Future temporal logic needs infinitely many modalities

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March 6, 2003

Abstract

Kamp's theorem states that there is a temporal logic with two modalities ("until" and "since") which is expressively complete for the first order monadic logic of order over the real line. In this paper we show that there is no temporal logic with finitely many modalities which is expressively complete for the future fragment of first order monadic logic of order over the real line (a future formula over the real time line is a formula whose truth value at a point is independent of what happened in the past).

1 Introduction

Predicate Logic has been accepted among logicians as the appropriate language for specification and verification in mathematics and elsewhere. It allows for a uniform treatment of a host of languages whose expressive power varies according to the use of different kinds of variables (individual variables, set variables etc.) and of different constant names (for objects, functions or relations). In particular the appropriate language to deal with the evolving of reactive systems (or any other system) in time is the Monadic Logic of Order (**MLO**): Its formulas are built using atomic propositions $t \in X$, atomic relations between elements $t_1 = t_2$, $t_1 < t_2$, Boolean connectives, first-order quantifiers $\exists t$ and $\forall t$. The relation $<$ is the natural order of time and the predicates X stand for each of the properties which may hold or not at any moment in time, and are of interest to us. This language is in fact the first order *MLO*. For more expressive power we could use second order *MLO* which has also second order (set) quantifiers $\exists X$ and $\forall X$

Outside of the mathematical community the introduction of variables and their interaction with the quantifiers was not unanimously embraced and attempts have been made to define formal languages which retain some of the flavor of natural speech. Notably among these languages are the temporal logics which avoid the individual variables and the quantification over these variables. This is done by (a) taking the order of time

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as a primitive relation that needs not be mentioned explicitly, (b) by restricting the properties defined by formulas to properties of single points (and not to relations among points)[1] (c) by taking as atoms *Atomic Propositions* and most important, (d) by the use of Modalities (called also Temporal Connectives) to combine formulas into more complex ones. A k-place modality C transforms statements $\varphi_1, \ldots, \varphi_k$ on points possibly other than the given point t_0 to a statement $C(\varphi_1, \ldots, \varphi_k)$ on the point t_0. The rule that specifies when the statement $C(\varphi_1, \ldots, \varphi_k)$ is true for the given point, is called the *Truth Table* of C. The choice of the particular modalities with their truth tables determines the different temporal logics. A Temporal Logic with modalities $C_1, \ldots, C_k$ is denoted by $TL(C_1, \ldots, C_k)$.

Of course there is not much interest in a modality C whose truth table is arbitrary and is defined ad-hoc in each and every structure where the logic is to be interpreted. To be of interest a truth table needs to have a uniform description in some metalanguage, that connects the set of points where $C(\varphi_1, \ldots, \varphi_k)$ holds to the sets of points where each of the φ_i holds. In all practical cases this truth table is itself a formula of some natural predicate logic and as a result all of the temporal logic embeds into the predicate logic. In this respect it is legitimate to say that "temporal logic is but syntactical sugaring of a fragment of predicate logic". This syntactical sugaring is all gain and no loss if the temporal logic is "expressively complete" for the predicate logic. I.e, if every formula (with single free variable) in the predicate logic is equivalent to some temporal formula. It may also happen that a temporal logic is expressively complete for a predicate logic with respect to one family of structures but not with respect to a larger family.

The simplest modality is the one place modality $\mathsf{F}X$ saying "X will hold some time in the future". Its truth table is usually formalized by the *MLO* formula $\varphi_{\mathsf{F}}(t_0, X) \equiv (\exists t > t_0)\ t \in X$. Two more natural modalities are the modalities U ("*Until*") and S ("*Since*"). $X\mathsf{U}Y$ means that X will hold from now until a time in the future when Y will hold. $X\mathsf{S}Y$ means that Y was true at some point of time in the past and since that point X was true until (not necessarily including) now. $TL(\mathsf{U}, \mathsf{S})$ is an important temporal logic and often it is referred to as "the temporal logic". Both modalities have truth tables in *MLO* (see example 1) and hence every temporal logic formula translates naturally into a formula of *MLO*.

The main models for time are $\langle \mathbb{N}, < \rangle$, the natural numbers as a model of *discrete time* and $\langle \mathbf{R}^+, < \rangle$, the non negative real line as the model for *continuous time*. These two models are called the *canonical models of time*. A major result concerning TL is Kamp's theorem [Kam68, GHR94] which says that $TL(\mathsf{U}, \mathsf{S})$, the temporal logic having "*Until*" and "*Since*" as only modalities, is expressively complete for (the first-order) *MLO* over the two canonical models.

This sets the scene for Amir Pnueli's influential declaration in [Pnu77] that Temporal Logic (TL) was the most suitable framework for the specification of properties of systems. TL became very popular in the Computer Science community and it enjoyed extensive research during the last 20 years to the exclusion of almost any other formalism, in particular that of *MLO*. Since $TL(\mathsf{U}, \mathsf{S})$ is expressively complete it may seem that

[1] this restriction is not essential.

concentrating on $TL(\mathbf{U},\mathbf{S})$ was harmless.

In fact we believe that much understanding, insight and technical tools were sacrificed when the robust and uniform classical logic was abandoned as the frame of reference. We addressed this point quite strongly in [HR99]. In this paper we answer a question that is almost as relevant to computer science as Kamp's theorem and yet it was ignored: What are the proper modalities to use for a temporal logic that deals only with future formulas.

