Selection and Uniformization in Generalized Product

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Abstract

We investigate the selection and uniformization properties for classes of structures constructed by the Feferman-Vaught generalized product. We show that if classes K_1 and K_2 have the selection (respectively, uniformization) property, then the generalized product of these classes has the selection (respectively, uniformization) property.

Keywords: Uniformization, Generalized Product, Feferman-Vaught theorem

1 Introduction

DEFINITION 1.1 (Selectors)
A class K of structures has the selection property if for every formula $\varphi(\vec{t})$ there is a formulas $\psi(\vec{t})$ such that

1. $K \models \exists^{\leq 1}\vec{t}.\ \psi(\vec{t})$
2. $K \models \forall\vec{t}\psi(\vec{t}) \rightarrow \varphi(\vec{t})$ and
3. $K \models \exists\vec{t}.\ \varphi(\vec{t}) \rightarrow \exists\vec{t}.\psi(\vec{t})$.

Here and below "$\exists^{\leq 1}$" stands for "there is at most one", and "$\exists!$" stands for "there is a unique" and $\vec{t}$ is a list of distinct variables. We say that $\psi(\vec{t})$ is a *selector* for $\varphi(\vec{t})$ over K if (1)-(3) of the above definition hold.

DEFINITION 1.2
A class of structures K has the uniformization property if for every formula $\beta(\vec{t},\vec{u})$ there is a formula $\alpha(\vec{t},\vec{u})$ such that

1. $K \models \forall\vec{t}.\exists^{\leq 1}\vec{u}.\ \alpha(\vec{t},\vec{u})$.
2. $K \models \forall\vec{t}.\forall\vec{u}.\ \alpha(\vec{t},\vec{u}) \rightarrow \beta(\vec{t},\vec{u})$ and
3. $K \models \forall\vec{t}.\ (\exists\vec{u}.\ \beta(\vec{t},\vec{u}) \rightarrow \exists\vec{u}\alpha(\vec{t},\vec{u}))$.

We say that $\alpha(\vec{t},\vec{u})$ *uniformizes* $\beta(\vec{t},\vec{u})$ over K if (1)-(3) of Definition 1.2 hold.

The selection and uniformization properties were studied for some important structures [6, 1, 3, 4, 5]. The uniformization problem for monadic second order logic of order over the structure of natural numbers plays a very important role in automata theory [1] and it has a natural game interpretation. Gurevich and Shelah [3] proved that the full binary tree has neither the uniformization nor selection properties. Lifsches and Shelah [4, 5] provided a full classification of trees, according to the selection (respectively uniformization) property. The selection problem is also tightly connected to the elimination of Hilbert's ϵ-symbol [6].

The aim of this paper is to investigate the inheritance of the selection and uniformization properties by algebraic constructions. The simplest of such construction is the direct product of structures.

Direct Product. Let A_i (for $i \in Ind$) be a family of structures for a relation vocabulary Σ. The direct product of A_i is the structure $G = \Pi_{i \in Ind} A_i$ for the vocabulary Σ defined as follows:

1. The universe $|G|$ of the structure is the Cartesian product $\Pi_{i \in Ind}|A_i|$ of the universes of A_i ($i \in Ind$); hence, the universe is the set of all functions g with the domain Ind, such that for each $i \in Ind$, $g(i)$ is an element of A_i.
2. The interpretation of an n-ary relation symbol R in the product is defined as follows:

$$R(g^1, \ldots, g^n) \text{ iff } R(g^1(i), \ldots, g^n(i)) \text{ holds in } A_i \text{ for every } i \in Ind.$$

It is easy to see that the class of well-ordered sets has the selection property. However the class of direct products of well-ordered sets does not have the selection property. In particular the formulas $\exists u.\, u < t$ does not have selector over this class.

Ultraproduct. Let A_i (for $i \in Ind$) be a family of structures for a relation vocabulary Σ. Let F be an ultrafilter of subsets of Ind. The ultraproduct of A_i over F is the structure $\bar{G} = \Pi_F A_i$ for the vocabulary Σ defined as follows. First, define an equivalence relation $\sim_F$ on the elements of the direct product $\Pi_{i \in Ind} A_i$:

$$g^1 \sim_F g^2 \text{ if } \{i \in Ind \ : \ g^1(i) = g^2(i)\} \in F.$$

We denote the $\sim_F$ equivalence class of g by $\widehat{g}$.

