SELECTION IN THE MONADIC THEORY OF A COUNTABLE ORDINAL

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Abstract. A monadic formula $\psi(Y)$ is a *selector* for a formula $\varphi(Y)$ in a structure $\mathcal{M}$ if there exists a unique subset P of $\mathcal{M}$ which satisfies ψ and this P also satisfies φ. We show that for every ordinal $\alpha \geq \omega^\omega$ there are formulas having no selector in the structure $(\alpha, <)$. For $\alpha \leq \omega_1$, we decide which formulas have a selector in $(\alpha, <)$, and construct selectors for them. We deduce the impossibility of a full generalization of the Büchi-Landweber solvability theorem from $(\omega, <)$ to $(\omega^\omega, <)$. We state a partial extension of that theorem to all countable ordinals. To each formula we assign a *selection degree* which measures "how difficult it is to select". We show that in a countable ordinal all non-selectable formulas share the same degree.

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§1. Introduction.

Definition 1.1 (Selection). Let $\varphi(\bar{Y})$, $\psi(\bar{Y})$ be formulas in some logic $\mathcal{L}$ and $\mathcal{M}$ a structure for $\mathcal{L}$. We say that ψ *selects* (or, is a *selector* for) φ *in* $\mathcal{M}$ iff:

1. $\mathcal{M} \models \exists^{\leq 1}\bar{Y}\psi(\bar{Y})$,
2. $\mathcal{M} \models \forall\bar{Y}(\psi(\bar{Y}) \to \varphi(\bar{Y}))$, and
3. $\mathcal{M} \models \exists\bar{Y}\varphi(\bar{Y}) \to \exists\bar{Y}\psi(\bar{Y})$.

Here $\bar{Y}$ is a tuple of distinct variables and "$\exists^{\leq 1}\bar{Y}\dots$" stands for "there exists at most one…".
We say that $\mathcal{M}$ has the *$\mathcal{L}$-selection property* iff every $\mathcal{L}$-formula φ has a selector ψ in $\mathcal{M}$.

We explore the question "Does there exist a $\psi(\bar{Y})$ that selects $\varphi(\bar{Y})$ in $\mathcal{M}$?" when φ and ψ are formulas in the *second-order monadic logic of order* (MLO), $\bar{Y}$ is a tuple of monadic variables ranging over *subsets* of the domain, and $\mathcal{M} := (\alpha, <)$ is some ordinal α equipped with its natural order.

MLO allows quantification over subsets of the domain (monadic predicates). The binary relation symbol '$<$' is its only non-logical constant. Since our structures are ordinals, we usually assume that it is interpreted as a *well*-order of the domain.

From now on, unless otherwise explicitly stated, "formula" means "MLO-formula", the "selection property" is the "MLO-selection property", etc.

In [LiS98], Lifsches and Shelah show that any $\alpha < \omega^\omega$ has the selection property (in fact, they prove a much stronger property called the *uniformization property*; see Definition 2.20).

In section 3, we show that the selection property fails for any $\alpha \geq \omega^\omega$. In particular, we show that the formula $\theta_{\omega\,\mathrm{ub}}(Y)$ saying "Y is an unbounded ω-sequence" has no selector in $(\omega^\omega, <)$. This naturally leads to the following algorithmic problem:

Definition 1.2 (Selection problem). The *selection problem* in a structure $\mathcal{M}$ is:

Input: an MLO formula $\varphi(\bar{Y})$.
Task: check whether there is a selector $\psi(\bar{Y})$ for φ in $\mathcal{M}$, and if so, construct one.

We show the solvability of the selection problem in $(\alpha, <)$ for every $\alpha \leq \omega_1$ (section 4).

In section 5, we try and compare formulas in terms, intuitively, of how difficult they are to select. The idea is, roughly, that a formula $\varphi_0(Y)$ is *easier to select* than a formula $\varphi_1(Y)$ in a structure $\mathcal{M}$ if, from any given subset P of (the domain of) $\mathcal{M}$ which satisfies φ_1 in $\mathcal{M}$, we can define a subset Q of (the domain of) $\mathcal{M}$ which satisfies φ_0 in $\mathcal{M}$. This is easily seen to be a partial preorder on the formulas, and its equivalence classes we call *selection degrees* (Definition 5.3). It turns out, interestingly, that in every countable $\alpha \geq \omega^\omega$ there are just two degrees: one containing all selectable formulas and one containing all non-selectable formulas. This implies, for instance, that with an unbounded ω-sequence as a parameter one can select every formula in $(\omega^\omega, <)$.

In section 6, we note that while there is a particular formula lacking a selector in each and every uncountable ordinal, no formula can be found which is non-selectable in every $\alpha \in [\omega^\omega, \omega_1)$.

An analysis of our proofs shows that for countable ordinals the above results concerning the selection property and the selection problem carry through to all logics between first-order and MLO. On the other hand, we show that every first-order formula has a selector in $(\omega_1, <)$. In fact, an interesting dichotomy holds:

(a) a first-order formula selectable in $(\omega^\omega, <)$ is also selectable in $(\omega_1, <)$, and we can choose for it a selector that works in both;

(b) a first-order formula which is not selectable in $(\omega^\omega, <)$ is not even *satisfied* in $(\omega_1, <)$.

For these results, see section 7.

We end by relating our work to the *Church synthesis problem* (Definition 8.3). In their seminal paper [BL69], after stating their main theorem, which proves the computability of the synthesis problem in $(\omega, <)$, Büchi and Landweber write:

> "We hope to present elsewhere a corresponding extension of [our main theorem] from ω to any countable ordinal."

We will see, however, that the formula $\theta_{\omega\,\mathrm{ub}}$ is a counter-example to a full extension of their theorem to $(\omega^\omega, <)$. We also provide a brief discussion of what is known and what remains to be studied with respect to its extension to ordinals above ω.

Finally, this paper focuses on matters related to selection in a *particular* structure (of the form $(\alpha, <)$). But, it is also interesting to consider selection over classes of structures:

DEFINITION 1.3. Let $\varphi(\bar{Y})$, $\psi(\bar{Y})$ be formulas and $\mathcal{C}$ a class of structures. We say that ψ *selects* (or, is a *selector* for) φ *over* $\mathcal{C}$ iff ψ selects φ in every $\mathcal{M} \in \mathcal{C}$. $\mathcal{C}$ is said to have the *selection property* iff every formula has a selector over $\mathcal{C}$.

In [RS] we employ the results of the current paper to explore the selection property and the selection problem over classes of countable ordinals. We also show there the solvability of a restricted version of the *uniformization problem* in a countable ordinal, which is a a natural generalization of the selection problem (see Definition 2.22).

§2. Preliminaries and background.

2.1. Notations and terminology. We use n, k, l, m, p, q for natural numbers, $\alpha, \beta, \gamma, \delta, \zeta, \mu$ for ordinals. Our ordinals are von Neumann ordinals: an ordinal is *identical* with the set of all ordinals below it. In particular, $0 = \varnothing$, $1 = \{0\} = \{\varnothing\}$, $2 = \{0, 1\}$, etc. $\omega = \{0, 1, 2, \dots\}$ is the set of natural numbers. ω_1 is the first uncountable ordinal. We write $\alpha + \beta$, $\alpha\beta$, α^β for the sum, multiplication and exponentiation, respectively, of ordinals α and β.