A future formula is a formula whose truth value at a point depends only on the point and its future and not on what happened or not in the past. For example the formula X *until* Y says that X will hold from now until (at least) a point in the future when Y will hold. The truth value of this formula at a point t_0 does not depend on the question whether $X(t)$ or $Y(t)$ are true for points t such that $t < t_0$. Note that the definition of a future formula is a semantical definition but the problem whether a given formula is a future formula or not is decidable (see remark 2.1).

Most of the temporal formalisms studied in computer science concern only future formulas. Is there also a natural future temporal logic that is expressively complete for the class of future *MLO* formulas with respect to the canonical models? Is it simply $TL(\mathbf{U})$ and if not can we add some naturally defined modalities to obtain an expressively complete future logic?

This important question was not answered before. In fact it was not even asked. Instead even the most advanced research in temporal logic treated the subject as if there was no problem and as if $TL(\mathbf{U})$ is the natural future fragment of $TL(\mathbf{U},\mathbf{S})$ (compare for example [AH92] and [Hen98]).

There is indeed no problem in the discrete case: Over the discrete model $\langle \mathbb{N}, < \rangle$ Kamp's theorem holds also for *future formulas* of *MLO* and the future fragment of *MLO* has the same expressive power as $TL(\mathbf{U})$ [GPSS80, GHR94].

The situation is radically different for the continuous time model $\langle \mathbf{R}, < \rangle$: In this paper we show that $TL(\mathbf{U})$ is not expressively complete and there is no easy way to remedy it. In fact we show:

Theorem: No future temporal logic with finite base is expressively complete; there is no temporal logic with a finite set of modalities which is expressively equivalent to the future fragment of *MLO* over the reals.

Almost the same proof shows there is no temporal logic with a finite set of modalities which is expressively equivalent to the future fragment of *MLO* over the rationals.

Related work: Similar methods produced negative results for other logics and models (see survey [Ra02]). It was shown in [RM00] that there is no temporal logic with finite base that is expressively complete for CTL^* [EH86] with respect to the class of (partially ordered) trees. In another direction it was shown in [BR00] that there is no temporal logic with finite base which is expressively complete for second order monadic logic of order, even with respect to the discrete model $\langle N, < \rangle$.

The rest of the paper is organized as follows: In section 2 we recall the definitions of the monadic logic, the temporal logics and Kamp's theorem. In section 3 we prove that over the class of ordinals there is no finite basis for a temporal logic which is expressively complete for future formulas. To this end we show that for every pair of distinct natural numbers k and n there is a future formula that distincts between the ordered sets ω^k and ω^n for $k \neq n$. On the other hand we use a theorem by Ehrenfeucht [Ehr61] to prove that this distinction cannot be made by a temporal logic whose modalities correspond to monadic logic formulas with bounded depth of quantifiers. Thus in particular not by a logic with finitely many modalities. In section 4 we embed the ordinals in the real line and deduce the main theorem for the real line. Finally in section 5 we discuss extensions of the theorem and compare it with the results on future logic in [GHR94].

2 Preliminaries

2.1 First-Order Monadic Logic of Order

The syntax of the monadic predicate logic of order - *MLO* has in its vocabulary *individual* (first order) variables $t_0, t_1, \ldots$, monadic *predicate* variables $X_0, X_1, \ldots$, and one binary relation $<$ (the order). We use lower case letters t, x, y to range over individual variables and upper case letters X, Y, Z to range over monadic variables. **Atomic formulas** are of the form $X(t), t_1 < t_2$ and $t_1 = t_2$. **Well formed formulas** of the monadic logic *MLO* are obtained from atomic formulas using Boolean connectives $\neg, \vee, \wedge, \rightarrow$ and the (first order) quantifiers $\exists t$ and $\forall t$.

The quantification depth is defined as usual: (1) if φ is atomic then its quantification depth, $qd(\varphi)$ is 0; (2) $qd(\neg(\varphi)) = qd(\varphi)$; (3) $qd(\varphi \vee \psi) = max\{qd(\varphi), qd(\psi)\}$; (4) $qd(\forall x \varphi) = qd(\exists x \varphi) = qd(\varphi) + 1$.

A structure for *MLO* is a tuple $M = \langle A, <, \mathcal{P}_1, \cdots, \mathcal{P}_n \rangle$ where $<$ is a linear order over A and $\mathcal{P}_1, \cdots, \mathcal{P}_n$ are monadic predicates. Such a structure is also called **colored chain**, or for brevity simply a **chain**.

The important examples for us are $\langle R^+, < \rangle$ and $\langle N, < \rangle$; the non-negative real line and the non-negative integers.

We shall not repeat the inductive definition saying when is a formula satisfied. We recall the notation:

$$M, \tau_1, \ldots \tau_k; \mathcal{P}_1, \ldots, \mathcal{P}_m \models \varphi(t_1, \ldots, t_k; X_1, \ldots, X_m)$$

which we also abbreviate to $M \models \varphi[\tau_1, \ldots, \tau_k; \mathcal{P}_1, \ldots, \mathcal{P}_m]$ or even to $M \models \varphi[\overline{\tau}, \overline{\mathcal{P}}]$ where M is a structure, $\tau_1, \ldots, \tau_k$ are elements of M, $\mathcal{P}_1, \cdots, \mathcal{P}_m$ are unary predicates (i.e sets) in M, φ is a formula and $t_1, \cdots, t_k; X_1, \ldots, X_m$ include all the variables of φ.

