Decidable Expansions of Labelled Linear Orderings

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Dedicated to Yuri Gurevich on the occasion of his seventieth birthday

Abstract. Let $M = (A, <, \overline{P})$ where $(A, <)$ is a linear ordering and $\overline{P}$ denotes a finite sequence of monadic predicates on A. We show that if A contains an interval of order type ω or $-\omega$, and the monadic second-order theory of M is decidable, then there exists a non-trivial expansion M' of M by a monadic predicate such that the monadic second-order theory of M' is still decidable.

Key words: monadic second-order logic, decidability, definability, linear orderings

1 Introduction

In this paper we address definability and decidability issues for monadic second order (shortly: MSO) theories of labelled linear orderings. Elgot and Rabin ask in [9] whether there exist maximal decidable structures, i.e., structures M with a decidable first-order (shortly: FO) theory and such that the FO theory of any expansion of M by a non-definable predicate is undecidable. This question is still open. Let us mention some partial results:

- Soprunov proved in [28] that every structure in which a regular ordering is interpretable is not maximal. A partial ordering $(B, <)$ is said to be regular if for every $a \in B$ there exist distinct elements $b_1, b_2 \in B$ such that $b_1 < a$, $b_2 < a$, and no element $c \in B$ satisfies both $c < b_1$ and $c < b_2$. As a corollary he also proved that there is no maximal decidable countable structure if we replace FO by weak MSO logic.
- In [2], Bès and Cégielski consider a weakening of the Elgot-Rabin question, namely the question of whether all structures M whose FO theory is decidable can be expanded by some constant in such a way that the resulting structure still has a decidable theory. They answer this question negatively by proving that there exists a structure M with a decidable MSO theory and such that any expansion of M by a constant has an undecidable FO theory.

– The paper [1] gives a sufficient condition in terms of the Gaifman graph of M which ensures that M is not maximal. The condition is the following: for every natural number r and every finite set X of elements of the base set $|M|$ of M there exists an element $x \in |M|$ such that the Gaifman distance between x and every element of X is greater than r.

We investigate the Elgot-Rabin problem for the class of labelled linear orderings, i.e., infinite structures $M = (A; <, P_1, \ldots, P_n)$ where $<$ is a linear ordering over A and the P_i's denote unary predicates. This class is interesting with respect to the above results, since on one hand no regular ordering seems to be FO interpretable in such structures, and on the other hand their associated Gaifman distance is trivial, thus they do not satisfy the criterion given in [1].

In this paper we focus on MSO logic rather than FO. The main result of the paper is that for every labelled linear ordering M such that $(A, <)$ contains an interval of order type ω or $-\omega$ and the MSO theory of M is decidable, then there exists an expansion M' of M by a monadic predicate which is not MSO-definable in M, and such that the MSO theory of M' is still decidable. Hence, M is not maximal. The result holds in particular when $(A. <)$ is order-isomorphic to the order of the naturals $\omega = (\mathbb{N}, <)$, or to the order $\zeta = (\mathbb{Z}, <)$ of the integers, or to any infinite ordinal, or more generally any infinite scattered ordering (recall that an ordering is scattered if it does not contain any dense sub-ordering).

The structure of the proof is the following: we first show that the result holds for ω and ζ. For the general case, starting from M, we use some definable equivalence relation on A to cut A into intervals whose order type is either finite, or of the form $-\omega$, ω, or ζ. We then define the new predicate on each interval (using the constructions given for ω and ζ), from which we get the definition of M'. The reduction from $MSO(M')$ to $MSO(M)$ uses Shelah's composition theorem, which allows to reduce the MSO theory of an ordered sum of structures to the MSO theories of the summands.

The main reason to consider MSO logic rather than FO is that it actually simplifies the task. Nevertheless we discuss some partial results and perspectives for FO logic in the conclusion of the paper.