We denote by `On` and `Lim` the classes of all ordinals and of limit ordinals, respectively.

For sets A, B, we denote by ${}^{B}A$ the set of all functions from B into A. Also, ${}^{\omega>}A := \bigcup_{n\in\omega} {}^{n}A$.

If $f : B \to A$ and $B' \subseteq B$, $f[B'] = f''B' := \{f(b') \mid b' \in B'\}$ is the image of B' under f.

We use the expressions "*chain*" and "*linear order*" interchangeably. We use standard notation for sub-intervals of a chain: if $(A, <)$ is a chain and $b < a$ are in A, we write $(b, a) := \{c \in A \mid b < c < a\}$, $[b, a) := (b, a) \cup \{b\}$, etc.

We use the symbol '$\cong$' for isomorphism.

2.2. The Monadic Logic of Order (MLO). The presentation of MLO given here is similar to the one given in [Th97], and is equivalent to the standard definition of MLO which allows both first-order and second-order variables over the signature '$<$'.

2.2.1. *Syntax.* The vocabulary of MLO consists of *monadic* second-order variables $\{X_i \mid i \in \omega\}$, binary relation symbols '$<$' and '$\subseteq$', and unary relation symbols 'Sing' and 'Emp'. Atomic formulas take the form $X_i < X_j, X_i \subseteq X_j, \mathrm{Sing}(X_i), \mathrm{Emp}(X_i)$. All other formulas are built up from these by means of the usual boolean connectives and second-order quantifiers $\exists X_i$, $\forall X_i$. The *quantifier depth* of a formula φ is defined as usual and denoted by $\mathrm{qd}(\varphi)$.

We use upper case letters X, Y, Z,... to denote variables; with an overline, $\bar{X}$, $\bar{Y}$, etc. to denote finite tuples of variables (always assumed distinct). We denote by $\lg(\bar{X})$ the length of a tuple $\bar{X}$, that is, the number of variables appearing in it.

2.2.2. *Semantics.* A *structure* is a tuple $\mathcal{M} := (A, <^{\mathcal{M}}, \bar{P}^{\mathcal{M}})$ where: A is a non-empty set, $<^{\mathcal{M}}$ is a binary relation on A, and $\bar{P}^{\mathcal{M}} := \langle P_0^{\mathcal{M}}, \dots, P_{l-1}^{\mathcal{M}} \rangle$ is a *finite* tuple of sub*sets* of A.

If $\lg(\bar{P}^{\mathcal{M}}) = l$, we call $\mathcal{M}$ an *l-structure*. If $<^{\mathcal{M}}$ linearly orders A, we call $\mathcal{M}$ an *l-chain*. When the specific l is unimportant, we simply say that $\mathcal{M}$ is a *labeled chain*.

Suppose $\mathcal{M}$ is an l-structure and φ a formula with free-variables among $X_0, \dots, X_{l-1}$. We define the relation $\mathcal{M} \models \varphi$ (read: $\mathcal{M}$ *satisfies* φ) as follows: $\mathcal{M} \models \mathrm{Emp}(X_i)$ iff $P_i^{\mathcal{M}} = \varnothing$; $\mathcal{M} \models \mathrm{Sing}(X_i)$ iff $P_i^{\mathcal{M}}$ is a singleton; $\mathcal{M} \models X_i \subseteq X_j$ iff $P_i^{\mathcal{M}} \subseteq P_j^{\mathcal{M}}$; and $\mathcal{M} \models X_i < X_j$ iff there are $b <^{\mathcal{M}} a$ in A with $P_i^{\mathcal{M}} = \{b\}$, $P_j^{\mathcal{M}} = \{a\}$. The boolean connectives are handled as usual and quantifiers range over *subsets* of A.

Let $\mathcal{M}$ be an l-structure. The *monadic theory* of $\mathcal{M}$, $\mathrm{MTh}(\mathcal{M})$, is the set of all formulas with free-variables among $X_0, \dots, X_{l-1}$ satisfied by $\mathcal{M}$.

From now on, we omit the superscript '$\mathcal{M}$' in '$<^{\mathcal{M}}$' and '$\bar{P}^{\mathcal{M}}$'. We often write $(A, <) \models \varphi(\bar{P})$ meaning $(A, <, \bar{P}) \models \varphi$. Note also the following notations endemic to this paper:

Notation 2.1. Let $\mathcal{M} := (A, <, \bar{P})$ be a structure, $\bar{Q} \in {}^{\omega>}\mathcal{P}(A)$. The *expansion* of $\mathcal{M}$ by $\bar{Q}$ is $\mathcal{M}^\frown\bar{Q} := (A, <, \bar{P}, \bar{Q})$, where we write '$\bar{P}, \bar{Q}$' meaning the tuple obtained by concatenating $\bar{P}$ and $\bar{Q}$.

Definition 2.2. (1) Let $l_1, l_2 \in \omega$, $\mathcal{M}$ an l_1-structure, $\varphi(\bar{X})$ a formula with $\lg(\bar{X}) = l_1 + l_2$. The *relation defined by φ in $\mathcal{M}$* is

$$\mathcal{D}(\varphi, \mathcal{M}) := \{\bar{Q} \in {}^{l_2}\mathcal{P}(\mathrm{dom}(\mathcal{M})) \mid \mathcal{M}^\frown\bar{Q} \models \varphi\}.$$

(2) Of every $\bar{Q} \in \mathcal{D}(\varphi, \mathcal{M})$ we say that it *satisfies* φ in $\mathcal{M}$.

(3) If $\mathcal{D}(\varphi, \mathcal{M}) \neq \varnothing$, we say that φ is *satisfied* in $\mathcal{M}$.

(4) If $\mathcal{D}(\varphi, \mathcal{M})$ is a singleton $\{\bar{Q}\}$, we say that φ *defines* $\bar{Q}$ in $\mathcal{M}$ and that $\bar{Q}$ is *definable* in $\mathcal{M}$.

2.2.3. *First-order formulas.*

DEFINITION 2.3. Let $\varphi(\bar{Y})$ be a formula. We call a quantifier appearing in φ a *first-order quantifier* iff it has the form $\exists X(\mathrm{Sing}(X) \wedge \dots)$ or $\forall X(\mathrm{Sing}(X) \rightarrow \dots)$.

We say that φ is a *first-order formula* iff all quantifiers appearing in φ are first-order.

Note that the *free* variables in φ are still second-order, so that φ is satisfied (or not) by (tuples of) subsets of the domain. When we want to indicate that a variable is to range only over elements of the domain (equivalently, over singletons), we use lower case letters x, y, etc. For instance, "Let $\varphi(x, Y)$ be a formula..." means that φ has the form "$\mathrm{Sing}(X) \wedge \varphi'(X, Y)$". In this context, we use obvious abbreviations such as "$x \in Y$" for "$\mathrm{Sing}(X) \wedge X \subseteq Y$".