Let $\mathcal{A} = \langle \mathbf{A}, < \rangle$ be a structure. We shall use $\mathcal{A}_{\geq a}$ to denote the substructure of $\mathcal{A}$ over the set $\mathbf{A}_{\geq a} = \{b \in \mathbf{A} | b \geq a\}$.

Definition 1 (Future *MLO* **Formula)** *A formula $\varphi(x_0, X_1, X_2, ..., X_m)$ of MLO with one free first-order variable x_0 is a future formula if for every chain $\mathcal{A} = (A, <)$ and*

$a \in A$ and every m subsets $S_1, S_2, .., S_m$ of A, the following holds:

$$(\mathcal{A}, a, S_1, S_2, .., S_m) \models \varphi(x_0, X_1, X_2, ..., X_m)$$
$$\textit{if and only if}$$
$$(\mathcal{A}_{\geq a}, a, S_1', S_2', .., S_m') \models \varphi(x_0, X_1, X_2, ..., X_m)$$

where $S_i' = \{b \in S_i \mid b \geq a\}$ for $i = 1, 2, ..., m$.

Remark 2.1 *Observe that the notion of a future formula is a semantic notion, not a syntactic one. However, it is possible to give a syntactic condition characterizing the future formulas: with the formula $\phi(x_0, X_1, \dots, X_k)$, we associate "the future version" ϕ' of ϕ, obtained by replacing all first-order quantifiers "$\forall x$" or "$\exists x$" in ϕ with relativized versions "$(\forall x)_{\geq x_0}$" and "$(\exists x)_{\geq x_0}$" which are shorthand for "$\forall x\ x \geq x_0 \rightarrow \dots$" and "$\exists x\ x \geq x_0 \wedge \dots$".*

Then for any ϕ, the relativized ϕ' is a future formula. Moreover,

$$\mathcal{A}, s, S_1, \dots, S_k \models \phi \ \ \textit{iff} \ \ \mathcal{A}_{\geq s}, s, S_1', \dots, S_k' \models \phi'$$

where, for $i = 1, \dots, k$, S_i' is the restriction of S_i to $\mid \mathcal{A}_{\geq s} \mid$. Hence, ϕ is a future formula iff ϕ and ϕ' are equivalent i.e., iff $\phi \leftrightarrow \phi'$ is valid. This implies that being a future formula is decidable since the validity of MLO formulae is decidable [Rab69].

2.2 Temporal Logics and Modalities

In this section we review basic definitions and results about temporal logics and modalities. We also introduce $\{TL\}_{k=1}^{\infty}$, an infinite sequence of temporal logics; these logics provide natural yardsticks by which to measure the expressive power of temporal logics.

The syntax of ***Temporal Logic (TL)*** has in its vocabulary a set of variables (sometimes called propositional names) and a set B of ***modality names*** (sometimes called "temporal connectives" or "temporal operators") with prescribed arity $B = \{\#_1^{l_1}, \#_2^{l_2}, \dots\}$ (we usually omit the arity notation). The set of modality names B might be infinite. A temporal logic based on a set of modalities B is denoted $TL(B)$; B is called the ***basis*** of $TL(B)$. Atomic temporal formulas are just variables and other formulas are obtained from the atoms using Boolean connectives and applying the modalities. The syntax of $TL(B)$ is given by the following grammar:

$$\varphi \ ::= \ \ \text{True} \ \ | \ \ \text{False} \ \ | \ P \ | \ \varphi_1 \wedge \varphi_2 \ | \ \neg\varphi_1 \ | \ \#_i^{l_i}(\varphi_1, \varphi_2, \dots, \varphi_{l_i}) \ ,$$

where P ranges over the variable names.

Temporal formulas are interpreted over linearly (or partially) ordered sets with monadic predicates. Every modality $\#^l$ is interpreted in every chain $\mathcal{A} = (A, <)$ as an operator $\#_T^l : [\mathbb{P}(A)]^l \rightarrow \mathbb{P}(A)$ which assigns "the set of points where $\#^l(Q_1, \dots, Q_k)$ holds" to the l−tuple $\langle Q_1, \dots, Q_l \rangle$ (Here $\mathbb{P}$ is the power set notation, and $\mathbb{P}(A)$ denotes the set of all subsets of A.) Formally, let $T = (A, <, P_1^T, P_2^T, \dots)$ be a chain with monadic predicates $P_1^T, P_2^T, \dots$ over A. The semantics of a formula $\varphi \in TL$ over T is defined inductively as follows. For atomic formulas $T, s \models P_i$ iff $s \in P_i^T$; the semantics of Boolean combinations is defined as usual, and the semantics of modalities is defined by: $T, s \models \#^l(\varphi_1, \varphi_2, \ldots, \varphi_l)$ iff $s \in \#^l_T(R_{\varphi_1}, R_{\varphi_2}, \ldots, R_{\varphi_l})$ where $R_{\varphi_i} = \{a | T, a \models \varphi_i\}$ for all $i, 1 \leq i \leq l$.