1. The universe of the ultraproduct are the $\sim_F$ equivalence classes.
2. The interpretation of an n-ary relation symbol R in the ultraproduct is defined as follows:

$$R(\widehat{g^1}, \ldots, \widehat{g^n}) \text{ iff } \{i \in Ind \ : \ R(g^1(i), \ldots, g^n(i)) \text{ holds in } A_i\} \in F.$$

It is well-known that for every first-order formula $\varphi(x_1, \ldots, x_n)$:

$$\varphi(\widehat{g^1}, \ldots, \widehat{g^n}) \text{ holds in the ultraproduct if and only if}$$

$$\{i \in Ind \ : \ \varphi(g^1(i), \ldots, g^n(i)) \text{ holds in } A_i\} \in F.$$

From this fact it is easy to infer that the selection and uniformiztion properties are inherited by the ultraproduct; i.e., if a class K of structures has the selection (respectively uniformization) property, then the class of the ultraproducts of structures from K has the selection (respectively uniformization) property.

In [2] Feferman-Vaught introduced a generalized product construct which encompasses a wide variety of algebraic constructions. This construct associates with a structure Ind (index structure) and a family A_i ($i \in Ind$) of factor structures the generalized product structure of A_i over Ind. The Feferman-Vaught theorem reduces the first-order theory of the generalized product to the first-order theory of the factors and the first-order theory of an algebra of subsets over indices.

Our main result shows that if classes K_1 and K_2 of structures have the selection (respectively, uniformization) property, then the generalized product of these classes has the selection (respectively, uniformization) property.

Our proofs are constructive. They reduce a construction of a selector for φ in the generalized product of K_1 over K_2 to a construction of (a finite set of) selectors in the class K_1 and a construction of (a finite set of) selectors in the class K_2.

The paper is organized as follows. In Section 2 we fix notations, recall the definition of the generalized product construct and state the Feferman-Vaught theorem. In Section 3 we show that the selection property is inherited by the generalized product construct. In Section 4 we show that the uniformization property is inherited by the generalized product construct. Section 5 contains final remarks and states open problems.

2 Generalized Product

Notations. Let A be a structure. We use $|A|$ for the universe of A and R^A for the interpretation of the relation symbol R in A. However, whenever there is no confusion we will also use A for the universe of A; sometimes we use "$a \in A$" instead of "$a \in |A|$". We write $\varphi(t_1, \ldots, t_k)$ to indicate that the free variables of φ are among $\{t_1, \ldots, t_k\}$. Let $\varphi(t_1, \ldots t_k)$ be a formula in a language L. We write $A, a_1, \ldots a_k \models \varphi(t_1, \ldots, t_k)$ if the formula $\varphi(t_1, \ldots t_k)$ is true in the structure A for L when the elements $a_1, \ldots, a_k$ in A are assigned to the variables names $t_1 \ldots t_k$. We also abbreviate these to $A \models \varphi(a_1 \ldots a_k)$. or to $A \models \varphi(\bar{a})$ where the bar denotes a tuple of the appropriate length; whenever $\bar{a}$ or A are clear from the context we even will use $A \models \varphi$ or $\varphi(\bar{a})$ respectively.

In a seminal paper [2] Feferman-Vaught introduced a generalized product; their composition theorem reduces first-order theory of the generalized product to the first-order theory of the factors and the first-order theory of an algebra of subsets over indices.

For the reader's convenience we recall the relevant definitions.

Let L_{ind} be a signature. A structure A is called an L_{ind} algebra of subsets over a set I if

1. The universe of A is the set of subsets of I,
2. A is a structure for the signature $L_{ind} \cup \{\Lambda,\ ^{-},\ \cup,\ \cap,\ \subseteq\}$, where Λ is a constant symbol interpreted as the empty set, $^{-}$ is a unary function symbol interpreted as the complementation relative to I, $\cup$ and $\cap$ are binary function symbols interpreted as the binary union and intersection operators and $\subseteq$ is a binary relation symbol interpreted as the inclusion relation. There is no restriction on the interpretation of L_{ind} symbols.