2.2.4. *Relativization.*

NOTATION 2.4. Let $\bar{P} := \langle P_0, \dots, P_{l-1} \rangle$ be a finite tuple and D any set. The $\check{\ }$*-intersection* of $\bar{P}$ and D is $\bar{P} \check{\cap} D := \langle P_0 \cap D, \dots, P_{l-1} \cap D \rangle$.

NOTATION 2.5. Let $\mathcal{M} := (A, <, \bar{P})$ be a structure and D a non-empty subset of A. The *restriction* of $\mathcal{M}$ to D is the structure $\mathcal{M}_{\restriction D} := (D, <, \bar{P} \check{\cap} D)$.

LEMMA 2.6 (Relativization). *Let $\varphi(\bar{Y})$ be a formula, U a variable not appearing in φ. We can compute a formula $\varphi_{\restriction U}(\bar{Y}, U)$ such that for every $\lg(\bar{Y})$-structure $\mathcal{M}$ and every non-empty subset D of its domain,*

$$\mathcal{M}^\frown D \models \varphi_{\restriction U}(\bar{Y}, U) \text{ iff } \mathcal{M}_{\restriction D} \models \varphi(\bar{Y}),$$

that is, $\varphi_{\restriction U}$ holds in $\mathcal{M}$ with U interpreted as D iff φ holds in the restriction of $\mathcal{M}$ to D.

When this is the case, we say that φ holds in $\mathcal{M}$ relativized *to D.*

We are mostly interested in the case where $\mathcal{M}$ is a labeled chain and D is an interval $[b, a)$ for some $b < a$ in $\mathcal{M}$.

2.3. Elements of the composition method. Our proofs make use of the technique known as the *composition method*, introduced in [FV59] and adapted (and ingeniously applied) to the case of MLO in [Sh75]. To fix notations and to aid the reader not familiar with this technique, we briefly review those definitions and results that we require. A more detailed presentation can be found in [Th97] or [Gu85], for instance.

2.3.1. *Hintikka formulas and n-types.*

NOTATION 2.7. Let $n, l \in \omega$. Denote by $\mathfrak{Form}_{n,l}$ the set of formulas of quantifier depth $\leq n$ and with free variables among $X_0, \dots, X_{l-1}$.

DEFINITION 2.8. Let $n, l \in \omega$ and $\mathcal{M}, \mathcal{N}$ be l-structures. We say that $\mathcal{M}$ and $\mathcal{N}$ are *n-equivalent*, denoted $\mathcal{M} \equiv^n \mathcal{N}$, iff for every $\varphi \in \mathfrak{Form}_{n,l}$, $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$.

Clearly, $\equiv^n$ is an equivalence relation. For any $n \in \omega$ and $l > 0$, the set $\mathfrak{Form}_{n,l}$ is infinite. However, it contains only finitely many semantically distinct formulas. So, there are finitely many $\equiv^n$-classes of l-structures. In fact, we can compute "representatives" for these classes:

Lemma 2.9 (Hintikka Lemma). *For $n, l \in \omega$, we can compute a* finite $H_{n,l} \subseteq \mathfrak{Form}_{n,l}$ *such that:*

(*a*) *For every l-structure $\mathcal{M}$, there is a* unique $\tau \in H_{n,l}$ *such that* $\mathcal{M} \models \tau$.[1]

(*b*) *If $\tau \in H_{n,l}$ and $\varphi \in \mathfrak{Form}_{n,l}$, then either $\tau \models \varphi$ or $\tau \models \neg\varphi$. Furthermore, there is an algorithm that, given such τ and φ, decides which of these two possibilities holds.*

*Any member of $H_{n,l}$ we call an (n, l)-*Hintikka formula.[2]

Definition 2.10 (n-type). For $n, l \in \omega$ and $\mathcal{M}$ an l-structure, we denote by $\text{type}^n(\mathcal{M})$ the unique member of $H_{n,l}$ satisfied by $\mathcal{M}$ and call it the *n-type* of $\mathcal{M}$.

Thus, $\text{type}^n(\mathcal{M})$ determines (effectively) which formulas of quantifier-depth $\leq n$ are satisfied by $\mathcal{M}$.

2.3.2. *The ordered sum of labeled chains and of n-types.*

Definition 2.11. (1) Let $l \in \omega$, let $\mathcal{I} := (I, <^{\mathcal{I}})$ be a linear order and let $\mathfrak{S} := \langle \mathcal{M}_\alpha \mid \alpha \in I \rangle$ be a sequence of l-chains. Write $\mathcal{M}_\alpha := (A_\alpha, <^\alpha, \bar{P}^\alpha)$ and assume $A_\alpha \cap A_\beta = \varnothing$ whenever $\alpha \neq \beta$ are in I. The *ordered sum* of $\mathfrak{S}$ w.r.t. $\mathcal{I}$ is the l-chain

$$\sum_{\mathcal{I}} \mathfrak{S} := (\bigcup_{\alpha \in I} A_\alpha, <^{\mathcal{I},\mathfrak{S}}, \langle \bigcup_{\alpha \in I} P_0^\alpha, \ldots, \bigcup_{\alpha \in I} P_{l-1}^\alpha \rangle)$$

where:

if $\alpha, \beta \in I$, $a \in A_\alpha$, $b \in A_\beta$, then $b <^{\mathcal{I},\mathfrak{S}} a$ iff $\beta < \alpha$ or $\beta = \alpha$ and $b <^\alpha a$.

If the domains of the $\mathcal{M}_\alpha$ are not disjoint, replace them with isomorphic l-chains that have disjoint domains, and proceed as before.

(2) If for all $\alpha \in I$, $\mathcal{M}_\alpha \cong \mathcal{M}$ for some *fixed* $\mathcal{M}$, we denote $\sum_{\mathcal{I}} \mathfrak{S}$ by $\mathcal{M} \otimes \mathcal{I}$.

(3) If $\mathcal{I} = (2, <)$ and $\mathfrak{S} = \langle \mathcal{M}_0, \mathcal{M}_1 \rangle$, we denote $\sum_{\mathcal{I}} \mathfrak{S}$ by $\mathcal{M}_0 + \mathcal{M}_1$.

The next proposition says that taking ordered sums preserves n-equivalence.

Proposition 2.12. *Let $n, l \in \omega$. Assume:*

1. *$(I, <)$ is a linear order,*
2. *$\langle \mathcal{M}_\alpha^0 \mid \alpha \in I \rangle$ and $\langle \mathcal{M}_\alpha^1 \mid \alpha \in I \rangle$ are sequences of l-chains, and*
3. *for every $\alpha \in I$, $\mathcal{M}_\alpha^0 \equiv^n \mathcal{M}_\alpha^1$.*

Then $\sum_{\alpha \in I} \mathcal{M}_\alpha^0 \equiv^n \sum_{\alpha \in I} \mathcal{M}_\alpha^1$.

This allows us to define the sum of formulas in $H_{n,l}$ with respect to any linear order.