Let us recall some important decidability results for MSO theories of linear orderings (the case of labelled linear orderings will be discussed later for ω and ζ). In his seminal paper [4], Büchi proved that languages of $\omega-$words recognizable by automata coincide with languages definable in the MSO theory of ω, from which he deduced decidability of the theory. The result (and the automata method) was then extended to the MSO theory of any countable ordinal [5], to ω_1, and to any ordinal less than ω_2 [6]. Gurevich, Magidor and Shelah prove [13] that decidability of MSO theory of ω_2 is independent of ZFC. Let us mention results for linear orderings beyond ordinals. Using automata, Rabin [19] proved decidability of the MSO theory of the binary tree, from which he deduces decidability of the MSO theory of $\mathbb{Q}$, which in turn implies decidability of the MSO theory of the class of countable linear orderings. Shelah [26] improved model-theoretical techniques that allow him to reprove almost all known decidability results about MSO theories, as well as new decidability results for the case of linear orderings, and in particular dense orderings. He proved in particular that the MSO theory of $\mathbb{R}$ is undecidable. The frontier between decidable and undecidable cases was specified in later papers by Gurevich and Shelah [11, 14, 15]; we refer the reader to the survey [12].

Our result is also clearly related to the problem of building larger and larger classes of structures with a decidable MSO theory. For an overview of recent results in this area see [3, 32].

2 Definitions, Notations and Useful Results

2.1 Labelled Linear Orderings

We first recall useful definitions and results about linear orderings. A good reference on the subject is Rosenstein's book [23].

A *linear ordering* J is a total ordering. We denote by ω (respectively ζ) the order type of $\mathbb{N}$ (respectively $\mathbb{Z}$). Given a linear ordering J, we denote by $-J$ the *backwards* linear ordering obtained by reversing the ordering relation.

Given two elements j, k of a linear ordering J, we denote by $[j; k]$ the interval $[\min(j, k), \max(j, k)]$. An ordering is *dense* if it contains no pair of consecutive elements. An ordering is *scattered* if it contains no dense sub-ordering.

In this paper we consider *labelled* linear orderings, i.e., linear orderings $(A, <)$ equipped with a function $f : A \to \Sigma$ where Σ is a finite nonempty set.

2.2 Logic

Let us briefly recall useful elements of monadic second-order logic, and settle some notations. For more details about MSO logic see e.g. [12, 31]. Monadic second-order logic is an extension of first-order logic that allows to quantify over elements as well as subsets of the domain of the structure. Given a signature L, one can define the set of (MSO) formulas over L as well-formed formulas that can use first-order variable symbols $x, y, \dots$ interpreted as elements of the domain of the structure, monadic second-order variable symbols $X, Y, \dots$ interpreted as subsets of the domain, symbols from L, and a new binary predicate $x \in X$ interpreted as "x belongs to X". A sentence is a formula without free variable. As usual, we often confuse logical symbols with their interpretation. Given a signature L and an $L-$structure M with domain D, we say that a relation $R \subseteq D^m \times (2^D)^n$ is (MSO) definable in M if a nd only if there exists a formula over L, say $\varphi(x_1, \dots, x_m, X_1, \dots, X_n)$, which is true in M if and only if $(x_1, \dots, x_m, X_1, \dots, X_n)$ is interpreted by an $(m+n)-$tuple of R. Given a structure M we denote by $MSO(M)$ (respectively $FO(M)$) the monadic second-order (respectively first-order) theory of M. We say that M is maximal if $MSO(M)$ is decidable and $MSO(M')$ is undecidable for every expansion M' of M by a predicate which is not definable in M.

We can identify labelled linear orderings with structures of the form $M = (A, <, P_1, \dots, P_n)$ where $<$ is a binary relation interpreted as a linear ordering over A, and the P_i's denote unary predicates. We use the notation $\overline{P}$ as a shortcut for the n-tuple $(P_1, \dots P_n)$. The structure M can be seen as a word indexed by A and over the alphabet $\Sigma_n = \{0,1\}^n$; this word will be denoted by $w(M)$. For every interval I of A we denote by M_I the sub-structure of M with domain I.