We will use upper case letters for the variables ranging over the universe of L_{ind} algebras over I. Note that every such variable is interpreted as a subset of I.

The generalized product construct will deal with L_{ind} - a language and Ind - an L_{ind} algebra of subset over $|Ind|$; L_{factor} - a language and a family A_i (for $i \in |Ind|$) of factor structures for L_{factor} and L_{result} - a language for a generalized product and A - a structure for L_{result}

The universe $|G|$ of the structure for the generalized product of A_i ($i \in |Ind|$) will be the Cartesian product $\Pi_{i \in |Ind|}|A_i|$ of the family $|A_i|$ ($i \in |Ind|$); hence, the universe is the set of all functions g with the domain $|Ind|$ such that for each $i \in |Ind|$, $g(i)$ is an element of $|A_i|$. The interpretation of symbols in L_{result} is defined by the Feferman-Vaught determining sequences. We recall the syntax and the semantics of these sequences.

DEFINITION 2.1 (Determining sequence for an n-nary predicate -Syntax)
An n-ary determining sequence consists of

1. A finite sequence $\varphi_1(x_1,\dots x_n), \varphi_2(x_1,\dots,x_n), \dots \varphi_m(x_1,\dots,x_n)$ of L_{factor} formulas (the free variables of these formulas are among $\{x_1,\dots,x_n\}$) and
2. A formula $\psi(Y_1,\dots,Y_m)$ in the language for L_{ind} algebras (the number of the free variables is at most as the length of the sequence of L_{factor} formulas).

Now we are ready to define the semantics of the generalized product. Let Ind be an L_{ind} algebra of subset and let A_i ($i \in |Ind|$) be a family of L_{factor} structures. The generalized product of A_i over Ind is denoted by $\Pi_{Ind}A_i$ and is defined as the following structure:

Signature: L_{result} has exactly one n-ary relation symbol for every n-ary determining sequence.

Universe: The Cartesian product of $|A_i|$ - the universes of A_i ($i \in |Ind|$).

Interpretation of relation symbols: To define the interpretation of a relation symbol that corresponds to a determining sequence

$$\langle \psi(Y_1,\dots,Y_m), \varphi_1(x_1,\dots x_n), \varphi_2(x_1,\dots,x_n),\dots,\varphi_m(x_1,\dots,x_n)\rangle$$

we need to introduce some notations.
Let $\langle a_1,\dots,a_n\rangle$ be an n-tuple of elements in $\Pi_{i\in|Ind|}|A_i|$. With such n-tuple and a formula $\varphi_l(x_1,\dots,x_n)$ in L_{factor} a subset P_l of $|Ind|$ (which depends on $\langle a_1,\dots,a_n\rangle$ and φ_l) is defined as follows:

$$P_l \triangleq \{i \in |Ind| \ : \ A_i, a_1(i),\dots a_n(i) \models \varphi_l(x_1,\dots,x_n)\}$$

The n-tuple $\langle a_1,\dots,a_n\rangle$ satisfies the determining sequence described above iff

$$Ind, P_1,\dots,P_m \models \psi(Y_1,\dots,Y_m)$$

Finally, a relation symbol that corresponds to the above determining sequence is interpreted as the set of tuples that satisfies the determining sequence.

For a class K_1 of L_{factor} structures and for a class K_2 of L_{ind} algebras, the generalized product of K_1 over K_2 is the class of structures $\{\Pi_{Ind}A_i \ : \ A_i \in K_1 \text{ and } Ind \in K_2\}$; we denote this class by $\Pi_{K_2}K_1$

DEFINITION 2.2 (Partitions)

1. A sequence $\alpha_1(\vec{t}),\dots,\alpha_m(\vec{t})$ of formulas is called a partitioning sequence if $\bigvee_{i=1}^m \alpha_i$ is valid and $\alpha_i \wedge \alpha_j$ is unsatisfiable for $i \neq j$.
2. A determining sequence $\langle \beta(X_1,\dots X_k), \alpha_1(\vec{t})\dots\alpha_k(\vec{t})\rangle$ is called a partitioning determining sequence if $\alpha_1,\dots\alpha_k$ is a partitioning sequence.
3. Subsets $P_1,\dots P_m$ of a set I partition I iff they are disjoint and their union is I.

