata [AD94]. The proof reduces the problem to the emptiness problem for Büchi ω-automata. This result was used in all the previous approaches to prove that the logic is decidable (we suspect that this is the main reason why all the authors ignored the standard model in favor of an ω sequence model of some kind).

The class of languages accepted by timed automata is not closed under complementation. On the other hand logic is closed under negation. Therefore the usual way to show that a logic was decidable was to introduce some variations and restriction of timed automata which is closed under complementation and providing an effective transformation from logical formulae to "equivalent" automata in this class. These classes were given nice descriptive names like "Recursive Event Clock Automata" [HRS98] or "Bounded Two Way Deterministic Timed Automata" [AH92a]. It seems that these classes of automata do not give any additional insight about the corresponding logic.

In contrast our decidability proof remains within the framework of logic first reducing the language to a normal form ("Timer normal form") and then reducing the problem to the known result about the decidability of pure MLO (or TL). In particular our proof applies just as well to the standard model with all the unary predicates (and not just finite variability predicates).

Complexity It may have been hoped that the use of automata theory for decidability would provide a less complex procedure and clearer bounds than decidability proofs based on general logical reductions. We found this not to be the case; our method easily yields the same upper bounds or better ones (see section 6).

2 Monadic Logic of Order (MLO) and Temporal Logic (TL)

We survey the basic facts about these logics introducing TL within the framework of MLO [GHR94].

The syntax of MLO has in its vocabulary *individual (first order) variables* $t_0, t_1, \ldots$, *monadic predicate names* $X_0, X_1, \ldots$, and one binary relation $<$ (the order). *Atomic formulas* are of the form $X(t), t_1 < t_2$ and $t_1 = t_2$. *MLO formulas* are obtained from atomic formulas using the Boolean connectives $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$ and the (first order) quantifiers $\exists t$ and $\forall t$.

Second order Monadic Logic of Order formulas are obtained using also second order quantifiers $\exists X$ and $\forall X$.

Warning on Terminology. In this work we are mainly interested in first-order monadic logic of order, unless explicitly stated otherwise.

As usual if φ is a formula we may write $\varphi(t_1, \ldots t_k; X_1, \ldots, X_m)$ to indicate that the free variables in φ are among $t_1, \ldots t_k$ and the monadic predicate names are among $X_1, \ldots X_m$.

A structure for MLO is a pair $M = \langle A, < \rangle$ where $<$ is a linear order over A. The important examples for us are: $\langle R^+, < \rangle$, and $\langle N, < \rangle$; the non-negative real line and the non-negative integers. Note that at this stage, we made the choice to deal only with linear order so that trees and other orders are not in the scope of the discussion at present.

We shall not repeat the inductive definition saying when is a formula satisfied. Recall that in order to check if the formula $\varphi(t_1, \ldots t_k; X_1, \ldots, X_m)$ is true we need to specify which model $M = \langle A, < \rangle$ is intended and what are the elements $\tau_1, \ldots, \tau_k$ in A and the predicates (subsets) $P_1 \cdots P_m$ over A which are assigned to the variables and predicate names $t_1, \ldots, t_k, X_1 \cdots X_m$. Hence, the notation will usually be

$$\langle M, \tau_1, \ldots \tau_k; P_1 \cdots, P_m \rangle \models_{MLO} \varphi(t_1, \cdots t_k; X_1, \ldots, X_m)$$

which we also abbreviate to

$$M \models \varphi[\tau_1, \ldots, \tau_k; P_1, \ldots, P_m]$$

or even to $M \models \varphi[\overline{\tau}, \overline{P}]$ where the bar denotes a tuple of appropriate length. When we define the semantics of a second order formula or when we deal with validity and satisfiability of a first order formula it is necessary to specify over which predicates should the variables X range. In *full MLO* they range over all unary predicates (i.e. – subsets).

A requirement that is often imposed in the literature is that in every bounded time interval a system can change its state only finitely many times. This requirement is called *finite variability* (or non-Zeno) requirement. We consider also finitely variability interpretation of second-order MLO. Under this interpretations monadic predicates range over predicates with finite variability. Observe that in the real model this property of a predicate X can be expresses by the pure first-order formula [Rab98]. It is worth noting that there is no first-order monadic formula that defines finite variability predicates over the rationals, however, the finite variability predicates over the rationals can be defined by a monadic second-order formula.