2.3 Composition Theorems

In this paper we rely heavily on composition methods, which allow to compute the theory of a sum of structures from the ones of its summands. For an overview of the subject see [3, 12, 16, 30]. In this section we recall useful definitions and results. For the whole section we consider signatures of the form $L = \{<, P_1, \dots, P_n\}$ where the P_i's denote unary predicate names, and deal only with $L-$structures where $<$ is interpreted as a linear ordering – that is, with labelled linear orderings. Given a formula φ over L, the quantifier depth of φ is denoted by $qd(\varphi)$. The $k-$type of an $L-$structure M, which is denoted by $T^k(M)$, is the set of sentences φ such that $M \models \varphi$ and $qd(\varphi) \leq k$. Given two structures M and M', the relation $T^k(M) = T^k(M')$ is an equivalence relation with finitely many classes. Let us list some fundamental and well-known properties of k-types. The proofs of these facts can be found in several sources, see e.g. [26, 31].

Proposition 1. *1. For every k there are only finitely k-types over a finite signature L*

2. *For each k-type t there is a sentence φ_t (called "characteristic sentence") which defines t, i.e., such that $M \models \varphi_t$ iff $T^k(M) = t$. For every k, a finite list of characteristic sentences for all the possible k-types can be computed. (We take the characteristic sentences as the canonical representations of k-types. Thus, for example, transforming a type into another type means to transform sentences.)*
3. *Each sentence φ is equivalent to a (finite) disjunction of characteristic sentences; moreover, this disjunction can be computed from φ.*

As a simple consequence, note that the MSO theory of a structure M is decidable if and only if the function $k \mapsto T^k(M)$ is recursive.

The sum of structures corresponds to concatenation; let us recall a general definition.

Definition 2. *Consider an index structure $Ind = (I, <^I)$ where $<^I$ is a linear ordering. Consider a signature $L = \{<, P_1, \dots, P_n\}$, where P_i are unary predicate names, and a family $(M_i)_{i \in I}$ of L-structures $M_i = (A_i; <^i, {P_1}^i, \dots, P_n^i)$ with disjoint domains and such that the interpretation $<^i$ of $<$ in each M_i is a linear ordering. We define the* ordered sum *of the family $(M_i)_{i \in I}$ as the L-structure $M = (A; <^M, {P_1}^M, \dots, P_n^M)$ where*

- *A equals the union of the A_i's*
- *$x <^M y$ holds if and only if ($x \in A_i$ and $y \in A_j$ for some $i <^I j$), or ($x, y \in A_i$ and $x <^i y$)*

– *for every $x \in A$ and every $k \in \{1, \ldots, n\}$, $P_k^M(x)$ holds if and only if $M_j \models P_k^j(x)$ where j is such that $x \in A_j$.*

If the domains of the M_i are not disjoint, replace them with isomorphic chains that have disjoint domains, and proceed as before.

We shall use the notation $M = \sum_{i \in I} M_i$ for the ordered sum of the family $(M_i)_{i \in I}$. If $I = \{1, 2\}$ has two element, we denote $\sum_{i \in I} M_i$ by $M_1 + M_2$.

We need the following composition theorem on ordered sums:

Theorem 3.

(a) The k-types of labelled linear orderings M_0, M_1 determine the k-type of the ordered sum $M_0 + M_1$, which moreover can be computed from the k-types of M_0 and M_1.

(b) If the labelled linear orderings $M_0, M_1, \ldots$ all have the same k-type, then this k-type determines the k-type of $\Sigma_{i \in \mathbb{N}} M_i$, which moreover can be computed from the k-type of M_0.

Part (a) of the theorem justifies the notation $s + t$ for the k-type of a linear ordering which is the sum of two linear orderings of k-types s and t, respectively. Similarly, we write $t \times \omega$ for the k-type of a sum $\Sigma_{i \in \mathbb{N}} M_i$ where all M_i are of k-type t.

3 The Case of $\mathbb{N}$

In this section we prove that there is no maximal structure of the form $(\mathbb{N}, <, \overline{P})$ with respect to MSO logic. The proof is based upon results from [20] . Let us first briefly review results related to the decidability of the MSO theory of expansions of $(\mathbb{N}, <)$. Büchi [4] proved decidability of $MSO(\mathbb{N}, <)$ using automata. On the other hand it is known that $MSO(\mathbb{N}, +)$, and even $MSO(\mathbb{N}, <, x \mapsto 2x)$, are undecidable [22]. Elgot and Rabin study in [9] the MSO theory of structures of the form $(\mathbb{N}, <, P)$, where P is some unary predicate. They give a sufficient condition on P which ensures decidability of the MSO theory of $(\mathbb{N}, <, P)$. In particular the condition holds when P denotes the set of factorials, or the set of powers of any fixed integer. The frontier between decidability and undecidability of related theories was explored in numerous later papers [7, 10, 25, 24, 21, 20, 27, 29]. Let us also mention that [25] proves the existence of unary predicates P and Q such that both $MSO(\mathbb{N}, <, P)$ and $MSO(\mathbb{N}, <, Q)$ are decidable while $MSO(\mathbb{N}, <, P, Q)$ is undecidable.