Note that for every $m > 0$ there is a formula $Part_m(X_1,\dots,X_m)$ in the signature $\{\Lambda, \ ^-, \cup, \cap, \subseteq\}$ that says "$X_1,\dots,X_m$ partitions the universe".

THEOREM 2.3 (Composition Theorem for Generalized Product [2])
For every formula $\varphi(u_1,\dots,u_n)$ (with n free variables) in L_{result} there is a partitioning n-ary determining sequence σ such that for every L_{ind} algebra Ind and every family of structures A_i ($i \in |Ind|$) for L_{factor} the relation definable in $A = \Pi_{Ind}A_i$ by φ is the same as the relation definable in A by σ. Moreover, σ is computable from φ.

3 Selectors

Observe that that we can restrict our discussion to formulas with one free variable. More precisely,

LEMMA 3.1
K has the selection property if and only if for every formula $\varphi(t)$ with one free variable there exists $\psi(t)$ such that $\psi(t)$ is a selector for $\varphi(t)$ over K.

PROOF. By induction on n we will show that for every $\varphi(t_0,\ldots,t_n)$ there exists $\psi(t_0,\ldots,t_n)$ such that ψ is a selector for φ over K.

The base $n=0$ is just the assumption of the lemma.

The inductive step $n \to n+1$. Let $\varphi(t_0,\ldots,t_{n+1})$ be a formula. By the inductive assumption there is a selector $\alpha(t_0,\ldots,t_n)$ for $\exists t_{n+1}\varphi(t_0,\ldots,t_{n+1})$. By the assumption of the lemma there exists a selector $\beta(t_{n+1})$ for $\exists t_0\ldots\exists t_n(\alpha(t_0,\ldots t_n)\wedge\varphi(t_0,\ldots,t_{n+1}))$. One can easy verify that $\alpha(t_0,\ldots,t_n)\wedge\beta(t_{n+1})$ is a selector for φ over K. ■

In this section we prove

THEOREM 3.2
Let K_1 be a class of L_{factor} structures and let K_2 be a class of L_{ind} algebras. If K_1 has the selection property and K_2 has the selection property, then the generalized product of K_1 over K_2 has the selection property.

PROOF. By lemma 3.1 it is sufficient to show that for every $\varphi(t)$ there is exists $\psi(t)$ which is a selector for φ over the generalized product of K_1 over K_2.

Let $\varphi(t)$ be a formula in L_{result} and let $\langle\beta(X_1,\ldots X_k),\alpha_1(t)\ldots\alpha_k(t)\rangle$ be the partitioning determining sequence that corresponds to $\varphi(t)$ by Theorem 2.3.

Let $\gamma_i(t)$ be selectors for $\alpha_i(t)$ over K_1.

Let $\delta(X_1,\ldots,X_k)$ be a selector for $Part(X_1,\ldots,X_k)\wedge\beta(X_1,\ldots,X_k)$ over the class K_2. Let $\psi(t)$ be defined by the determining sequence $\langle\delta,\gamma_1(t),\ldots\gamma_k(t)\rangle$. In the rest of this section we show that $\psi(t)$ is a selector for $\varphi(t)$ over the generalized product of K_1 and K_2.

Let Ind be in K_2 and let A_i ($i\in|Ind|$) be a family of structures from K_1. Let M be the generalized product of A_i over Ind. First, we will show that $M\models\exists^{\le 1}t\psi(t)$. Assume that for some $a,b\in M$

$$M,a\models\psi(t) \text{ and } M,b\models\psi(t) \tag{3.1}$$

Notations For $a\in M$ and $j\in Ind$ we will use a_j for the j-th coordinate of a.