The syntax of Temporal Logic (TL) has in its vocabulary *Predicate variables* $X_1, X_2, \ldots$ and some *modality names* with prescribed arity $O_1^{(k_1)} \cdots O_n^{(k_n)}$ (the arity notation is usually omitted). For example the "sometime in the future" modality $\diamondsuit$ has arity 1. The "_until_" modality has
arity 2.

Atomic formulas are just variables X_i and *temporal formulas* are obtained from the atoms using Boolean connectives and applying the modalities: if $\varphi_1, \ldots, \varphi_{k_i}$ are temporal formulas then so is $O^{(k_i)}(\varphi_1, \ldots, \varphi_{k_i})$. As usual we write φ *until* ψ instead of *until* (φ, ψ).

Structures for TL are again linear orders $M = \langle A, < \rangle$. Every modality $O^{(k)}$ is interpreted in every structure M as an operator $O_M^{(k)} : [P(M)]^k \rightarrow P(M)$ which assigns "the set of points where $O[P_1, \ldots, P_k]$ holds" to the k-tuple $\langle P_1, \ldots, P_k \rangle$. We consider only modalities which are defined in MLO: we assume that for every modality $O^{(k)}$ there is a formula *(truth table)* $\overline{O}(t_0, X_1, \ldots X_k)$ of MLO such that in every structure M:

$$O_M(P_1, \ldots, P_k) = \{\tau \mid M \models_{MLO} \overline{O}[\tau; P_1, \ldots, P_k]\} .$$

Example (Tables for Modalities)

- The modality $\diamondminus X$, "X has happened before" is defined by $\varphi(t_0, X) \equiv \exists t < t_0$ $X(t)$.
- The modality X *until* Y is defined by $\psi(t_0, X, Y) \equiv \exists t_1(t_0 < t_1 \wedge Y(t_1) \wedge \forall t(t_0 < t < t_1 \rightarrow X(t)))$.
- The modality X *since* Y is defined by $\psi(t_0, X, Y) \equiv \exists t_1(t_0 > t_1 \wedge Y(t_1) \wedge \forall t(t_1 < t < t_0 \rightarrow X(t)))$.

Satisfaction of a formula at a point τ in the model M is defined inductively starting with:

$$\langle M, \tau, P \rangle \models_{TL} X \quad \text{iff} \quad \tau \in P$$

and

$$\langle M, \tau, P_1, \ldots P_k \rangle \models_{TL} O(X_1, \ldots X_k) \quad \text{iff} \quad \langle M, \tau, P_1, \ldots P_k \rangle \models_{MLO} \overline{O}(t_0, X_1, \ldots X_k) .$$

This extends easily to:

Proposition. *For every formula $\varphi(X_1, \ldots X_n)$ of TL there is a formula $\overline{\varphi}(t_0, X_1, \ldots X_n)$ of MLO such that for every M, $\tau \in M$ and predicates $P, \ldots P_n$*

$$\langle M, \tau, P_1, \ldots P_n \rangle \models_{TL} \varphi \quad \text{iff} \quad \langle M, \tau, P_1, \ldots P_n \rangle \models_{MLO} \overline{\varphi} .$$

MLO supplies us with a yardstick by which to measure if a temporal logic (i.e. a choice of modalities) is as expressive as can be hoped for:

Definition. Let $\mathcal{C}$ be a class of structures and L a temporal logic. L is *expressively complete* with respect to $\mathcal{C}$ if every formula φ of MLO with single first order free variable t_0 is equivalent in $\mathcal{C}$ to some formula φ' of the logic. A list of modalities $O_1^{(k_1)} \cdots O_n^{(k_n)}$ is *expressively complete* if the temporal logic with these modalities is expressibly complete.