2.2.1 Truth Tables

In this paper, we consider only temporal modalities which are defined in (*first order*) *MLO*: we assume that for every l place modality $\#$ there is a formula (*truth table*) $\bar{\#}(x_0, X_1, X_2, \ldots, X_l)$ of *MLO* with one free first-order variable x_0 and l set variables, such that for every chain $T = \langle A, < \rangle$ and subsets $R_i \subseteq A$:

$$\#_T(R_1, R_2, \ldots, R_l) = \{s | (T, s, R_1, R_2, \ldots, R_l) \models \bar{\#}(x_0, X_1, X_2, \ldots, X_l)\}.$$

Let M be a temporal modality defined by a truth table $\phi_M \in MLO$. We say that M has quantifier-depth k if $\mathrm{qd}(\phi_M) = k$.

Example 1 (Some common modalities and their truth tables)

- *The one place modality* F *(“*eventually*”); its truth table is* $\phi_{\mathsf{F}}(x_0, X_1) \triangleq \exists y(y > x_0 \wedge y \in X_1)$.
- *The one place modality* G *(“*globally*”); its truth table is* $\phi_{\mathsf{G}}(x_0, X_1) \triangleq \forall y(y > x_0) \rightarrow (y \in X_1)$.
- *The two place modality* U *(“*until*”); its truth table is* $\phi_{\mathsf{U}}(x_0, X_1, X_2) \triangleq \exists y(y > x_0 \wedge y \in X_1 \wedge \forall z(x_0 < z < y) \rightarrow z \in X_2)$. *In the literature, sometimes a “non strict” definition of Until is given: the “non-strict until”* U_{ns} *modality has truth table* $\phi_{\mathsf{U}_{ns}}(x_0, X_1, X_2) \triangleq \exists y(y \geq x_0 \wedge y \in X_1 \wedge \forall z(x_0 \leq z < y) \rightarrow z \in X_2)$. *Clearly,* U_{ns} *can be defined using* U.
- *The two place modality* S *(“*since*”); its truth table is* $\phi_{\mathsf{S}}(x_0, X_1, X_2) \triangleq \exists y(y < x_0 \wedge y \in X_1 \wedge \forall z(x_0 > z > y) \rightarrow z \in X_2)$.

The choice of the particular modalities with their truth tables determines the different temporal logics.

Definition 2 (Future Modality)

- *A temporal modality is a first-order future modality if its truth table is a future formula of MLO.*
- *A temporal logic is a future temporal logic if all its modalities are future modalities.*

In the example above all modalities except S are first-order future modalities.

2.2.2 The logics TL_k

We define a hierarchy of future temporal logics. For each k the temporal logic TL_k has infinitely many modalities but each of them has a truth table that has quantifier depth at most k. Formally:

Definition 3 (Logics TL_k) *For every $k \geq 1$, let TL_k be the temporal logic with the set of basic modalities defined by the following set M_k of truth tables:*

$$M_k = \{\varphi : \mathrm{qd}(\varphi(x_0, X_1, \ldots, X_l)) \leq k \textit{ and } \varphi \textit{ is a first-order future formula}\}.$$

The basic modalities of TL_k have a truth table of quantifier depth k, but formulas of TL_k permit arbitrarily deep nesting of modalities. Properties of "an arbitrary quantifier depth" can be expressed even in TL_1. More precisely, for every k, there is a TL_1 formula ϕ_k which is equivalent to no *MLO* formula of quantifier-depth $\leq k$ [RM00]. To see this observe that the modality F of example 1 has quantifier depth one. We define φ_n to be the formula of $TL(\mathsf{F})$ claiming that in the future there will be at least n points where P holds. Formally $\varphi_1 = \mathsf{F}P$ and $\varphi_{n+1} = \mathsf{F}(P \wedge \varphi_n)$ then it is known that any formula equivalent to φ_n must have at least quantifier depth $log(n)$. Hence, the formulas in the temporal logic with this single modality F are not all equivalent to formulas in any fragment of *MLO* which has bounded quantifier depth.

The main technical result of the paper is that the logics TL_k form a proper hierarchy for linear continuous time. From this it will follow that no future temporal logic with finite base is expressively complete.

2.3 Kamp's Theorem

The different temporal logics are a convenient way to use fragments of *MLO*. To discuss the strength of a given temporal logic, we use the following standard definition.

A temporal logic formula $\varphi(X_1, \ldots X_k)$ is *equivalent* to an *MLO* formula $\psi(x_0, X_1, \ldots, X_k)$ (or to a temporal logic formula ψ) over a structure $\mathcal{A} = \langle A, < \rangle$ iff for every $a \in A$ and $S_1, \ldots S_k \subseteq A$:

$$\langle A, < \rangle, a \models \varphi(S_1, \ldots, S_k) \text{ if and only if } (A, <), a \models \psi(a, S_1, \ldots, S_k).$$

φ is equivalent to ψ (over a set C of structures) iff for every $\mathcal{A}$ (respectively, for every $\mathcal{A} \in C$), φ is equivalent to ψ over $\mathcal{A}$.