Let

$$Q_i^a \triangleq \{j\in|Ind| \ : \ A_j,a_j\models\gamma_i(t)\} \tag{3.2}$$

and

$$Q_i^b \triangleq \{j\in|Ind| \ : \ A_j,b_j\models\gamma_i(t)\}. \tag{3.3}$$

Observe that by the definition of $\psi(t)$ and (3.1) we obtain

$$Ind\models\delta(Q_1^a,\ldots,Q_k^a) \tag{3.4}$$

and

$$Ind\models\delta(Q_1^b,\ldots,Q_k^b) \tag{3.5}$$

Since, δ is a selector, it follows

$$Ind\models\exists^{\le 1}X_1\ldots X_k.\,\delta(X_1,\ldots,X_k) \tag{3.6}$$

Hence, by (3.4)-(3.6) we obtain

$$Q_i^a = Q_i^b. \tag{3.7}$$

Moreover, since $K_1 \models \exists^{\leq 1} t.\ \gamma_i(t)$, it follows by (3.2), (3.3) and (3.7)

$$a_j = b_j \text{ for } j \in Q_i^a \text{ and } i = 1, \ldots, k \tag{3.8}$$

Recall that δ is a selector for $Part(X_1, \ldots, X_k) \wedge \beta$. Therefore from (3.4) and (3.8) we obtain that Q_i^a $(i = 1, \ldots, k)$ partitions $|Ind|$ and

$$a_j = b_j \text{ for } j \in |Ind| \tag{3.9}$$

Hence $a = b$ and this establishes $K \models \exists^{\leq 1} t.\ \psi(t)$.

Now we will show that $K \models \forall t. \psi(t) \rightarrow \varphi(t)$. Let $M = \Pi_{i \in Ind} A_i$ for $Ind \in K_2$ and $A_i \in K_1$. Assume that

$$M, a \models \psi(t) \tag{3.10}$$

Let Q_i^a for $i = 1, \ldots, k$ be defined as in (3.2). Then by (3.10) and by the definitions of $\psi(t)$ and of δ

$$Ind \models Part(Q_1^a, \ldots, Q_k^a) \wedge \beta(Q_1^a, \ldots, Q_k^a) \tag{3.11}$$

Let P_i^a for $i = 1, \ldots, k$ be defined as

$$P_i^a \triangleq \{j \in |Ind| \ : \ A_j, a_j \models \alpha_i(t)\} \tag{3.12}$$

Since, $\gamma_i(t)$ is a selector for $\alpha_i(t)$, it follows from (3.2) and (3.12) that $Q_i^a \subseteq P_i^a$ for $i = 1, \ldots, k$. Observe that P_i^a is a partition of $|Ind|$, because $\alpha_i(t)$ is a partitioning sequence. Q_i^a is a partition of $|Ind|$ by (3.11). Therefore, none of the above inclusions can be proper. Hence, $Q_i^a = P_i^a$ for $i = 1, \ldots, k$. This together with (3.11), (3.12) and the definition of φ imply that $M, a \models \varphi(t)$. This completes the proof of $K \models \forall t. \psi(t) \rightarrow \varphi(t)$.

It remains to show that $K \models \exists t \varphi(t) \rightarrow \exists t \psi(t)$.

Assume that $M, a \models \varphi(t)$. We are going to define $b \in M$ such that $M, b \models \psi(t)$. Let P_i^a be defined as in (3.12). By the definition of φ we have

$$Ind \models \beta(P_1^a, \ldots, P_k^a) \tag{3.13}$$

Recall that $\alpha_i(t)$ is a partitioning sequence, Hence, by (3.12) we obtain

$$Ind \models Part(P_1^a, \ldots, P_k^a) \tag{3.14}$$

Since $\gamma_i(t)$ is a selector for $\alpha_i(t)$ it follows from (3.12) that for every $j \in P_i^a$ there is a unique $b_j \in A_j$ such that

$$A_j, b_j \models \gamma_i(t) \tag{3.15}$$

Define b as $\langle b_j \ : j \in |Ind| \rangle$. Note that this is well defined because P_i^a is a partition of $|Ind|$. Let Q_i^b be defined as in (3.3). From the definition of b it follows that $P_i^a \subseteq Q_i^b$ for $i = 1, \ldots, k$. None of the inclusion can be proper, because P_i^a is a partition and Q_i^b are disjoint (indeed, if $j \in Q_{i_1}^b \cap Q_{i_2}^b$ then $A_j, b_j \models \gamma_{i_1}(t) \wedge \gamma_{i_2}(t)$; this implies that $A_j, b_j \models \alpha_{i_1}(t) \wedge \alpha_{i_2}(t)$; since $\alpha_i(t)$ is a partitioning sequence we will obtain that $i_1 = i_2$.) Therefore, $P_i^a = Q_i^b$, hence, by (3.13) and (3.14) we obtain that

$$Ind \models Part(Q_1^b, \ldots, Q_k^b) \wedge \beta(Q_1^b, \ldots, Q_k^b) \tag{3.16}$$

The last equations together with the definition of ψ imply $M, b \models \psi(t)$. This completes the proof of $\exists t \varphi(t) \rightarrow \exists t \psi(t)$.