Definition 2.13. (1) Let $n, l \in \omega$, $\mathcal{I} := (I, <)$ a chain, $\mathfrak{H} := \langle \tau_\alpha \mid \alpha \in I \rangle$ a sequence of (n, l)-Hintikka formulas. The *ordered sum* of $\mathfrak{H}$ w.r.t. $\mathcal{I}$, denoted $\sum_{\mathcal{I}} \mathfrak{H}$, is *an* element of $H_{n,l}$ such that:

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[1]Equivalently, $\bigvee H_{n,l}$ is a tautology and for *distinct* $\tau, \tau' \in H_{n,l}$, $\tau \wedge \tau'$ is *not* satisfiable.

[2]Hintikka formulas made their first appearance in [Hi53], in the framework of first-order logic.

if $\mathfrak{S} := \langle \mathcal{M}_\alpha \mid \alpha \in I \rangle$ is a sequence of l-chains and $\text{type}^n(\mathcal{M}_\alpha) = \tau_\alpha$ for all $\alpha \in I$, then

$$\text{type}^n(\sum_{\mathcal{I}} \mathfrak{S}) = \sum_{\mathcal{I}} \mathfrak{H}.^3$$

(2) If for all $\alpha \in I$, $\tau_\alpha = \tau$ for some *fixed* $\tau \in H_{n,l}$, we denote $\sum_{\mathcal{I}} \mathfrak{H}$ by $\tau \otimes \mathcal{I}$.
(3) If $\mathcal{I} = (2, <)$ and $\mathfrak{H} = \langle \tau_0, \tau_1 \rangle$, we denote $\sum_{\mathcal{I}} \mathfrak{H}$ by $\tau_0 + \tau_1$.

We are usually interested in cases (2) and (3) above. For *addition* (case 3) we have:

LEMMA 2.14 (Addition Lemma). *$\lambda n, l \in \omega . \lambda \tau_0, \tau_1 \in H_{n,l} . \tau_0 + \tau_1$ is recursive.*

For multiplication, we have the following fundamental result of Shelah's ([Sh75]):

THEOREM 2.15. *There is a recursive $\mathfrak{c} : \omega \times \omega \to \omega$ such that for any $n, l \in \omega$ and chain $\mathcal{I}$, given $\tau \in H_{n,l}$ and $\text{type}^{\mathfrak{c}(n,l)}(\mathcal{I})$, we can compute $\tau \otimes \mathcal{I}$.*

In particular, $\lambda n, l \in \omega . \lambda \tau \in H_{n,l} . \tau \otimes \mathcal{I}$ is recursive in the monadic theory of $\mathcal{I}$.

2.4. The monadic theory of countable ordinals. Büchi (for instance [BS73]) has shown that there is a *finite* amount of data concerning any ordinal $\leq \omega_1$ which determines its monadic theory:

THEOREM 2.16. *Let $\alpha \in [1, \omega_1]$. Write $\alpha = \omega^\omega \beta + \zeta$ where $\zeta < \omega^\omega$ (this can be done in a unique way). Then the monadic theory of $(\alpha, <)$ is determined by:*

1. *whether α is countable or $\alpha = \omega_1$,*
2. *whether $\alpha < \omega^\omega$, and*
3. *ζ.*

We can associate with every $\alpha \leq \omega_1$ a finite *code* which holds the data required in the previous theorem. This is clear with respect to (1) and (2). As for (3), if $\zeta \neq 0$, write

$\zeta = \sum_{i \leq n} \omega^{n-i} \cdot a_{n-i}$, where $a_i \in \omega$ for $i \leq n$ and $a_n \neq 0$

(this, too, can be done in a unique way), and let the sequence $\langle a_n, \dots, a_0 \rangle$ encode ζ. The following is then implicit in [BS73]:

THEOREM 2.17 (Code Theorem). *There is an algorithm that, given a sentence φ and the* code *of an $\alpha \in [1, \omega_1]$, determines whether $(\alpha, <) \models \varphi$.*

AGREEMENT. From now on, we shall simply say that an algorithm is "given an ordinal..." or "returns an ordinal...". We always mean the *code* of the ordinal.

From the last theorem and 2.15, one obtains:

THEOREM 2.18 (Multiplication Theorem). *There exists an algorithm that, given $n, l \in \omega$, $\tau \in H_{n,l}$ and $\alpha \in [1, \omega_1]$, computes $\tau \otimes \alpha$.*

Finally, we often use the following facts concerning definability below ω^ω.

[3] The reader may wonder why we do not say: "$\sum_{\mathcal{I}} \mathfrak{H}$ is the *unique* element of $H_{n,l}$ such that...". This is because there are in $H_{n,l}$ formulas that are not satisfied in any structure. If one of these apppears in $\mathfrak{H}$, then any $\tau \in H_{n,l}$ could serve as the sum.

Lemma 2.19 (Definability-below-ω^ω Lemma). *For any $\alpha < \omega^\omega$, we can compute* first-order *sentences $\theta^{\text{def}}_\alpha$ and $\theta^{\text{def}}_{<\alpha}$ such that for every $\beta \in \mathtt{On} \setminus 1$:*

(*a*) $(\beta, <) \models \theta^{\text{def}}_\alpha$ *iff* $\beta = \alpha$.
(*b*) $(\beta, <) \models \theta^{\text{def}}_{<\alpha}$ *iff* $\beta < \alpha$.

2.5. Uniformization.

Definition 2.20 (Uniformization). Let $\varphi(\bar{X}, \bar{Y})$, $\psi(\bar{X}, \bar{Y})$ be formulas and $\mathcal{M}$ a structure. We say that ψ *uniformizes* (or, is a *uniformizer* for) φ *in* $\mathcal{M}$ iff:

1. $\mathcal{M} \models \forall \bar{X} \exists^{\leq 1} \bar{Y} \psi(\bar{X}, \bar{Y})$,
2. $\mathcal{M} \models \forall \bar{X} \forall \bar{Y} (\psi(\bar{X}, \bar{Y}) \rightarrow \varphi(\bar{X}, \bar{Y}))$, and
3. $\mathcal{M} \models \forall \bar{X} \big(\exists \bar{Y} \varphi(\bar{X}, \bar{Y}) \rightarrow \exists \bar{Y} \psi(\bar{X}, \bar{Y})\big)$.

$\mathcal{M}$ has the *uniformization property* iff every formula φ has a uniformizer ψ over $\mathcal{M}$.

Note that selection is the special case of uniformization where $\bar{X}$ is the empty tuple. In [LiS98], Lifsches and Shelah show that an ordinal α has the uniformization property iff $\alpha < \omega^\omega$. Thus, in particular, any ordinal $\alpha < \omega^\omega$ has the selection property. Moreover, from their proof an algorithm can be extracted as follows:

Proposition 2.21 (Uniformization below ω^ω). *There is an algorithm that, given $k \in \omega$ and $\varphi(\bar{X}, \bar{Y})$, computes a $\psi(\bar{X}, \bar{Y})$ that uniformizes φ in $(\alpha, <)$ for every $\alpha \in \omega^k \setminus 1$.*

Now, uniformization, too, naturally leads to a decision problem:

Definition 2.22 (Uniformization problem). The *uniformization problem* in a structure $\mathcal{M}$ is:

Input: a formula $\varphi(\bar{X}, \bar{Y})$.
Task: determine whether φ has a uniformizer in $\mathcal{M}$, and if so, construct one for it.