There are natural choices for expressively complete sets of modalities:

Theorem 1. *a) ([Kamp68], reproved in [GPSS80]): The pair of modalities $X\,until\,Y$* and
$X\,since\,Y$ is expressively complete for the canonical structures: the real line and the natural numbers.

b) [GHR94] There is a pair of modalities $X\,until_s Y$ and *$X\,since_s Y$ ("Stavi's modalities") which together with the since and until is expressively complete for the class of all linear orders.*

3 Quantitative Temporal Logic and Quantitative MLO

The logics MLO and TL are not suitable to deal with quantitative statements like "X will occur within one unit of time". This can be easily remedied in predicate logic by using the function $t+1$:

Definition. MLO_1, the monadic logic of order with a $+1$ function is the monadic logic built from the primitive relation $<$ *and* one unary function symbol which we denote either by $S(t)$ or by $t+1$. The *standard (canonical) model* for this logic is the non-negative real line with the usual $+1$ function. *General structures* for MLO_1 are ordered structures $M = \langle A, <, 0, S\rangle$ with first element 0 and with a unary function S that satisfies some natural requirements: like strict monotonicity, non-archimedian property.

We will deal here only with the canonical model and leave out the axiomatization of more general models. We use the standard notation $t+1$ for $S(t), n$ for $0+1+\cdots+1$ (n times) and $t-1$ for 0 when $t<1$ and for the unique t_1 such that $t_1+1=t$ for $t \geq 1$.

MLO_1 is a broad language and everything that we do is inside MLO_1. But it is too strong: the problem of validity and satisfiability in the standard model is undecidable for MLO_1 (there is a natural encoding of Turing computations that shows it but it can also be deduced from the undecidability proof in [AFH96]). We shall, therefore, start at the other end: introduce the simplest quantitative modalities to TL and check what the corresponding fragment of MLO_1 is. We then check if the result is expressive enough and if it is decidable.

Definition: Quantitative Temporal Logic (QTL) is the temporal logic constructed from an expressively complete set of modalities for MLO *and* two new modalities $\diamondsuit_1 X$ and $\overleftarrow{\diamondsuit}_1 X$ defined by the tables (in t_0):

$$\diamondsuit_1 X: \qquad \exists t((t_0 < t < t_0 + 1) \wedge X(t)) \tag{3}$$

$$\overleftarrow{\diamondsuit}_1 X: \qquad \exists t((t_0 - 1 < t < t_0) \wedge X(t)) \ . \tag{4}$$

Next we intend to identify the fragment of MLO_1 that corresponds to QTL. This fragment will use the function $t+1$ only in a very restricted form as indicated in (3) and (4). We introduce some syntactical sugar to MLO_1 – the "bounded quantifiers" $(\exists t)^{<t_0+1}_{>t_0}$ and $(\exists t)^{<t_0}_{>t_0-1}$ as follows: if φ is a formula of MLO_1 then we use the shorthand:

$$(\exists t)^{<t_0+1}_{>t_0} \varphi \equiv \exists t(t_0 < t < t_0 + 1 \wedge \varphi(t)) \tag{5}$$

$$(\exists t)^{<t_0}_{>t_0-1} \varphi \equiv \exists t(t_0 - 1 < t < t_0 \wedge \varphi(t)) \tag{6}$$

Definition. Quantitative Monadic Logic of Order (QMLO) is the fragment of MLO_1 which is built from the atomic formulas $t_1 < t_2, t_1 = t_2, X(t)$ $(t, t_1, t_2$ variables) using Boolean connectives, first order quantifiers and the following rule: if $\varphi(t)$ is a formula of QMLO *with t its only first order free variable* then $(\exists t)^{<t_0+1}_{>t_0} \varphi$ and $(\exists t)^{<t_0}_{>t_0-1} \varphi$ are formulas of QMLO.

The following observation characterizes the expressive power of QTL.

Theorem 2. *Let Γ be an expressive complete set of modalities (over the reals) for first-order MLO. For every formula p over the modalities $\{\Gamma, \diamondsuit_1, \overleftarrow{\diamondsuit}_1\}$ there is a formula $\phi(t_0)$ of QMLO effectively computable from p such that $\phi(t_0)$ is equivalent (over the reals) to p. For every formula $\phi(t_0)$ of QMLO there is a formula p over the modalities $\{\Gamma, \diamondsuit_1, \overleftarrow{\diamondsuit}_1\}$ effectively computable from $\phi(t_0)$ such that $\phi(t_0)$ is equivalent (over the reals) to p.*

Proof. Straightforward induction.