Therefore $\psi(t)$ is a selector for $\varphi(t)$. ∎

4 Uniformization

THEOREM 4.1
Let K_1 be a class of L_{factor} structures and let K_2 be a class of L_{ind} algebras. If K_1 has the uniformization property and K_2 has the uniformization property, then the generalized product of K_1 over K_2 has the uniformization property.

PROOF. We will show that for every $\varphi(t,u)$ there exists $\psi(t,u)$ which uniformizes $\varphi(t,u)$ over the generalized product. The general case when t and u are lists of variables is proved exactly in the same way.

Let $\varphi(t,u)$ be a formula in L_{result} and let $\langle \beta(X_1,\dots X_k), \alpha_1(t,u)\dots\alpha_k(t,u)\rangle$ be the partitioning determining sequence that corresponds to $\varphi(t,u)$ by Theorem 2.3. We have to construct $\psi(t,u)$ that uniformizes φ. Below we first describe the construction of ψ and then prove that the construction is correct.

Construction. For every $M \subseteq \{1,\dots,k\}$ we define a formula $\theta_M(t)$ as

$$\theta_M(t) \triangleq \bigwedge_{i\in M} \exists u.\alpha_i(t,u) \wedge \bigwedge_{i\notin M} \neg\exists u.\alpha_i(t,u) \tag{4.1}$$

The sequence $\theta_M(t)$ ($M \subseteq \{1,\dots,k\}$) is a partitioning sequence.

Let $\gamma_i(t,u)$ be a formula that uniformizes $\alpha_i(t,u)$ ($i = 1,\dots,k$) over K_1.

Let $\overline{Z}$ be the set of variable $\{Z_M \ : \ M \subseteq \{1,\dots,k\}\}$. Let $\Delta(\overline{Z},Y_1,\dots,Y_k)$ be defined as

$$\begin{aligned}\Delta(\overline{Z},Y_1,\dots Y_k) \triangleq Part(\overline{Z}) \to \quad & \Big(\beta(Y_1,\dots,Y_k) \wedge Part(Y_1,\dots,Y_k)\wedge \\ & \wedge \textstyle\bigwedge_{m=1}^{k}(\,Y_m \subseteq \bigcup_{m\in M} Z_M)\Big).\end{aligned} \tag{4.2}$$

Let $\delta(\overline{Z},Y_1,\dots,Y_k)$ be a formula that uniformizes Δ over K_2, i.e.,

$$K_2 \models \forall\overline{Z}.\exists^{\leq 1}\overline{Y}.\ \delta(\overline{Z},\overline{Y}). \tag{4.3}$$

$$K_2 \models \forall\overline{Z}.\forall\overline{Y}.\ \delta(\overline{Z},\overline{Y}) \to \Delta(\overline{Z},\overline{Y}) \tag{4.4}$$

$$K_2 \models \forall\overline{Z}.\ (\exists\overline{Y}.\ \Delta(\overline{Z},\overline{Y}) \to \exists\overline{Y}\delta(\overline{Z},\overline{Y})). \tag{4.5}$$

Let $\psi(t,u)$ be defined by the determining sequence

$$\langle\delta(\overline{Z},\overline{Y}),\dots\theta_M(t)\dots,\gamma_1(t,u),\dots,\gamma_k(t,u)\rangle.$$

In the rest of this section we show that $\psi(t,u)$ uniformizes $\varphi(t,u)$ over the generalized product of K_1 and K_2.