Most decidability proofs for $MSO(\mathbb{N}, <, P)$ are related somehow to the possibility of cutting $\mathbb{N}$ into segments whose $k-$type is ultimately constant, from which one can compute the $k-$type of the whole structure (using Theorem 3). This connection was specified in [20] (see also [21]) using the notion of homogeneous sets.

Definition 4 (k-homogeneous set). *Let $k \geq 0$. A set $H = \{h_0 < h_1 < \ldots\} \subseteq \mathbb{N}$ is called* k-homogeneous *for $M = (\mathbb{N}, <, \overline{P})$, if all sub-structures $M_{[h_i, h_j)}$ for $i < j$ (and hence all sub-structures $M_{[h_i, h_{i+1})}$ for $i \geq 0$) have the same k-type.*

This notion can be refined as follows.

Definition 5 (uniformly homogeneous set). *A set* $H = \{h_0 < h_1 < \ldots\} \subseteq \mathbb{N}$ *is called* uniformly homogeneous for $M = (\mathbb{N}, <, \overline{P})$ *if for each* k *the set* $H_k = \{h_k < h_{k+1} < \ldots\}$ *is* k*-homogeneous.*

The following result [20] settles a tight connection between $MSO(\mathbb{N}, <, \overline{P})$ and uniformly homogeneous sets.

Theorem 6. *For every* $M = (\mathbb{N}, <, \overline{P})$*, the MSO theory of* M *is decidable if and only if (the sets* $\overline{P}$ *are recursive and there exists a recursive uniformly homogeneous set for* M*).*

One can use this theorem to show that no structure $M = (\mathbb{N}, <, \overline{P})$ is maximal. Let us give the main ideas. Starting from M such that $MSO(M)$ is decidable, Theorem 6 implies the existence of a recursive uniformly homogeneous set $H = \{h_0 < h_1 < \ldots\}$ for M.

Let M' be an expansion of M by a monadic predicate P_{n+1} defined as $P_{n+1} = \{h_{n!} \mid n \in \mathbb{N}\}$.

By definition of H, the structures $M_{[h_{k!}, h_{(k+j)!}[}$ have the same $k-$type for all $j, k \geq 0$. If we combine this with the fact that successive elements of P_{n+1} are far away from each other, we can prove that P_{n+1} is not definable in M. For all $i, k \geq 0$ let us define the interval $I(i,k) = [h_{(k+i)!}, h_{(k+i+1)!}[$. In order to prove that $MSO(M')$ is decidable, we exploit the fact that all structures $M_{I(i,k)}$ have the same $k-$type for all $i, k \geq 0$, and that only the first element of each interval $I(i,k)$ belongs to P_{n+1}. This allows to compute easily the $k-$type of structures $M'_{I(i,k)}$ from the one of $M_{I(i,k)}$, and then the $k-$type of the whole structure M'. This provides a reduction from $MSO(M')$ to $MSO(M)$.

The above construction, which we described for a fixed structure M, can actually be defined uniformly in M. This leads to the following result.

Proposition 7. *There exists a function* E *and two recursive function* g_1, g_2 *such that* E *maps every structure* $M = (\mathbb{N}, <, \overline{P})$ *to an expansion* M' *of* M *by a predicate* P_{n+1} *such that*

1. P_{n+1} *is not definable in* M*;*
2. g_1 *computes* $T^k(M')$ *from* k *and* $T^{g_2(k)}(M)$*.*

Hence $MSO(M')$ *is recursive in* $MSO(M)$*. In particular, if* $MSO(M)$ *is decidable, then* $MSO(M')$ *is decidable.*

Let us discuss item (2). In the proof of the general result (see Sect. 5), we start from a labelled linear ordering $M = (A, <, \overline{P})$ with a decidable MSO theory and try to expand it while keeping decidability. In some case the (decidable) expansion M' of M will be defined by applying the above construction to infinitely many intervals of A of order type ω. In order to get a reduction from $MSO(M')$ to $MSO(M)$, we need that the reduction algorithm for such intervals is uniform, which is what item (2) expresses.