Definition 4 *Let L be a fragment of MLO and let C be a set of structures. A temporal logic TL' is expressively complete for L over C if for every formula $\psi(x_0, X_1, ..., X_n) \in L$ with single free first order variable x_0, there is a formula $\varphi \in TL'$ which is equivalent to ψ over C, and for every formula $\psi \in TL'$, there is a formula $\psi(x_0, X_1, ..., X_n) \in L$ with single free first order variable x_0 which is equivalent to φ over C.*

A major result concerning TL and *MLO* is Kamp's theorem. In this paper we are interested in the expressive power of logics relative to the real line. There is a technical property which unifies in the general case the treatment of real and discrete time: Recall that a chain is Dedekind complete iff every non empty bounded subset has a lowest upper bound. The natural numbers are trivially Dedekind complete since every bounded set is finite and has a maximal element. The real line is also Dedekind complete. Kamp's result applies to both these models:

Theorem 2.2 (Kamp [Kam68, GHR94]) *The temporal logic with two modalities* $\mathsf{U}(q_1, q_2)$ *("q_2 until q_1") and* $\mathsf{S}(q_1, q_2)$ *("q_2 since q_1") is expressively complete for MLO over the class of Dedekind complete chains.*

If we restrict our attention to future logic we obtain a dichotomy for which only one side of the coin was observed before:

Theorem 2.3 ([GPSS80]) *The temporal logic with the single modality* $\mathsf{U}(q_1, q_2)$ *is expressively complete for the future fragment of MLO over natural numbers.*

Thus in the discrete case $TL(\mathsf{U})$ is as strong as first order future *MLO*. We will see that for continuous time the situation is radically different.

3 No Finite Base over Ordinals

The ordinal numbers may not be to the taste of every computer scientist, but we need them for the proof that finite base future temporal logics are never expressively complete over real time. Not much set theory is needed for our purpose; suffices it to say that every ordinal is a chain. We use the following ordinals which we define directly:

- The ordinal ω is the set of natural numbers with its natural order.
- The ordinal ω^2 is an ω-sequence of blocks, each isomorphic to ω. We also declare each point to be bigger than every point in a previous block.
- More generally, ω^{n+1} is an ω-sequence of blocks, each isomorphic to ω^n. Each point is declared larger than all points in previous blocks.

An alternative definition of the ordered set ω^n is the set of n-tuples of natural numbers ordered lexicographically. The element which corresponds to a tuple $\langle m_{n-1}, m_{n-2}, \ldots, m_0\rangle$ is denoted by $\omega^{n-1}m_{n-1} + \omega^{n-2}m_{n-2} + \cdots + m_0$.

Easy induction proves the following useful feature of these ordinals:

Lemma 3.1 *Every suffix of the ordinal* ω^k *is isomorphic to* ω^k.

It follows immediately that:

Lemma 3.2 (Future Indiscernibility) ω^k *is homogeneous for future formulas:*

- *Every future formula in the predicate logic of order, with one free first order variable (and no predicate variables) hold at any point of* ω^k *iff it holds at every point in* ω^k.

- *In particular if $\varphi(t)$ is a future formula with no predicate variables then*

$$\omega^k, 0 \models \varphi(t) \quad \textit{iff} \quad \omega^k \models \forall t \varphi(t)$$

- *Every future TL formula without predicate variables holds at one point of ω^k iff it holds at every point of ω^k.*

Note that a TL formula without free predicate variables is built out of the constant predicates $True$ and $False$, the modalities and the Boolean operators.

We introduce the notation $\equiv_n$ to say that two models cannot be distinguished by a first order sentence of quantifier depth n. More precisely, let M and M' be two structures in the signature of MLO. We write $M \equiv_n M'$ if and only if for any sentence φ with $\mathrm{qd}(\varphi) \leq n$ we have $M \models \varphi$ iff $M' \models \varphi$.

Theorem 3.3 (Ehrenfeucht Theorem [Ehr61]) *$\omega^k \equiv_n \omega^m$ for $k, m \geq n$.*

Lemma 3.4 (Main Lemma) *Let φ be a TL_n formula with no predicate variables. if $k, m > n+1$ then*

$$\omega^k, 0 \models \varphi \textit{ if and only if } \omega^m, 0 \models \varphi$$

Proof The proof of this lemma combines the Ehrenfeucht's theorem for general formulas with the facts that were established for future formulas. The proof proceeds by structural induction.

For the atomic formulas *True* and *False* the claim is obvious.

For Boolean combinations the result follows immediately from the induction assumption.

It remains to deal with the case where $\varphi = C(\varphi_1, \cdots, \varphi_l)$ where C is an l place modality with future truth table of quantifier depth n. We assume without loss of generality that $\omega^k, 0 \models \varphi$.

By induction assumption for $i = 1, \cdots, l$:

$$\omega^k, 0 \models \varphi_i \quad \text{iff} \quad \omega^m, 0 \models \varphi_i$$

By lemma 3.2 and by induction assumption for each $i = 1, \cdots, l$ the formula φ_i either holds for every t in ω^k and for every t in ω^m or it holds for none of these points. Accordingly we shall denote by T_{φ_i} the formula *True* if $\omega^k, 0 \models \varphi_i$ and the formula *False* otherwise. Since $\omega^k, 0 \models \varphi$ it follows that $\omega^k, 0 \models \Psi(t, T_{\varphi_1}, \cdots, T_{\varphi_l})$, where $\Psi(t, X_1, \cdots, X_l)$ is the truth table of $C(X_1, \cdots, X_l)$. Again by lemma 3.2 $\omega^k \models (\forall t)\Psi(t, T_{\varphi_1}, \cdots, T_{\varphi_l})$. By Ehrenfeucht theorem $\omega^m \models \forall t \Psi(t, T_{\varphi_1}, \cdots, T_{\varphi_l})$, so that $\omega^m, 0 \models \Psi(t, T_{\varphi_1}, \cdots, T_{\varphi_l})$. It now follows that $\omega^m, 0 \models \varphi$. $\square$