Let Ind be in K_2 and let A_i ($i \in |Ind|$) be a family of structures from K_1. Let A be the generalized product of A_i over Ind. First, we will show that $A \models \forall t\exists^{\leq 1}u.\ \psi(t,u)$. Assume that for some $a,b,c \in A$

$$A \models \psi(a,b) \tag{4.6}$$

$$A \models \psi(a,c) \tag{4.7}$$

For $M \subseteq \{1,\dots,k\}$ let

$$P^a_M = \{i \in Ind \ : \ A_i \models \theta_M(a)\ \} \tag{4.8}$$

For $m \in \{1, \dots, k\}$ let

$$Q_m^{a,b} = \{i \in Ind \ : \ A_i \models \gamma_m(a_i, b_i) \ \} \tag{4.9}$$

$$Q_m^{a,c} = \{i \in Ind \ : \ A_i \models \gamma_m(a_i, c_i) \ \} \tag{4.10}$$

From the definition of ψ and (4.6)-(4.10), it follows that

$$Ind \models \delta(\overline{P^a}, Q_1^{a,b}, \dots Q_k^{a,b}) \tag{4.11}$$

$$Ind \models \delta(\overline{P^a}, Q_1^{a,c}, \dots Q_k^{a,c}) \tag{4.12}$$

(Here and below we use $\overline{P^a}$ for the sequence $\langle P_M^a \ : \ M \subseteq \{1, \dots, k\}\rangle$.) Hence, by (4.3), for $m \in \{1, \dots, k\}$

$$Q_m^{a,b} = Q_m^{a,c} \tag{4.13}$$

Since $\gamma_m(t, u)$ uniformizes $\alpha_m(t, u)$, by (4.9), (4.10) and (4.13), we obtain

$$b_i = c_i \text{ for } i \in Q_m^{a,b} \tag{4.14}$$

$Q_m^{a,b}$ ($m = 1, \dots, k$) is a partition of $|Ind|$, by (4.11) and the definitions of Δ and δ. Therefore (4.14) implies that $b = c$. Hence, $A \models \forall t \exists^{\leq 1} u.\ \psi(t, u)$.

Inclusion of ψ in φ. Let us show that $A \models \forall t \forall u \psi(t, u) \to \varphi(t, u)$. Assume that

$$A \models \psi(a, b) \tag{4.15}$$

Let P_M^a for $M \subseteq \{1, \dots, k\}$. be defined like in (4.8) and let $Q_m^{a,b}$ for $m \in \{1, \dots, k\}$ be defined like in (4.9). From (4.15) and the definition of ψ we obtain

$$\begin{aligned} Part(\overline{P^a}) \to \quad & \Big(\beta(Q_1^{a,b}, \dots, Q_k^{a,b}) \wedge Part(Q_1^{a,b}, \dots, Q_k^{a,b}) \wedge \\ & \wedge \textstyle\bigwedge_{m=1}^{k} (\ Q_m^{a,b} \subseteq \bigcup_{m \in M} P_M^a)\Big). \end{aligned} \tag{4.16}$$

Recall that θ_M is a partitioning sequence. Therefore, $Part(\overline{P^a})$, and by (4.16), we obtain

$$Ind \models \beta(Q_1^{a,b}, \dots, Q_k^{a,b}) \tag{4.17}$$

Since $\gamma_m(t, u)$ uniformizes $\alpha_m(t, u)$ we obtain

$$A_i \models \alpha_m(a_i, b_i) \text{ for } i \in Q_m^{a,b} \tag{4.18}$$

For $m \in \{1, \dots, k\}$ let

$$R_m^{a,b} = \{i \in Ind \ : \ A_i \models \alpha_m(a_i, b_i)\} \tag{4.19}$$

By (4.18) and (4.19) we obtain that $Q_m^{a,b} \subseteq R_m^{a,b}$ for $m \in \{1, \dots, k\}$. Both $R_m^{a,b}$ ($m \in \{1, \dots, k\}$) and $Q_m^{a,b}$ ($m \in \{1, \dots, k\}$) are partitions ($R_m^{a,b}$, because α_m is a partition sequence, and $Q_m^{a,b}$ by (4.16)). Hence none of the above inclusions can be proper. Therefore, $R_m^{a,b} = Q_m^{a,b}$ for $m = 1, \dots, k$. This together with (4.17), (4.19) and the definition of φ implies that $A \models \varphi(a, b)$. This establishes that $\forall t \forall u \psi(t, u) \to \varphi(t, u)$ holds over the generalized product.