4 The Expressive Power of QMLO

4.1 General bounded quantifiers

At first glance the modalities $\overleftarrow{\Diamond}_1$ and $\Diamond_1$ may seem insufficient to express more general modalities like $\Diamond_{[5,7)}$ defined by the table $\exists t.\ t_0 + 5 \leq t < t_0 + 7 \wedge X(t)$. However this is not the case.

We shall first show that one can use more general quantifiers in QMLO: $(\exists t)^{<t_0+n+m}_{>t_0+n}$, $(\exists t)^{<\infty}_{>t_0+n}$ and quantifiers with weak inequality replacing the strict inequality in one or both ends of the interval, where n is an integer and m is a positive natural number.

(a) $(\exists t)^{<t_0+1}_{\geq t_0} X(t) \equiv X(t_0) \vee (\exists t)^{<t_0+1}_{>t_0} X(t)$

(b) $(\exists t)^{\leq t_0+1}_{>t_0} X(t) \equiv (\exists t)^{<t_0+1}_{>t_0} X(t) \vee [\text{First } (t_0, X) \wedge (\forall t_1)^{<t_0+1}_{>t_0} (\exists t)^{<t_1+1}_{>t_1} X(t)]$

when First (t_0, X) says that there is a first point past t_0 for which $X(t)$ holds:

$$\text{First } (t_0, X) \equiv \exists t_1 [t_0 < t_1 \wedge X(t_1) \wedge \forall t (t_0 < t < t_1 \rightarrow \neg X(t))] \tag{7}$$

it is not difficult to see that once a first solution to $X(t)$ is granted then the last conjunct in (b) is equivalent to $X(t_0+1)$ (note, however, that $X(t_0+1)$ by itself is not definable in QMLO and its addition would lead to undecidability).

Hence quantifiers $(\exists t)^{<t_0+1}_{\geq t_0}$ and $(\exists t)^{\leq t_0+1}_{>t_0}$ are definable in QMLO.

Let us list some more laws

(c) $(\exists t)^{<\infty}_{>t_0+1} X(t) \equiv (\forall t)^{\leq t_0+1}_{>t_0} \exists t_1 . t_1 > t \wedge X(t_1)$

(d) $(\forall t)^{<t_0+n+1}_{>t_0+n} X(t) \equiv (\forall t)^{<t_0+n}_{>t_0+n-1} (\exists t_2)^{<t+1}_{>t} (\forall t_1)^{<t_2+1}_{>t_2} X(t_1)$ for $n > 0$.

Theorem 3. *The extension L of QMLO by the following rules is expressive equivalent to QMLO over the canonical model.*

if $\phi(t)$ is an L formula with the only free variable t then the following are L formulae:

1. $(\exists t)^{<t_0+n+m}_{>t_0+n} \phi(t)$, *where n is an integer and m a positive natural number.*
2. $(\exists t > t_0 + n)\phi(t)$ *(denoted also by $(\exists t)^{<\infty}_{>t_0+n} \phi(t)$) and $(\exists t < t_0 + n)\phi(t)$ (denoted by $(\exists t)^{<t_0+n}_{>-\infty} \phi(t)$), where n is an integer.*
3. *Similar to (1) or (2) above with weak inequality replacing one or both occurrences of the strong inequality.*

Henceforth, we will freely use these generalized quantifiers which are definable in QMLO and also the corresponding modalities like $\Box_{\leq n}$ which is defined by $(\forall t)^{\leq t_0+n}_{\geq t_0} X_1(t)$ and $\Diamond_{[n,m]}$ which is defined by $(\exists t)^{\leq t_0+m}_{\geq t_0+n} X_1(t)$.