Next we note that:

Lemma 3.5 *For every n there is a future formula $\phi_n(x_0)$ such that ω^n does not satisfy $\phi_n(0)$, but $\omega^k \models \phi_n(0)$ for every $k > n$.*

Proof
For every formula $\psi(t_0)$ in the language of order we denote by $LIM_\psi(t_0)$ the formula:

$$LIM_\psi(t_0) \quad = \quad (\forall t_1 < t_0)\exists t((t_1 < t < t_0) \wedge \psi(t))$$

Let now ψ_0 be the formula $t_0 = t_0$ and for every n, $\psi_{n+1} = LIM_{\psi_n}$. It is a main feature of the ordinals chosen that for $n > 0$ no element in ω^n satisfies ψ_n while unboundedly many elements in ω^{n+1} do satisfy it. More precisely, in ω^n only the (limit points) elements of the form $\omega^{n-1}m_{n-1} + \omega^{n-2}m_{n-2} + \cdots + \omega m_1 + 0$ satisfy ψ_1; only the points of the form $\omega^{n-1}m_{n-1} + \omega^{n-2}m_{n-2} + \cdots + \omega^2 m_2 + \omega 0 + 0$ satisfy ψ_2 and only the points of the form $\omega^{n-1}m_{n-1} + \omega^{n-2}m_{n-2} + \cdots + \omega^l m_l + \omega^{l-1}0 + \cdots + \omega 0 + 0$ satisfy ψ_l.

Note that ψ_n is not a future formula but if we define ϕ_n to be $(\exists t > t_0)\psi_{n+1}(t)$ then ϕ_n is a future formula which holds in ω^k at 0 iff $n < k$. $\square$

Theorem 3.6 *There is no finite base temporal logic which is expressively equivalent to the future fragment of MLO over the class of ordinals.*

Proof Let $TL(C_1, \cdots, C_s)$ be a temporal logic all whose formulas are equivalent to future predicate formulas. We shall show that there are formulas of future predicate logic which are not equivalent to any formula of $TL(C_1, \cdots, C_s)$. For $i = 1, \cdots, s$ let C_i be a j_i place modality. The formula $C_i(X_1, \cdots, X_{j_i})$ is equivalent to a future predicate formula $\varphi_i(t, X_1, \cdots, X_{j_i})$ which in turn may serve as a truth table for the modality C_i. If we choose n_0 to be large enough so that $\varphi_1, \cdots \varphi_s$ have quantifier depth less than n_0 then by lemma 3.4 no formula of $TL(C_1, \cdots, C_s)$ can tell apart ω^k and ω^{n_0} for $n_0 < k$. On the other hand by lemma 3.5 there is a future predicate formula which does. Hence $TL(C_1, \cdots, C_s)$ is not expressively complete. $\square$

4 No Finite Base over the Reals

In this section we prove

Theorem 4.1 *There is no temporal logic with a finite set of modalities which is expressively equivalent to the future fragment of MLO over the reals.*

We intend to use the results from the previous section and to that end we construct a systematic embedding of the ordinals under discussion into the real line. For a fixed natural number k we denote by R_k the following set:

$$R_k \quad = \quad \omega^k \ \times \ [0, 1)$$

where [0,1) is a half open interval. We order R_k lexicographically:

$$\langle \alpha, a \rangle \ < \ \langle \alpha_1, a_1 \rangle \quad iff \quad \alpha < \alpha_1 \ \ \textit{or} \ (\alpha = \alpha_1 \ \textit{and} \ a < a_1)$$

Relatively simple arguments from set theory show that:

1. There is an order isomorphism σ_k between R_k and R^+, the non negative real line.
2. We denote by P_k the image under σ_k of the set $\{ \langle \alpha, 0 \rangle \mid \alpha \in \omega^k \}$. Then $0 \in P_k$, P_k is unbounded in R^+ and P_k is order isomorphic to ω^k.
3. Every point $t \in P_k$ has an immediate successor in P_k, i.e, a point t' which is in P_k and such that no point between t and t' is in P_k. Moreover the open interval (t, t') is order isomorphic to the unit interval $(0, 1)$
4. Any bounded increasing sequence of elements in P_k converges to an element of P_k.

Note that this construction amounts to a uniform order preserving embedding of each ω^k as an unbounded subset of the real line.

Similarly to lemma 3.2 this construction yields model which is very homogeneous with respect to future formulas:

Lemma 4.2 *Let $M_k = \langle R^+, <, P_k \rangle$ be a continuous time model with an embedding of the set ω^k onto the set P_k as described above.*

- *Any suffix of M_k that has a first point which is in P_k is isomorphic to M_k.*
- *Any two suffixes that have first points which are not in P_k are isomorphic to each other.*
- *Let $\varphi(t, X)$ be a future formula of the first order monadic language of order, such that X is its only predicate variable. Then under the assignment of P_k to X in M_k the formula $\varphi(t, X)$ is equivalent to either $True$ or $False$ or $X(t)$ or $\neg X(t)$.*

Proof The first two facts are easy set theory and the third follows directly from the first two. □

To generalize the method of the previous section we need the generalization of Ehrenfeucht's theorem 3.3.