It remains to show that $\forall t\,.(\exists u\ \varphi(t,u) \to \exists u\ \psi(t,u))$ holds over the generalized product. Assume that

$$A \models \varphi(a,b) \tag{4.20}$$

We are going to construct $c \in A$ such that $A \models \psi(a,c)$

Let $R_m^{a,b}$ for $m \in \{1,\dots,k\}$ be defined like in (4.19). From the definition of φ and (4.20), it follows that

$$Ind \models \beta(R_1^{a,b},\dots,R_k^{a,b}). \tag{4.21}$$

Since α_m is a partitioning sequence, it follows that $Part(R_1^{a,b},\dots,R_k^{a,b})$. Let P_M^a for $M \subseteq \{1,\dots,k\}$. be defined like in (4.8). From the definition of θ_M, it follows that $R_m^{a,b} \subseteq \bigcup_{m\in M} P_M^a$. Therefore, $\Delta(\overline{P^a}, R_1^{a,b},\dots,R_k^{a,b})$ holds. Since, δ uniformizes Δ it follows by (4.5) that there are $S_m^{a,b}$ $(m = 1,\dots,k)$ such that

$$Ind \models \delta(\overline{P^a}, S_1^{a,b},\dots,S_k^{a,b}) \tag{4.22}$$

Since P_M^a partitions $|Ind|$, it follows from (4.2) and (4.22) that

$$Ind \models \bigwedge_{m=1}^{k} (\ S_m^{a,b} \subseteq \bigcup_{m\in M} P_M^a). \tag{4.23}$$

$$Ind \models \beta(S_1^{a,b},\dots,S_k^{a,b}) \wedge Part(S_1^{a,b},\dots,S_k^{a,b}) \tag{4.24}$$

Note that, by (4.8), $A_i \models \theta_M(a_i)$ for $i \in P_M^a$. Therefore by the definition of θ_M, for every $m \in M$ there is d_i such that $A_i \models \alpha_m(a_i,d_i)$. Since γ_m uniformizes α_m, this implies

$$\text{For every } i \in P_M^a \text{ and } m \in M \text{ there is } e_{i,m} \text{ such that } A_i \models \gamma_m(a_i,e_{i,m}) \tag{4.25}$$

Let c_i be defined as follows

$$c_i = e_{i,m} \text{ for } i \in S_m^{a,b} \tag{4.26}$$

Since P_M^a partitions $|Ind|$, it follows from (4.23)-(4.25) that the c_i in (4.26) are well defined, moreover, (4.26) defines c_i for every $i \in |Ind|$. We are going to show that $A \models \psi(a,c)$.

For $m \in \{1,\dots,k\}$ let $Q_m^{a;c}$ be defined as in (4.10). Note that (4.10), (4.25) and (4.26) imply that $Q_m^{a,c} \supseteq S_m^{a,b}$. None of these inclusions can be proper. Indeed, γ_m uniformizes α_m and α_m $(m = 1,\dots,k)$ is a partitioning sequence, therefore $Q_m^{a;c}$ are disjoint sets. Hence, if one of the inclusion is proper, it would contradict (4.24). Therefore, $Q_m^{a,c} = S_m^{a,b}$ for $m \in \{1,\dots,k\}$. This together with (4.22) imply that $Ind \models \delta(\overline{P^a}, Q_1^{a,c},\dots,Q_k^{a,c})$. Therefore, by the definition of ψ, it follows that $A \models \psi(a,c)$. This completes the proof that $\forall t\,.(\exists u\ \varphi(t,u) \to \exists u\ \psi(t,u))$ holds

Therefore, $\psi(t,u)$ uniformizes $\varphi(t,u)$. ∎

5 Final Remarks

The generalized product, as was defined here following [2], has infinitely many relation symbols; one symbol for every determining sequence. If one takes a set Δ of determining sequences one can define a notion of product which has in the vocabulary only symbols that correspond to the sequences in Δ. Such a notion of product was referred in [2] as a generalized product. We will also refer to it as Δ-product.

The direct product considered in the Introduction is an instance of a generalized product. As it was mentioned there, the direct product does not inherit the selection and uniformization properties.

It is interesting to characterize those generalized products that inherit selection (or uniformization) property. A related algorithmic problem can be stated as follows. Assume that the classes of structures K_1 and K_2 are decidable. Given a finite set Δ of determining sequences. Is it decidable whether the Δ-product of K_1 over K_2 has the selection (respectively uniformization) property?

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