4 The Case of $\mathbb{Z}$

Decidability of the MSO theory of structures $M = (\mathbb{Z}, <, \overline{P})$ was studied in particular by Compton [8], Semënov [25, 24], and Perrin and Schupp [18] (see also [17, chapter 9]). These works put in evidence the link between decidability of $MSO(M)$ and computability of occurrences and repetitions of finite factors in the word $w(M)$. Let us state some notations and definitions. A set X of finite words over a finite alphabet Σ is said to be regular if it is recognizable by some finite automaton. Given a $\mathbb{Z}-$word w and a finite word u, both over the alphabet Σ, we say that u occurs in w if $w = w_1 u w_2$ for some words w_1 and w_2. We say that w is *recurrent* if for every regular language X of finite words over Σ, either no element of X occurs in w, or in every prefix and every suffix of w there is an occurrence of some element of X. In particular in a recurrent word w, every finite word u either has no occurrence in w, or occurs infinitely often on both sides of w. We say that w is *rich* if every finite word occurs infinitely often on both sides of w. Given $M = (\mathbb{Z}, <, \overline{P})$, we say that M is *recurrent* if $w(M)$ is.

We have the following result.

Theorem 8. *([25, 18]) Given $M = (\mathbb{Z}, <, P_1, \ldots, P_n)$,*

1. *If M is not recurrent, then every $c \in \mathbb{Z}$ is definable in M.*
2. *If M is recurrent, then no element is definable in M, and $MSO(M)$ is computable relative to an oracle which, given any regular language X of finite words over $\Sigma_n = \{0,1\}^n$, tells whether some element of X occurs in $w(M)$.*

Let $c \in \mathbb{Z}$, and let M_1 be defined as $M = M_{]-\infty,c[}$ and M_2 be defined as $M_{[c,\infty[}$. Then $M = M_1 + M_2$.

Let M_1' be the expansion of M_1 by the empty predicate P_{n+1} and let M_2' be obtained by apply the construction of Proposition 7 to M_2. Let $M' = M_1' + M_2'$.

Note that the above construction of M' from M depends on c. We denote by E_c the function described above that maps every $M = (\mathbb{Z}, <, P_1, \ldots, P_n)$ to its expansion M' by P_{n+1}.

It is easy to show that P_{n+1} is not definable in M, hence M' is a non-trivial expansion of M.

We claim that if M is not recurrent, then MSO(M') is recursive in MSO(M). Indeed, in this case, by Theorem 8, c is definable in M. Hence, M_1 and M_2 can be interpreted in M, which yields that MSO(M_1) and MSO(M_2) are recursive in MSO(M). Therefore, MSO(M_1') and MSO(M_2') are recursive in MSO(M). Finally, applying Theorem 3(a) we obtain that MSO(M') is recursive in MSO(M).

Hence, we have the following.

Proposition 9 (Expansion of non-recurrent structures). *There are two recursive function g_1, g_2 such that if $M = (\mathbb{Z}, <, P_1, \ldots, P_n)$ is not recurrent, and $c \in \mathbb{Z}$ is definable in M by a formula of quantifier depth m, then E_c maps M to an expansion M' by a predicate P_{n+1} such that*

1. *P_{n+1} is not definable in M;*

2. g_1 computes $T^k(M')$ from k and $T^{g_2(k+m)}(M)$.

Hence $MSO(M')$ is recursive in $MSO(M)$. In particular, if $MSO(M)$ is decidable, then $MSO(M')$ is decidable.