Using results from this paper, we show in [RS] the solvability of the uniformization problem in $(\omega^\omega, <)$ when the $\bar{X}$ variables are restricted to range over subsets of ω^ω of order-type $< \delta$ for some fixed $\delta < \omega^\omega$. Solving (or showing the insolvability of) the full uniformization problem in $(\omega^\omega, <)$ remains an interesting open problem.

§3. The selection property. In this section we show that an ordinal α has the selection property iff $\alpha < \omega^\omega$. Of course, by Proposition 2.21, it suffices to show that selection fails in each $\alpha \geq \omega^\omega$. We begin by looking at the ordinals ω^ω and ω_1. In subsection 3.1, we show that a tuple definable in either of these ordinals is *periodic* (Definition 3.3). Since a formula φ (which is satisfied in $(\alpha, <)$) is selectable in $(\alpha, <)$ iff it satisfied by a definable tuple, we infer that if φ is selectable in $(\omega^\omega, <)$ or $(\omega_1, <)$ then it is satisfied by a periodic tuple. We note that every unbounded periodic sub*set* of ω^ω must have order-type ω^ω, to obtain the non-selectability of an unbounded ω-sequence in $(\omega^\omega, <)$. Using similar considerations, we present a formula $\theta_{\text{stat}}(Y)$ not selectable in $(\omega_1, <)$. Since ω_1 is a definable ordinal, we deduce that θ_{stat} itself is not selectable in *every* $\alpha \geq \omega_1$. Finally, to show $\alpha \in (\omega^\omega, \omega_1)$ lacks the selection property, we note that any such α is monadically equivalent to a β of the form $\beta = \omega^\omega + \zeta$ with $\zeta < \omega^\omega$, and that ω^ω is definable in $\omega^\omega + \zeta$.

3.1. The Periodicity Lemma. The next few definitions and lemmas (up to and including 3.5) are repeatedly used in the sequel.

DEFINITION 3.1 (Part and tail). Let μ, α be ordinals with $\mu > 0$. Write $\alpha = \mu\beta + \zeta$ with $\zeta < \mu$ (this can be done in a unique way). We call $\mu\beta$ the *μ-part* of α and ζ its *μ-tail*. We also write $(\alpha \bmod \mu)$ for ζ.

DEFINITION 3.2 (Periodic functions). Let $\alpha_0, \alpha_1, \alpha' \in \mathtt{On}$, $t_0 : \alpha_0 \to T$ and $t_1 : \alpha_1 \to T$ where T is any set.

1. Define $t_0 + t_1 : \alpha_0 + \alpha_1 \to T$. For all $\beta < \alpha_0 + \alpha_1$,
$$(t_0 + t_1)(\beta) := \begin{cases} t_0(\beta) & \text{if } \beta < \alpha_0 \\ t_1(\beta - \alpha_0) & \text{if } \beta \geq \alpha_0 \end{cases} .$$
2. Define $(t_1 \otimes \alpha') : \alpha_1 \cdot \alpha' \to T$ by: $(t_1 \otimes \alpha')(\beta) := t_1(\beta \bmod \alpha_1)$, for all $\beta < \alpha_1 \cdot \alpha'$.
3. Suppose $\alpha \in \mathtt{On}$, $t : \alpha \to T$ and there exist $\alpha_0 < \alpha$, $\alpha_1 \in \alpha \setminus 1$, $\alpha' \leq \alpha$, $t_0 : \alpha_0 \to T$ and $t_1 : \alpha_1 \to T$ such that $t = t_0 + t_1 \otimes \alpha'$. Then we say that t is *periodic*.
4. We call (α_0, α_1) a *phase and period* for t.[4]
5. If μ is an ordinal such that $\alpha_0, \alpha_1 < \mu$, we say that t is *μ-periodic*. □

In (5), we will usually have $\mu = \omega^\omega$. Next, we define the notion of a periodic *tuple* of subsets of an ordinal.

DEFINITION 3.3 (Periodic tuple). Let $l \in \omega$, $\alpha \in \mathtt{On}$ and $\bar{P} \in {}^l\mathcal{P}(\alpha)$. The *characteristic function of $\bar{P}$ in α* is $\Im^\alpha_{\bar{P}} : \alpha \to {}^l 2$ given by:
$$\Im^\alpha_{\bar{P}}(\beta) := \langle \sigma_0(\beta), \dots, \sigma_{l-1}(\beta) \rangle \text{ where for all } \beta < \alpha \text{ and } i < l,$$
$$\sigma_i(\beta) := \begin{cases} 1 & \text{if } \beta \in P_i \\ 0 & \text{else} \end{cases} .$$
We shall apply to $\bar{P}$ all terms and notations of Definition 3.2, referring to $\Im^\alpha_{\bar{P}}$. Thus, we speak of $\bar{P}$ being *periodic*, of a *phase and period* for $\bar{P}$, etc.

The following is a special case of Theorem 3.5(B) of [Sh75] (only our language is different):

LEMMA 3.4 ($\mathfrak{p}$-Lemma). *For every $n \in \omega$, there is a natural number $\mathfrak{p}(n) < |H_{n,0}|$ such that the function $\lambda\alpha \in \omega_1 . \operatorname{type}^n(\alpha, <)$ is periodic with phase and period $(\omega^{\mathfrak{p}(n)}, \omega^{\mathfrak{p}(n)})$. In particular, any two non-0 countable multiples of $\omega^{\mathfrak{p}(n)}$ are n-equivalent.*

Now, every multiple of ω^ω is a multiple of $\omega^{\mathfrak{p}(n)}$ for each $n \in \omega$. Thus,

LEMMA 3.5 (Tail Lemma). *If α,$\beta \in [\omega^\omega, \omega_1)$ have the same ω^ω-tail, then* $\mathrm{MTh}(\alpha, <) = \mathrm{MTh}(\beta, <)$.

The following is a standard fact about n-types.

[4] Thus, t may have many phases and periods.

LEMMA 3.6. *Let $n, l \in \omega$ and let be $\mathcal{M}$ a structure. Then* $\mathrm{type}^n(\mathcal{M})$ *determines* $\mathrm{type}^n(\mathcal{M}^\frown\langle \underbrace{\varnothing, \dots, \varnothing}_{l \text{ times}} \rangle)$. *In fact, the latter is computable from the former.*

We need the following lemma only for the case $\alpha = \omega^\omega$ or $\alpha = \omega_1$, but since this involves no significant effort, we prove something slightly more general.