4.2 Modalities in Real Time Logics

Here are the definitions of some modalities that were used before:
(e) Pnueli's $\mathrm{Age}_1(X)$ modality is dual to $\overleftarrow{\Diamond}_1$

$$\mathrm{Age}_1(X) \equiv (\forall t)^{<t_0}_{>t_0-1} X(t)$$

(f) Wilke's relative distance construct "the first time $X(t)$ occurs after t_0 is in distance smaller than larger than or equal to n"[Wilke94]

$$d_{<n}(X) \quad \text{is just} \quad \mathrm{First}(t_0, X) \wedge (\exists t)^{\leq t_0+n}_{>t_0} X(t)$$

$$d_{=n}(X) \quad \text{is} \quad (\exists t)^{\leq t_0+n}_{>t_0} X(t) \wedge (\forall t)^{\leq t_0+n}_{>t_0} \neg X(t)$$

(g) The $\triangleright$ modality of [HRS98]: $\triangleright_{(n,n+m)} X$ "there is a first instance of $X(t)$ among the points in the interval $(t_0 + n, t_0 + n + m)$". For $n = 0$ this is $\mathrm{First}(t_0, X) \wedge (\exists t)^{\leq t_0+m}_{>t_0} X(t)$. For $n > 0$ this is $[(\forall t_1)^{\leq t_0+n}_{>t_0+n-1} (\exists t)^{\leq t_1+1}_{>t_1} \mathrm{First}(t, X)] \wedge (\exists t)^{<t_0+n+m}_{>t_0+n} X(t)$.

The logic MITL [AFH96] is based on an infinite set of modalities $until_I$ where I is a non-singular interval with integer endpoints - these modalities are called constrained *until* modalities. The MLO_1 truth table for example for the formula $X\ until_{[5,8)}\ Y$ is $\exists t. t_0 + 5 \leq t < t_0 + 8 \wedge Y(t) \wedge (\forall t'.\ t_0 < t' < t \rightarrow X(t'))$.

It is shown in [AFH96] that the operators like our $\Diamond_1$ are definable in this logic. Unfortunately the fundamental operator $\overleftarrow{\Diamond}_1$ is not discussed there and in fact we can show that it cannot be expressed using constrained $until_I$ modalities.

What is called MITL in [HRS98] is the logic based on $until_I$ operators and the since operators $since_I$. The modality $\overleftarrow{\Diamond}_1$ is easily expressed using these since operators.

In the sequel we will refer only to the version with both until and since operators.

(h) $\mathrm{until}_{(n,n+m)}$ can be defined as $P\ \mathrm{until}_{(n,n+m)} Q \equiv \Box_{(0,n]}(P \wedge (P\ until\ Q)) \wedge \Diamond_{(n,m)} Q$.

Similar definitions work for since and for half open or closed intervals. Hence, MITL and QTL are expressive equivalent.

5 Decidability

We want to show that there is an algorithm which given a formula $\varphi(\overline{Z})$ of QTL determines if φ is valid in $\langle R^+, 0, <, +1\rangle$. We prove the equivalent claim that there is an algorithm which given ψ of QMLO determines if ψ is satisfiable in $\langle R^+, 0, <, +1\rangle$.

Notation. For every n we define the formula:

$$\mathrm{Timer}_n(X_1, \ldots X_n, Y_1, \cdots Y_n) \equiv \bigwedge_i \forall t(Y_i(t) \longleftrightarrow (\forall t_1)^{<t}_{>t-1} X_i(t_1))$$

i.e. each Y_i is a timer that measures if X_i persisted for at least one unit of time.

Theorem 4 (Timer Normal form). *There is an algorithm which associates with any formula $\varphi(t_0, \overline{Z})$ of QMLO variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and a formula $\overline{\varphi}(t_0, X_1, \ldots, X_n, Y_1, \ldots, Y_n, \overline{Z})$ of pure MLO such that $\varphi(t_0, \overline{Z})$ is satisfiable iff the following formula is satisfiable:*

$$\mathrm{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \wedge \overline{\varphi}(t_0, \overline{X}, \overline{Y}, \overline{Z})$$

The proof is by routine normalization (note that the two formulas are equivalent only with respect to satisfiability).