Lemma 4.3 (Generalized Ehrenfeucht theorem) $\langle R^+, <, P_k \rangle \equiv_n \langle R^+, <, P_n \rangle$ *for* $n < k$.

Proof The proof adapts Ehrenfeucht's original proof for ordinals [Ehr61] to the new context and it will be given as an appendix. □

Our main theorem 4.1 follows immediately from Lemma 4.5 and Lemma 4.4 below, exactly as theorem 3.6 follows from lemma 3.5 and 3.4.

Lemma 4.4 *For every TL_n formula $\varphi(X)$ if $k, m > n$ then*

$$\mathbf{R}, P_k, 0 \models \varphi \textit{ if and only if } \mathbf{R}, P_m, 0 \models \varphi$$

Proof The proof is similar to the proof of lemma 3.4. The proof is again reduced to the case of a formula $C(\varphi_1, \cdots, \varphi_l)$, where C is a l-place modality with truth table $\Psi(t, X_1, \cdots, X_l)$. By the induction assumption and lemma 4.2 each φ_i is equivalent to *True*, *False*, X or $\neg X$ and when these are substituted in the truth table they yield the same value in $\langle R, P_k\rangle$ and in $\langle R, P_n\rangle$ by lemma 4.3.

□

Lemma 4.5 *For every n there is a future formula $\psi_n(x_0, X)$ such that the chain of reals $\mathbf{R}$ does not satisfy $\psi_n(0, P_n)$, but $\mathbf{R} \models \psi_n(0, P_k)$ for every $k > n$.*

Proof We relativize the formula ϕ_n of lemma 3.5 to the variable X; specifically, starting with innermost quantified formulas we replace every formula $\exists t\psi$ by the formula $\exists t(X(t) \wedge \psi)$, and every formula $\forall t\psi$ by the formula $\forall t(X(t) \rightarrow \psi)$. Working our way outwards we relativize in this way all the quantifiers to the predicate X. This technique is due to Tarski and can be found for example in [Rab69]. It results in a formula ψ_n which has the property [Rab69] that

$$\langle \mathbf{R}, P\rangle \models \psi_n \;\text{ iff }\; \langle P, <\rangle \models \phi_n$$

Recall that $\langle P_k, <\rangle$ is isomorphic to $\langle \omega^k, <\rangle$. Lemma 3.5, the definition of ψ_n and the above equation implies the conclusion of Lemma 4.5. □

5 Concluding Remarks

1. We proved that over the real time line there is no temporal logic with only finitely many modalities which describes only future formulas and is expressively complete for future first order monadic logic of order. A closer look at the proof of Theorem 4.1 shows that it applies almost word for word to the rational line. The only changes are (1) replace reals by the rationals and (2) in the definition of R_k replace the interval $[0, 1)$ of reals by the interval $[0, 1)$ of rationals. Hence we obtain

   Theorem 5.1 *There is no temporal logic with a finite set of modalities which is expressively equivalent to the future fragment of MLO over the rationals.*

   Recall that a subset B of a chain is dense in itself it it contains at least two points and between any two distinct points of B there is a point of B. A more careful version of the proof of Theorem 4.1 would yield the following:

   Theorem 5.2 *Let $\langle A, <\rangle$ be a totally ordered set which contains (not necessarily as a subinterval) a subset B that is dense in itself. Then there is no finite expressively complete future temporal logic over the model.*

2. In computer science we often concentrate on behaviors that have finite variability rather than on all possible behaviors in time; A predicate X has finite variability if there is an unbounded sequence $\langle t_n \rangle_{n \in N}$ such that $t_0 = 0$ and the truth value of X does not change on the open intervals (t_i, t_{i+1}). If we restrict our interest to the subclass of models that have only predicates with finite variability then it can be shown that the logic $TL(\mathbf{U})$ with only the "Until" modality is expressively complete for future formulas. We do not prove it here; it can be done directly along the line of Kamp's proof, with some sensitive adaptation, or we can combine Kamp's result for the discrete case with decomposition arguments. This is certainly not something that can be taken as self evident.

3. Theorem 10.3.20 in [GHR94] identifies a fragment of $TL(\mathbf{U}, \mathbf{S})$ which comprises future formulas and is expressively complete for future monadic logic of order. The description of this "future temporal logic" is very simple and it looks like the best positive result that is consistent with our negative result. The authors define the one place modality $\mathbf{K}^-$, saying that there are on the left arbitrary close points where the property holds. To be specific, $\mathbf{K}^-$ is a one place modality; its truth table is
$$\phi(x_0, X_1) \triangleq \forall y\, ((y < x_0) \rightarrow \exists z (y < z < x_0 \wedge z \in X_1))\,.$$
Note that $\mathbf{K}^-$ is not a future modality and $\mathbf{K}^-(X)$ is equivalent to $\neg(\neg X \,\mathbf{S}\, \mathit{True})$. They look at the fragment of $TL(\mathbf{U}, \mathbf{K}^-)$ which includes the formulas in which the modality $\mathbf{K}^-$ occurs only inside the scope of an occurrence of the modality $\mathbf{U}$. This fragment is then shown to be expressively complete for future formulas. Note however that this logic is not a temporal logic according to the definition and in particular it does not have the property that any subformula of a formula is in itself a formula in the logic.