LEMMA 3.7 (Periodicity Lemma). *Let $\alpha \in [\omega^\omega, \omega_1]$ and suppose $\bar{P} \in {}^l\mathcal{P}(\alpha)$ is definable in $(\alpha, <)$ by a formula of quantifier depth $\leq n$. If α is a multiple of ω^ω, then $\bar{P}$ is periodic with phase and period $(\omega^{\mathfrak{p}(n+l)}, \omega^{\mathfrak{p}(n+l)})$.*

PROOF. We show that for every $\beta \in \alpha \setminus 1$, the $(n+l)$-type of $(\beta, <)$ determines $\Im^\alpha_{\bar{P}}(\beta)$.[5] Since, by the $\mathfrak{p}$-Lemma, $\lambda\beta \in \alpha.\, \mathrm{type}^{n+l}(\beta, <)$ is periodic with phase and period $(\omega^{\mathfrak{p}(n+l)}, \omega^{\mathfrak{p}(n+l)})$, this will finish the proof.

Let $\psi(\bar{Y})$ be a formula defining $\bar{P}$ in $(\alpha, <)$ with $\mathrm{qd}(\psi) \leq n$. Let $\sigma := \langle \sigma_0, \dots, \sigma_{l-1} \rangle$ be a binary l-tuple (that is, $\sigma : l \to 2$). Then there is a formula $\theta_\sigma(x)$ of quantifier depth $\leq n+l$ such that for every $\beta < \alpha$, $(\alpha, <) \models \theta_\sigma(\beta)$ iff $\Im^\alpha_{\bar{P}}(\beta) = \sigma$. Indeed,

$$\theta_\sigma(x) := \exists \bar{Y}\big(\psi(\bar{Y}) \wedge \bigwedge_{i<l, \sigma_i=0} x \notin Y_i \wedge \bigwedge_{i<l, \sigma_i=1} x \in Y_i\big).$$

Of course, whether or not $(\alpha, <) \models \theta_\sigma(\beta)$ is determined by the $(n+l)$-type of $(\alpha, <, \{\beta\})$. But, for any $\beta \in \alpha \setminus 1$ we have, by definition of the sum of $(n+l)$-types (see 2.13),

$$\mathrm{type}^{n+l}(\alpha, <, \{\beta\}) = \mathrm{type}^{n+l}(\beta, <, \varnothing) + \mathrm{type}^{n+l}([\beta, \alpha), <, \{\beta\}).$$

We claim that $\mathrm{type}^{n+l}([\beta, \alpha), <, \{\beta\})$ is independent of β. Indeed, fix $\beta \in \alpha \setminus 1$. Assume first α is countable. Since α is a multiple of ω^ω, so is the order-type of $[\beta, \alpha)$. By the Tail Lemma, $([\beta, \alpha), <)$ is monadically equivalent to $(\alpha, <)$. For $\phi(X)$ any formula, let $\theta^{\min}_\phi$ be a sentence saying: "if x is the minimal element of the domain, then $\phi(\{x\})$ holds." Then $([\beta, \alpha), <, \{\beta\}) \models \phi$ iff $([\beta, \alpha), <) \models \theta^{\min}_\phi$ iff $(\alpha, <) \models \theta^{\min}_\phi$. But, $\mathrm{MTh}(\alpha, <)$ is, of course, independent of β. If $\alpha = \omega_1$, we even have $([\beta, \alpha), <, \{\beta\}) \cong (\omega_1, <, \{0\})$.

Thus, $\Im^\alpha_{\bar{P}}(\beta)$ is determined by $\mathrm{type}^{n+l}(\beta, <, \varnothing)$, which Lemma 3.6 tells us is in turn determined by $\mathrm{type}^{n+l}(\beta, <)$. This finishes the proof. ⊣

3.2. Failure of the selection property in $\alpha \geq \omega^\omega$.

NOTATION 3.8. Let P be a set of ordinals. Then P is well-ordered by the usual ordering of ordinals. The *order-type* of P, denoted $\mathrm{otp}(P)$, is the unique ordinal isomorphic to $(P, <)$.

LEMMA 3.9. *If α is a non-0 multiple of ω^ω and $P \subseteq \alpha$ is ω^ω-periodic, then* $\sup(P) < \omega^\omega$ *or* $\mathrm{otp}(P) = \alpha$.

PROOF. Suppose $\alpha_0, \alpha_1 < \omega^\omega$ are a phase and period for P. Since α is a multiple of ω^ω, $\alpha - \alpha_0 = \alpha$ and $\alpha_1\alpha = \alpha$. Thus, $(\alpha, <, P) \cong (\alpha, <, P)_{\restriction[0,\alpha_0)} + (\alpha, <, P)_{\restriction[\alpha_0, \alpha_0+\alpha_1)} \otimes \alpha$. If $P \cap [\alpha_0, \alpha_0 + \alpha_1) = \varnothing$, then $\sup(P) \leq \alpha_0 < \omega^\omega$. If $\beta_1 \in P \cap [\alpha_0, \alpha_0 + \alpha_1)$, then $\{\alpha_0 + \alpha_1\gamma + \beta_1 \mid \gamma \in \alpha\}$ has order-type α and is a subset of P, so $\mathrm{otp}(P) \geq \alpha$. But, $P \subseteq \alpha$, so $\mathrm{otp}(P) = \alpha$. ⊣

[5]Recall that $\Im^\alpha_{\bar{P}}$ is the characteristic function of $\bar{P}$ in α.

From the last lemma and the Periodicity Lemma, we immediately obtain:

COROLLARY 3.10. *The formula* $\theta_{\omega\,\mathrm{ub}}(Y)$ *which says "Y is an unbounded* ω*-sequence" has no selector in* $(\omega^\omega, <)$.

To handle ω_1 recall the following definitions:

DEFINITION 3.11. (1) Let $C \subseteq \omega_1$. C is called:
closed iff for every limit $\beta < \omega_1$, if $\sup(C \cap \beta) = \beta$, then $\beta \in C$.[6]
a *club* iff C is closed and unbounded in ω_1.
(2) $S \subseteq \omega_1$ is called *stationary* iff for every club $C \subseteq \omega_1$, $S \cap C \neq \varnothing$.

Note that being a club and being stationary are definable properties of a subset of ω_1. In defining clubs we do not even use second order-quantification, and in defining stationary subsets we quantify over clubs.

COROLLARY 3.12. *Let* $\theta(Y)$ *say: "both* Y *and its complement are stationary". Then* θ *is* not *selectable in* $(\omega_1, <)$.

PROOF. It takes an amount of the Axiom of Choice to prove that $\mathcal{D}(\theta, \omega_1) \neq \varnothing$ (that is, that there are stationary co-stationary subsets of ω_1).[7] But, using AC it can even be shown that there is a family of ω_1 *disjoint* stationary subsets of ω_1. For a proof see, for instance, [HJ99]. By the Periodicity Lemma, it suffices to show that any periodic $P \subseteq \omega_1$ either contains a club or is non-stationary. Indeed, let $\alpha_0, \alpha_1 < \omega_1$ be a phase and a period for P, respectively. Assume $\alpha_0 \in P$. Then, by periodicity, $C := \{\alpha_0 + \alpha_1\beta \mid \beta < \omega_1\} \subseteq P$. It is obvious C is unbounded. Also, if $\beta < \omega_1$ is a limit point of C then we can pick an increasing $\langle \beta_i \mid i \in \omega \rangle \subseteq \beta$ such that $\beta = \sup\{\alpha_0 + \alpha_1\beta_i \mid i \in \omega\}$. But, then $\beta = \alpha_0 + \alpha_1 \cdot \sup\{\beta_i \mid i \in \omega\}$ is a member of C, so C is also closed. Similarly, if $\alpha_0 \notin P$, then $\{\alpha_0 + \alpha_1\beta \mid \beta < \omega_1\} \subseteq \omega_1 \setminus P$ is a club. ⊣

Now, the reduction from any other ordinal $\alpha \geq \omega^\omega$ to the case of ω^ω or ω_1 is based on the following simple observation.