Definition A formula is said to be in first (second) order **timer normal form** if it has the form

$$\mathrm{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \wedge \phi,$$

where ϕ is a first (second) order monadic formula.

Next we associate with the formula $\mathrm{Timer}_n(X_1, \ldots X_n, Y_1, \cdots Y_n)$ a formula $\overline{\mathrm{Timer}}_n(X_1, \ldots, X_n, Y_1, \ldots Y_n)$ in pure monadic logic. It is the conjunction of the properties A_i, $B_{i,j}$, C_i below and some technical properties D_i which we omit. We introduce first the following notation: *the duration of X at t_0* is the largest interval (t_1, t_0) such that $X(t)$ holds for every point t in the interval.

A_i : "If $Y_i(t_0)$ holds then the duration of X_i at t_0 is not empty.

$B_{i,j}$:"If $Y_i(t_0)$ holds and if the duration of X_j at t_0 is at least as big as the duration of X_i then $Y_j(t_0)$ holds. Moreover; if the duration of X_j at t_0 is strictly larger than the duration of X_i then $Y_j(t)$ started to hold already at a point before t_0.

C_i : "Y_i has finite variability. This is expressible in R by the pure first order monadic formula that says: Every point is the left end of some open interval on which Y_i does not change its value and every point except 0 is the right end of an open interval on which Y_i does not change its value.

D_i deals with the behavior of Y_i at 0 and when approaching infinity. It also asserts that Y_i holds on a topologically closed set. The definition will appear in the full version of the paper.

The formula Timer of QMLO and the formula $\overline{\text{Timer}}$ of MLO are related by the following main Lemma:

Lemma 5 (Reduction of Quantitative Properties to Pure Monadic Properties). *The predicates $P_1, \ldots, P_n Q_1, \ldots, Q_n$ over $\langle R^+, 0, < \rangle$ satisfy the formula $\overline{\text{Timer}_n}$ if and only if there is an order preserving bijection $\rho : R^+ \to R^+$ such that $P_1\rho, \ldots, P_n\rho,\ Q_1\rho, \ldots, Q_n\rho$ satisfy* Timer_n.

In the above lemma the predicate $P_i\rho$ is obtained from the predicate P_i by stretching according to the bijection ρ, i.e $(P\rho)(\tau)$ iff $P(\rho(\tau))$. Observe that P satisfies a monadic formula $\phi(X)$ iff $P\rho$ satisfies $\phi(X)$. Therefore,

Corollary 6. *For every first-order or second-order monadic formula ϕ $\overline{\text{Timer}_n} \wedge \phi$ is satisfiable iff $\text{Timer}_n \wedge \phi$ is satisfiable.*

We recall

Theorem 7. *1. The satisfiability of monadic first order logic of order over the reals is decidable [BG85].*

2. The satisfiability of monadic second-order logic of order over the reals is undecidable [She75]

3. The satisfiability of monadic second order logic over the real finitely variable interpretation is decidable [Rab98].

Therefore,

Corollary 8. *1. The satisfiability of the formulas in first-order timer normal form in the canonical model is decidable.*

2. The satisfiability of the formulas in the second-order timer normal form under the finite variability interpretation is decidable.

Theorem 9. *Satisfiability in the canonical model is decidable for QMLO and for QTL. Satisfiability in the finite variability canonical model is decidable for QMLO and for QTL.*

6 On Complexity of QTL

In this section we show somewhat informally how one can derive easily the complexity bounds without appealing to automata theoretical techniques.

We observe that the monadic formula $\overline{Timer}$ (see section 5) is expressible in Temporal Logic.

Lemma 10. *There is a TL formula $timer_n(X_1, \ldots X_n, Y_1, \ldots Y_n)$ of size $O(n^2)$ over the modalities Until and Since such that $\overline{Timer}_n(X_1, \ldots X_n, Y_1, \ldots Y_n)$ is equivalent over the canonical model to $timer_n(X_1, \ldots X_n, Y_1, \ldots Y_n)$.*

Proof. Just observe that every clause A_i, $B_{i,j}, C_i$ and D_i in the definition of $\overline{Timer_n}$ (see section 5) is expressible by a temporal logic formula which is independent of n.