Remark 10. Let us discuss uniformity issues related to Proposition 7 and Proposition 9. Proposition 7 implies that there is an algorithm which reduces $MSO(M')$ to $MSO(M)$. This reduction algorithm is independent of M; it only uses an oracle for $MSO(M)$. Proposition 9 implies a weaker property. Namely, it implies that for every non-recurrent M there is an algorithm which reduces $MSO(M')$ to $MSO(M)$. However, this reduction algorithm depends on M.

Consider a recurrent structure M and let $M' = E_c(M)$ for some $c \in \mathbb{Z}$. We claim that it is possible that $MSO(M')$ is not recursive in $MSO(M)$. Indeed, we can prove that there exists a recurrent structure M over $\mathbb{Z}$ such that $MSO(M)$ is decidable, and $MSO(M_{[c',\infty[})$ is undecidable for every $c' \in Z$. Now let c' be the minimal element of P_{n+1}. Observe that c' is definable in M' and therefore, $M_{[c',\infty[}$ can be interpreted in M'. Since, $MSO(M_{[c',\infty[})$ is undecidable, we derive that $MSO(M')$ is undecidable. Hence, E_c does not preserves decidability of recurrent structures, and we need a different construction for the recurrent case.

To describe our construction for the recurrent case let us introduce first some notations.

For every word w over the alphabet $\Sigma_{n+1} = \{0,1\}^{n+1}$ which is indexed by some linear ordering $(A, <)$ we denote by $\pi(w)$ the word w' indexed by A and over the alphabet $\Sigma_n = \{0,1\}^n$, which is obtained from w by projection over the n first components of each symbol in w. The definition and notation extend to $\pi(X)$ where X is any set of words over the alphabet Σ_{n+1}. Given $M = (\mathbb{Z}, <, \overline{P})$ where $\overline{P}$ is an $n-$tuple of sets, and any expansion M' of M by a predicate P_{n+1}, by definition $w(M)$ and $w(M')$ are words over Σ_n and Σ_{n+1}, respectively, and we have $\pi(w(M')) = w(M)$.

Lemma 11. *If $M = (\mathbb{Z}, <, \overline{P})$ is recurrent, then there is an expansion M' of M by a predicate P_{n+1} which has the following property:*

(*) *for every $u \in \Sigma_n^*$, if u occurs infinitely often on both sides of $w(M)$, then the same holds in $w(M')$ for every word $u' \in \Sigma_{n+1}^*$ such that $\pi(u') = u$.*

The proof of Lemma 11 is similar to the proof of Proposition 2.8 in [1], which roughly shows how to deal with the case when $w(M)$ is rich.

Now $w(M')$ has a finite factor in some regular language $X' \subseteq \Sigma_{n+1}^*$ iff $w(M)$ has a finite factor in $\pi(X') \subseteq \Sigma_n^*$. The set $\pi(X')$ is regular, and a sentence which defines $\pi(X')$ is computable from a sentence that defines X', thus we obtain, by Theorem 8(2), that if $MSO(M)$ is decidable then $MSO(M')$ is decidable.

One can show that if M' is any expansion of M which has property (*), then P_{n+1} is not definable in M. This implies that no recurrent structure is maximal.

From a more detailed analysis of the proof of Theorem 8(2) we can derive the following proposition.

Proposition 12 (Expansion of recurrent structures). *There are two recursive function g_1, g_2 such that if $M = (\mathbb{Z}, <, \overline{P})$ is recurrent and M' is an expansion of M which has property (*), then*

1. *P_{n+1} is not definable in M;*
2. *g_1 computes $T^k(M')$ from k and $T^{g_2(k)}(M)$.*

Hence $MSO(M')$ is recursive in $MSO(M)$. In particular, if $MSO(M)$ is decidable, then $MSO(M')$ is decidable.

Remark 13. Proposition 12 implies that there is an algorithm which reduces $MSO(M')$ to $MSO(M)$. This reduction algorithm (like the algorithm from Proposition 7) is independent of M; it only uses an oracle for $MSO(M)$.

Proposition 9, Lemma 11 and Proposition 12 imply the following corollary.

Corollary 14. *Let $M = (\mathbb{Z}, <, \overline{P})$. There exists an expansion M' of M by some unary predicate P_{n+1} such that P_{n+1} is not definable in M, and $MSO(M')$ is recursive in $MSO(M)$. In particular, if $MSO(M)$ is decidable, then $MSO(M')$ is decidable.*

5 Main Result

The next theorem is our main result.