DEFINITION 3.13. Let $(A, <)$ be a linear order. Call $S \subseteq A$ a *segment* of $(A, <)$ if $(b, a) \subseteq S$ whenever $b < a$ are in S.

LEMMA 3.14 (Segment Lemma). *Let* $\mathcal{M}$ *be a labeled chain and* ψ *a formula which defines a tuple* $\bar{P}$ *in* $\mathcal{M}$. *Let* S *be a segment of* $\mathcal{M}$. *Then* $\mathrm{type}^{\mathrm{qd}(\psi)}((\mathcal{M}^\frown \bar{P})_{\restriction S})$ *defines* $\bar{P}\check{\cap}S$ *in* $\mathcal{M}_{\restriction S}$.

PROOF. Set $n := \mathrm{qd}(\psi)$ and let $\bar{P}'$ satisfy $\mathrm{type}^n((\mathcal{M}^\frown \bar{P})_{\restriction S})$ in $\mathcal{M}_{\restriction S}$. We must show that $\bar{P}' = \bar{P}\check{\cap}S$.

Write $S^- := \{b \in \mathrm{dom}(\mathcal{M}) \mid \forall a \in S. b < a\}$ and $S^+ := \mathrm{dom}(\mathcal{M}) \setminus (S^- \cup S)$. Both are segments of $\mathcal{M}$. Assume S^- and S^+ are nonempty. Then

$$\mathrm{type}^n(\mathcal{M}^\frown \bar{P}) = \mathrm{type}^n((\mathcal{M}^\frown \bar{P})_{\restriction S^-}) + \mathrm{type}^n((\mathcal{M}^\frown \bar{P})_{\restriction S}) + \mathrm{type}^n((\mathcal{M}^\frown \bar{P})_{\restriction S^+}).$$

By assumption, $\mathrm{type}^n((\mathcal{M}^\frown \bar{P})_{\restriction S}) = \mathrm{type}^n(\mathcal{M}_{\restriction S}{}^\frown \bar{P}')$, so

[6] That is, C is closed under taking sup.

[7] It is consistent with Zeremelo-Frankel set theory that $\mathcal{D}(\theta, \omega_1) = \varnothing$, but ZF+"$\mathcal{D}(\theta, \omega_1) = \varnothing$" is a rather strange set theory. In section 5 of [BS73] the reader will find an enlightening discussion of the relation between "$\mathcal{D}(\theta, \omega_1) = \varnothing$" and definability in $(\omega_1, <)$.

$$(\mathcal{M}^\frown \bar{P})_{\restriction S^-} + \mathcal{M}_{\restriction S}{}^\frown \bar{P}' + (\mathcal{M}^\frown \bar{P})_{\restriction S^+} \models \psi.$$

But, this structure equals $\mathcal{M}^\frown(\bar{P}\check{\cap}S^- \cup \bar{P}' \cup \bar{P}\check{\cap}S^+)$. Since ψ defines $\bar{P}$ in $\mathcal{M}$, it follows $(\bar{P}\check{\cap}S^- \cup \bar{P}' \cup \bar{P}\check{\cap}S^+) = \bar{P}$, so $\bar{P}' = \bar{P}\check{\cap}S$.

Finally, if $S^- = \varnothing$ ignore the leftmost "summand" above, and if $S^+ = \varnothing$, ignore the rightmost "summand". ⊣

Note that for every $\alpha \geq \omega_1$, we can define the segment $[0, \omega_1)$ in $(\alpha, <)$ by saying: "x is in this segment iff for every limit ordinal y smaller than or equal to x, there is an unbounded ω-sequence in $[0, y)$". Therefore, the formula $\theta_{\mathrm{stat}}(Y)$ appearing below does exist.

Proposition 3.15. *Let $\theta_{\mathrm{stat}}(Y)$ say: "Both $Y \cap \omega_1$ and $\omega_1 \setminus Y$ are stationary in ω_1". Then θ_{stat} has no selector in $(\alpha, <)$ for* every *$\alpha \geq \omega_1$.*

Proof. Suppose $P \subseteq \alpha$ satisfies θ_{stat} in $(\alpha, <)$. Then $P \cap \omega_1$ satisfies the θ of Corollary 3.12 in $(\omega_1, <)$, hence is not definable in $(\omega_1, <)$. By the Segment Lemma (applied to the initial segment $\omega_1 = [0, \omega_1)$), P is not definable in $(\alpha, <)$. ⊣

To handle $\alpha \in (\omega^\omega, \omega_1)$ note first that the monadic theory of α "knows" exactly which formulas select which others in $(\alpha, <)$:

Definition 3.16 (Selection axiom). For formulas $\varphi(\bar{Y}), \psi(\bar{Y})$, the (ψ, ϕ)-*selection axiom*, denoted sel-ax(ψ, φ), is the conjunction of the sentences appearing in the definition of selection (Definition 1.1).

Clearly, ψ selects φ in $(\alpha, <)$ iff $(\alpha, <) \models$ sel-ax(ψ, φ). By the Tail Lemma, this means that to show that in $\alpha \in [\omega^\omega, \omega_1)$ selection fails, it suffices we handle α of the form $\alpha = \omega^\omega + \zeta$ where $\zeta < \omega^\omega$.

Proposition 3.17. *Let $k \in \omega \setminus 1$. There is a formula $\theta_k(Y)$ such that for all $\zeta \in \omega^k \setminus 1$, θ_k is* not *selectable in $(\omega^\omega + \zeta, <)$.*

Proof. Let $\theta_k(Y)$ say:

> "If x is the *least* such that the order-type of $[x, \infty)$ is smaller than ω^k, then $Y \cap [0, x)$ is an unbounded ω-sequence in [0,x)."

Suppose $\zeta \in [1, \omega^k)$. Then ω^ω is the least $\beta < \omega^\omega + \zeta$ such that otp$([\beta, \omega^\omega + \zeta)) < \omega^k$. Therefore, if P satisfies θ_k in $(\omega^\omega + \zeta, <)$, then $P \cap \omega^\omega$ is an unbounded ω-sequence in ω^ω. By Corollary 3.10, $P \cap \omega^\omega$ is not definable in $(\omega^\omega, <)$. By the Segment Lemma, neither is P definable in $(\alpha, <)$. ⊣

Summing up, we have indeed shown:

Theorem 3.18 (Selection property). *An ordinal α has the selection property iff $\alpha < \omega^\omega$.*

In [LiS98], Lifsches and Shelah present a particular formula $\varphi(X, Y)$ not *uniformizable* in every $\alpha \geq \omega^\omega$ (whether countable or not). Above, a formula not *selectable* in every $\alpha \geq \omega_1$ was presented. Is there a *single* formula not selectable in *every* $\alpha \in [\omega^\omega, \omega_1)$? The answer turns out to be negative:

Proposition 3.19. *Let $k \in \omega$ and $\varphi(\bar{Y})$ a formula with* qd$(\exists \bar{Y} \varphi) \leq k$. *Then for every $\zeta \in [\omega^{\mathfrak{p}(k)}, \omega^\omega)$, φ is selectable in $(\omega^\omega + \zeta, <)$.*[8]

[8]The selector may depend on ζ, of course.