Unnesting: We define a process of unnesting by a generic example. Let $Always(X)$ be the modality defined by the table $t_0 = t_0 \wedge \forall t.\ X(t)$. Let OP_1 and OP_2 be any two-place modalities over a linear order A. It is easy to see that that the formulas $OP_1(\neg X_1, OP_2(X_2, X_3 \vee X_4)$ is unsatisfiable over A iff the conjunction of the following formulas is unsatisfiable over A

1. Z
2. $Always(Z \leftrightarrow OP_1(Z_1, Z_2))$
3. $Always(Z_1 \leftrightarrow \neg X_1)$
4. $Always(Z_2 \leftrightarrow OP_2(X_2, Z_3,))$

5. $Always(Z_3 \leftrightarrow (X_3 \vee X_4))$

A simple generalization of this observation shows

Lemma 11. *For every QTL formula ϕ with k metrical quantifiers there is a TL formula ψ and a subset F of $\{1 \dots k\}$ such that the size of ψ is linear in the size of ϕ and ϕ is satisfiable if and only if the conjunction of ψ with $\bigwedge_{i \in F} Always(X_i \leftrightarrow \Diamond_1 Z_i) \wedge \bigwedge_{i \in \{1,\dots,k\} \setminus F} Always(X_i \leftrightarrow \overleftarrow{\Diamond}_1 Z_i)$ is satisfiable.*

Lemma 12. *There is a formula over $\{until,\ since,\ \overleftarrow{\Diamond}_1\}$ which is equivalent to $Always(X_i \leftrightarrow \Diamond_1 Z_i)$.*

Lemma 13. *The formula $Always(X_i \leftrightarrow \overleftarrow{\Diamond}_1 Z_i)$ is equivalent to $Timer_1(\neg X_i, \neg Z_i)$.*

The above lemmas and Lemma 5 imply

Theorem 14. *There is a polynomial time algorithm that for every QTL formula ϕ of size m with k metrical modalities constructs a TL formula ψ of size $O(k^2 \times m)$ such that (1) ψ is satisfiable in the canonical model iff ϕ is satisfiable in the canonical model. (2) ψ is satisfiable in the finite variability canonical model iff ϕ is satisfiable in the finite variability canonical model.*

Theorem 15. *[Rab98a] Let Γ be any finite set of modalities definable in monadic first-order logic of order. The satisfiability problem for TL formulas over Γ in the finite variability canonical model is in PSPACE.*

We note here that the proof of the above theorem is automata free and relies on the compositional method.

Open Question: What is the complexity of the satisfiability for TL formulas in the general canonical model.

Corollary 16. *The satisfiability problem for QTL in the finite variability canonical model is in PSPACE.*

Let ϕ be an MITL formula with m boolean and temporal connectives and let c be the largest constant that appears as a subscript of the bounded until or since operator in ϕ. From the laws of Section 4 and an unnesting procedure one can construct a QTL formula ψ of size $O(m \times c)$ such that ψ is satisfiable (in either of the canonical or finite variability canonical models) iff ϕ is satisfiable in that model. Hence, we reprove the result of [AFH96] that satisfiability for MITL is decidable in space polynomial in $m \times c$.

7 About Two Popular Real Time Models

In this paper we are dealing with the canonical model for QTL or with the finitely variable canonical model (in this model predicates are restricted to finitely variable predicates). In the survey [AH92] sixteen models for real time logics are provided. Surprisingly, neither the canonical model nor the finite variability canonical model is among these models.

All the models considered in the literature are ω-sequences of something. Probably because the only technique to show the decidability was the reduction to Alur-Dill timed automata which in turn are reduced to ω-automata. Our proof of decidability is automata free and works for general predicates as well as finitely variable predicates.

We discuss two of these model that gained popularity.

7.1 The State Sequence Model

The most popular model is the *state sequence model.* A state sequence is an ω-sequence (t_0, σ_0), ..., (t_n, σ_n), ... where t_i is an unbounded increasing sequence of reals and σ_i gives the values of the monadic predicates at t_i.