Theorem 15. *Let $M = (A, <, P_1, \ldots, P_n)$ where $(A, <)$ contains an interval of type ω or $-\omega$. There exists an expansion M' of M by a relation P_{n+1} such that P_{n+1} is not definable in M, and $MSO(M')$ is recursive in $MSO(M)$. In particular, if $MSO(M)$ is decidable, then $MSO(M')$ is decidable.*

As an immediate consequence we obtain the following corollary.

Corollary 16. *Let $M = (A, <, P_1, \ldots, P_n)$ where $(A, <)$ is an infinite scattered linear ordering. There exists an expansion M' of M by some unary predicate P_{n+1} not definable in M such that $MSO(M')$ is recursive in $MSO(M)$.*

We present a sketch of proof for Theorem 15. Let $M = (A, <, \overline{P})$ where $(A, <)$ contains an interval of type ω or $-\omega$.

Consider the binary relation defined on A by $x \approx y$ iff $[x, y]$ is finite. The relation $\approx$ is a *condensation*, i.e., an equivalence relation such that every equivalence class is an interval of A. Moreover the relation $\approx$ is definable in M. If A_i and A_2 are $\approx$-equivalence classes, we say that A_1 precedes A_2 if all elements of A_1 are less than all elements of A_2. Let I be the linear order of the $\approx$-equivalence classes for $(A, <)$. Then $M = \sum_{i \in I} M_{A_i}$ where the A_i's correspond to equivalence classes of $\approx$. Using the definition of $\approx$ and our assumption on A, it is easy to check that the A_i's are either finite, or of order type ω, or $-\omega$, or ζ, and that not all A_i's are finite.

We define the interpretation of the new predicate P_{n+1} in every interval A_i. The definition proceeds as follows:

1. if some A_i has order type ω or $-\omega$, then we apply to each substructure M_{A_i} of order type ω the construction given in Proposition 7, and add no element of P_{n+1} elsewhere. If there is no A_i of order type ω, we proceed in a similar way with each substructure M_{A_i} of order type $-\omega$, but using the dual of Proposition 7 for $-\omega$.
2. if no A_i has order type ω or $-\omega$, then at least one $\approx -$equivalence class A_i has order type ζ. We consider two subcases:
   (a) if all $\approx -$equivalence classes A_i with order type ζ are such that $w(M_{A_i})$ is recurrent, then we apply to each substructure M_{A_i} of order type ζ the construction given in Proposition 12. For other $\approx -$equivalence classes A_i we set $P_{n+1} \cap A_i = \varnothing$.
   (b) otherwise there exist $\approx -$equivalence classes A_i with order type ζ and such that $w(M_{A_i})$ is not recurrent. Let $\varphi(x)$ be a formula with minimal quantifier depth such that $\varphi(x)$ defines an element in some M_{A_i} where A_i has order type ζ. For every M_{A_i} such that A_i has order type ζ and $\varphi(x)$ defines an element c_i in M_{A_i}, we apply the construction E_{c_i} from Proposition 9 to M_{A_i}. For other $\approx -$equivalence classes A_i we set $P_{n+1} \cap A_i = \varnothing$.

The fact that the set P_{n+1} is not definable in M follows rather easily from the construction, which ensures that there exists some A_i such that the restriction of P_{n+1} to A_i is not definable in the substructure M_{A_i}.

Let M' be the expansion of M by the predicate P_{n+1}. In order to prove that $MSO(M')$ is recursive in $MSO(M)$, we use Shelah's composition method [26, Theorem 2.4] (see also [12, 30]) which allows to reduce the MSO theory of a sum of structures to the MSO theories of the components and the MSO theory of the index structure.

Theorem 17 (Composition Theorem [26]). *There exists a recursive function f and an algorithm which, given $k, l \in \mathbb{N}$, computes the k-type of any sum $M = \sum_{i \in I} M_i$ of labelled linear orderings over a signature $\{<, P_1, \dots, P_l\}$ from the $f(k, l)$-type of the structure $(I, <, Q_1, \dots, Q_p)$ where*

$$Q_j = \{i \in I : T^k(M_i) = \tau_j\} \qquad j = 1, \dots, p$$

and $\tau_1, \dots, \tau_p$ is the list of all formally possible k-types for the signature L.