4. It is a natural conjecture that the logic $TL(\mathbf{U}, \mathbf{K}^-)$ is expressively complete for a somewhat more general class than that of future formulas:

   Definition: A formula $\varphi(t, X_0, \cdots, X_n)$ of the logic of order is called an *almost future formula* if whenever φ holds at a point c of a model $\langle \mathbf{A}, < \rangle$ then it also holds at c in every submodel $\langle \mathbf{A}_{\geq \mathbf{b}}, < \rangle$, provided $b < c$.

   Conjecture: The temporal logic $TL(\mathbf{U}, \mathbf{K}^-)$ is expressively complete for almost future formulas of the monadic logic of order.

A Appendix: proof of Lemma 4.3

The proof of the lemma, like Ehrenfeucht's proof for ordinals, uses a characterization of logical equivalence in terms of "Ehrenfeucht Fraisse games". For simplicity of notations we define these games only in the context of the models addressed to in lemma 4.3. A k-step game on chains $M = \langle A, <, C_1, \ldots C_l \rangle$ and $M' = \langle A', <, C'_1, \ldots, C'_l \rangle$ is played between two players *Spoiler* and *Duplicator* and it involves k moves of each player; *Spoiler* plays first. He chooses one of the structures and an element in that structure. *Duplicator* responds by choosing an element in the other model, which he believes 'matches' the element chosen by *Spoiler*. This pair of moves is repeated for k rounds. *Spoiler* may or may not alter the model he chooses the next element from and *Duplicator* always reacts by choosing from the other model.

After k pairs of choices there are two sequences of elements: $a_1, \cdots, a_k$ in M and $b_1, \cdots, b_k$ in M'. *Duplicator* is deemed the winner if the two sequences include similar elements, i.e., if for all $i, j \leq k$

1. $a_i < a_j$ iff $b_i < b_j$
2. $C_m(a_i)$ iff $C'_m(b_i)$ for $m = 1, \ldots, l$.

Otherwise *Spoiler* wins.

A ***winning strategy***, for either player, is a strategy to follow, which will guarantee him or her a win, no matter what moves the other player chooses to play.

The following theorem by Ehrenfeucht characterizes the equivalence $\equiv_k$ in terms of these games [Ehr61]:

Theorem A.1 *$M \equiv_k M'$ iff Duplicator has a winning strategy for the k-step game.*

Ehrenfeucht proved that *Duplicator* has a winning strategy for the k-step game on $\langle \omega^k, < \rangle$ and $\langle \omega^n, < \rangle$, when $k < n$. We shall use his strategy to construct a winning strategy for Duplicator in the k-step game on M_k and M_n. We recall the construction of the models under discussion: $M_k = \langle R^+, <, P_k \rangle$ is an isomorphic image of $M'_k = \langle \omega^k \times [0,1), <, Q_k \rangle$ where $Q_k = \omega^k \times \{0\}$. We shall define the winning strategy for duplicator as played for the models M'_k: Use Ehrenfeucht's strategy on the first coordinate and pick the exact element that your opponent chooses in the second coordinate. Specifically, assume that sequences of points $a_1, \cdots, a_i$ and $b_1, \cdots, b_i$ were already played with $a_j = \langle \alpha_j, r_j \rangle$ and $b_j = \langle \beta_j, r_j \rangle$ for $j = 1, \cdots, i$. It will be evident from the description of our strategy that indeed the element in the second coordinate is the same in a_j and b_j, and that indeed $\alpha_1, \cdots, \alpha_i$ and $\beta_1, \cdots, \beta_i$ represent a game confirming with Ehrenfeucht strategy for ordinals. On a move by Spoiler (without loss of generality) $a_{i+1} = \langle \alpha_{i+1}, r_{i+1} \rangle$ Duplicator makes the move $\langle \beta_{i+1}, r_{i+1} \rangle$ where β_{i+1} is the move that Ehrenfeucht strategy suggests in the ordinal game $\alpha_1, \cdots, \alpha_{i+1}$ and $\beta_1, \cdots, \beta_i$.

It is now easy to see that if the game results in the sequences $\langle \alpha_1, r_1 \rangle, \cdots, \langle \alpha_k, r_k \rangle$ and $\langle \beta_1, r_1 \rangle, \cdots, \langle \beta_k, r_k \rangle$ then these two sequences satisfy the requirements for winning in the definition above: Since $\alpha_1, \cdots, \alpha_n$, $\beta_1, \cdots, \beta_n$ is won by Duplicator in the ordinal game they are order isomorphic. By the definition of the order in M_k and M_n the sequence in our game with similar numbers in the second coordinate of corresponding points are also order isomorphic. Also because the second number in corresponding pairs is the same it is either 0 in both or it is 0 in none. By the definition of Q_k and Q_n we have $\langle \alpha_i, r_i \rangle \in Q_k$ iff $\langle \beta_i, r_i \rangle \in Q_n$. This completes the proof that Duplicator has a winning strategy for the k-game and hence that $M_k \equiv_k M_n$ if $k \leq n$

Acknowledgments

We would like to thank Danièle Beauquier and Anatol Slissenko for the fruitful discussions and their hospitality during preparation of this paper.

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