The state sequence model does not faithfully reflect the real line. For example the very basic law $\Diamond_2 X \leftrightarrow \Diamond_1 X \vee \Diamond_1 \Diamond_1 X$ fails in this model. In fact already at the pure temporal logic level (without the metric) it is clear that sequences can not represent faithfully the canonical model as any formula is satisfiable over real sequences iff it is satisfiable over the discrete model of the natural numbers. Clearly we expect TL to differentiate between the discrete and the continuous model of time. another price paid for the use of state sequences is the proliferation of modal operators; while connectives like $\Diamond_{[m,n]}$ can be defined in the canonical model using the connective $\Diamond_1$ this is not true for state sequences. Most of the laws from section 4 fail in this model. For example $\Diamond_{(0,1]}$ is not expressible from $\Diamond_1$.

A remarkable work that achieves impressive results using state sequences is Wilke's [Wilke94]. He introduced metrical properties into second order monadic logic and proved the decidability of an extensive fragment. He embedded all the known decidable formalisms (and much more) into this fragment thus reproving uniformly the decidability.

7.2 The Trace Interval Model

Another popular model is trace interval model. A trace interval is an ω-sequence $(I_0, \sigma_0), \ldots (I_n, \sigma_n) \ldots$ where I_i are disjoint intervals that cover R^+ and I_i precedes I_{i+1}.

A trace interval represents a finite variability structure for MLO. Indeed, every $t \in R^+$ is in exactly one of the intervals I_j and σ_j determines the value of the monadic predicates at t. This clearly defines a finite variability model. However, the same finite variability model has many distinct representations. So the formalisms that distinguish between representations of the same model are unreasonable.

Even though a trace interval encodes all the information about the corresponding finite variability canonical model it may still be misleading if one tries to use it for a different model like the rationals Q. In the words of [AFH96] "MITL cannot distinguish the time domain of the reals from the time domain of the rationals". In fact theorem 2.4.1.4 in [AFH96] says that an MITL formula ϕ is satisfiable in the rational interval traces iff it is satisfiable in the real interval traces. (By the rational interval trace it is meant there an ω-sequence $(I_0, \sigma_0), \ldots (I_n, \sigma_n) \ldots$ where I_i are disjoint intervals of the rationals with the rationals end-points that cover Q^+ and I_i precedes I_{i+1}.)

But the following TL formula is satisfiable in the reals and not in the rationals.

$$(X \wedge \Diamond\neg X) \wedge \Box(\neg X \rightarrow \Box\neg X) \wedge ((X \rightarrow \Diamond X) \wedge (\neg X \rightarrow \overleftarrow{\Diamond}\neg X))$$

(By the first two clauses X holds for a prefix of the model and $\neg X$ holds in a suffix of the model; by the third clause X has no maximal element and $\neg X$ has no minimal element). Does theorem 2.4.1.4 in [AFH96] say that MITL is two weak to express the above formula. Of course not; it only shows that the rationals are not modeled adequately by the rational trace intervals.

8 Conclusion

We believe that we proved the case for the quantitative logic QTL and its predicate logic twin QMLO.

It is easy to modify the theory to deal with more general time lines. For the rational line Q we would need to require that the time unit is Archimedian: the sequence $0, 0+1, 0+1+1, \ldots$ must be unbounded for every $+1$ function. A more general approach will replace the $+1$ function by a relation between pairs of points in time "t_2 is_not_too_far_ahead_of t_1" such that some obvious axioms hold.

Future research must look for *natural* stronger modalities. A. Pnueli noticed that the following modality

$$\varphi(t_0, X, Y) \equiv (\exists t_1 t_2)[(t_0 < t_1 < t_2 < t_0 + 1) \wedge X(t_1) \wedge Y(t_2)]$$

is not definable in QMLO [HR99]. We can show that one may add modalities of the following form without losing decidability

$$(\exists t_1 \cdots t_n)[(t_0 < t_1 < \cdots < t_n < t_0 + 1) \wedge \bigwedge_i \varphi_i(t_i)]$$

(note that φ_i speaks only of t_i). But it is difficult to see from this what should be *the* next modality in the logic.

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