Let us explain the reduction from $MSO(M')$ to $MSO(M)$. We can apply Theorem 17 to $M' = \sum_{i \in I} M'_{A_i}$, which allows to show that for every k, the $k-$type of M' can be computed from $f(k, n+1)-$type of the structure $N' = (I, <, Q'_1, \dots, Q'_p)$ where the Q'_i's correspond to the $k-$types of structures M'_{A_i} over the signature $\{<, P_1, \dots, P_{n+1}\}$. Using the definition of P_{n+1} and Propositions 7, 9 and 12, one can prove that the $k-$type of M'_{A_i} can be computed form the $g(k)-$type of M_{A_i} for some recursive function g (note that g depends on M, namely whether we used case 1, 2(a) or 2(b) to construct M'). This allows to prove that N' is interpretable in the structure $N = (I, <, Q_1, \dots, Q_q)$ where the Q_i's correspond to the $g(k)-$types of structures M_{A_i} over the signature

$\{<, P_1, \ldots, P_n\}$. It follows that $MSO(N')$ is recursive in $MSO(N)$. Now using the fact that the equivalence relation $\approx$ is definable in M, we can prove that N is interpretable in M, thus $MSO(N)$ is recursive in $MSO(M)$.

Remark 18. Let us discuss uniformity issues related to Theorem 15.

- The choice to expand "uniformly" all $\approx$ −equivalence classes is crucial for the reduction from $MSO(M')$ to $MSO(M)$. For example, if some A_i has order type ω and we choose to expand only A_i then $MSO(M')$ might become undecidable. This is the case for the structure M considered in [2] (Definition 2.4), which has decidable MSO theory, and is such that the MSO theory of any expansion of M by a constant is undecidable. For this structure all A_i's have order type ω. If we consider the structure M' obtained from M by an expansion of only one A_i, then P_{n+1} has a least element, which is definable in M', thus MSO(M') is undecidable.
- The definition of P_{n+1} in case (2) depends on whether all components A_i with order type ζ are such that $w(M_{A_i})$ is recurrent, which is not a MSO definable property. Thus that the reduction algorithm from $MSO(M')$ to $MSO(M)$ depends on M.

6 Further Results and Open Questions

Let us mention some possible extensions and related open questions.

First of all, most of our results can be easily extended to the case when the signature contains infinitely many unary predicates.

Our results can be extended to the Weak MSO logic. In the case M is countable this follows from Soprunov result [28]. However, our construction works for labelled orderings of arbitrary cardinality.

An interesting issue is to prove uniform versions of our results in the sense of items (2) in Propositions 7 and 12. A first step would be to generalize Proposition 12 to all structures $(\mathbb{Z}, <, \overline{P})$.

One can also ask whether the results of the present paper hold for FO logic. Let us emphasize some difficulties which arise when one tries to adapt the main arguments. A FO version of Theorem 6 (about the recursive homogeneous set) was already proven in [21]. Moreover, using ideas from [25] one can also give a characterization of structures $M = (\mathbb{Z}, <, \overline{P})$ with a decidable FO theory, in terms of occurrences and repetitions of finite words in $w(M)$. This allows to give a FO version of our non-maximality results for labelled orders over ω or ζ. However for the general case of $(A, <, \overline{P})$, two problems arise: (1) the constructions for $\mathbb{N}$ and $\mathbb{Z}$ cannot be applied directly since they are not uniform, and (2) the equivalence relation $\approx$ used in the proof of Theorem 15 to cut A into small intervals is not FO definable. We currently investigate these issues.

Finally, we also study the case of labelled linear orderings $(A, <, \overline{P})$ which do not contain intervals of order types ω or $-\omega$. In this case the construction presented in Sect. 5 does not work since the restriction of P_{n+1} to each A_i will be empty, i.e., our new relation is actually empty. In a forthcoming paper we show that it is possible to overcome this issue for the countable orders, and prove that no infinite countable structure $(A, <, \overline{P})$ is maximal.

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