Because it requires some preparation, we postpone the proof to section 6.

§4. The selection problem. In this section we show the solvability of the selection problem in $(\alpha, <)$ for $\alpha \leq \omega_1$. For $\alpha < \omega^\omega$ this is handled by Proposition 2.21. In the previous section we have seen that any formula which is (satisfied and) selectable in $(\omega^\omega, <)$ or $(\omega_1, <)$ is satisfied by a periodic tuple. Here we show that: (1) conversely, a formula satisfied by a periodic tuple is selectable; (2) it is decidable whether a formula is satisfied by a periodic tuple. We handle (2) in a way that allows us to construct a selector for φ when one exists.

4.1. Solvability of the selection problem in $(\alpha, <)$ for $\alpha \in [\omega^\omega, \omega_1]$. The following is not hard to derive from the $\mathfrak{p}$-Lemma.

LEMMA 4.1 (Satisfiability Lemma). *(a) Let $k \in \omega$ and $\varphi(\bar{Y})$ a formula such that $\mathrm{qd}(\exists \bar{Y} \varphi) \leq k$. If φ is satisfiable in some countable ordinal, then φ is also satisfied in some $\alpha < \omega^{\mathfrak{p}(k)} \cdot 2$.*

(b) There is an algorithm that, given k and φ as above, decides whether φ is satisfied in a countable ordinal, and if so, returns an $\alpha < \omega^{\mathfrak{p}(k)} \cdot 2$ in which φ is satisfied.

PROPOSITION 4.2. *Let $n, l \in \omega$, $\varphi \in \mathfrak{Form}_{n,l}$ and $\alpha \in [\omega^\omega, \omega_1]$ a multiple of ω^ω. Suppose φ is satisfied in $(\alpha, <)$. Then the following are equivalent:*

(a) φ is selectable in $(\alpha, <)$.
(b) There exists an ω^ω-periodic $\bar{P} \in \mathcal{D}(\varphi, \alpha)$.
(c) There are $\tau_0, \tau_1 \in H_{n,l}$ such that:
 (c_1) τ_0, τ_1 are satisfiable in a countable ordinal,
 (c_2) $\tau_0 + \tau_1 \otimes \alpha \models \varphi$.[9]

Furthermore, there is an algorithm that, given τ_0, τ_1 as in (c), constructs a selector $\psi_{(\tau_0, \tau_1)}$ for φ in $(\alpha, <)$.

PROOF. $(a) \Rightarrow (b)$ is the Periodicity Lemma.

$(b) \Rightarrow (c)$: Let $\bar{P} \in \mathcal{D}(\varphi, \alpha)$ be periodic and assume α_0, α_1 are a phase and period $< \omega^\omega$ for $\bar{P}$. Write

$$\tau_0 := \mathrm{type}^n((\alpha, <, \bar{P})_{\restriction [0, \alpha_0)}), \tau_1 := \mathrm{type}^n((\alpha, <, \bar{P})_{\restriction [\alpha_0, \alpha_0 + \alpha_1)}).$$

As in the proof of Lemma 3.9,

$$(\alpha, <, \bar{P}) \cong (\alpha, <, \bar{P})_{\restriction [0, \alpha_0)} + (\alpha, <, \bar{P})_{\restriction [\alpha_0, \alpha_0 + \alpha_1)} \otimes \alpha,$$

so $\mathrm{type}^n(\alpha, <, \bar{P}) = \tau_0 + \tau_1 \otimes \alpha$. Since $(\alpha, <, \bar{P}) \models \varphi$, this means $\tau_0 + \tau_1 \otimes \alpha \models \varphi$.

$(c) \Rightarrow (a)$: Fix τ_0, τ_1 as in (c). We construct $\psi_{(\tau_0, \tau_1)}$ as follows. Note that the construction depends only on τ_0, τ_1 and *not* on α. Fix $i \in 2$:

1. By (c_1), τ_i is satisfiable in a countable ordinal. By the Satisfiability Lemma, we can compute an $\alpha_i \in \omega^\omega \setminus 1$ in which τ_i is satisfiable.
2. Apply Proposition 2.21 to compute a selector $\psi_i(\bar{Y})$ for τ_i in $(\alpha_i, <)$. Denote by $\bar{P}_i$ the unique element of $\mathcal{D}(\psi_i, \alpha_i)$.
3. By the Definability-below-ω^ω Lemma, we can compute a sentence $\theta^{\mathrm{def}}_{\alpha_i}$ which is satisfied only in $(\alpha_i, <)$.

[9] Multiplication by α is a bit misleading here. It only matters whether α is countable or ω_1. Indeed, if α is countable, then by the Tail Lemma, α is monadically equivalent to ω^ω. Thus, by Theorem 2.15, $\tau_1 \otimes \alpha = \tau_1 \otimes \omega^\omega$.

Now,

4. $M := \{\alpha_0 + \alpha_1\beta \mid \beta < \alpha\}$ is definable in $(\alpha, <)$. Indeed, M is the unique unbounded subset of α such that:
   $\min(M) = \alpha_0$,
   for any $\beta \in M$, if β' is the successor of β in M, then $\mathrm{otp}([\beta, \beta')) = \alpha_1$, and
   M is closed under limits.
   Therefore, M is defined by the formula $\theta_M(W)$ saying:
   "W is unbounded; $\theta^{\mathrm{def}}_{\alpha_0}$ holds relativized to $[0, \min(W))$; for every $x \in W$, $\theta^{\mathrm{def}}_{\alpha_1}$ holds relativized to $[x, y)$, where y is the successor of x in W; and W is closed under limits."
5. Finally, let $\psi_{(\tau_0,\tau_1)}(\bar{Y})$ say: "for every (i.e. the unique) W that satisfies $\theta_M(W)$,
   ψ_0 holds relativized to the segment $[0, \min(W))$, and
   for every $x \in W$, ψ_1 holds relativized to $[x, y)$, where y is the successor of x in W."

Then, by choice of ψ_0, ψ_1, we have that $\psi_{(\tau_0,\tau_1)}$ defines in $(\alpha, <)$ the unique $\bar{P}$ such that

$$(\alpha, <, \bar{P}) \cong (\alpha_0, <, \bar{P}_0) + (\alpha_1, <, \bar{P}_1) \otimes \alpha.$$

But, $\mathrm{type}^n(\alpha_0, <, \bar{P}_0) = \tau_0$, $\mathrm{type}^n(\alpha_1, <, \bar{P}_1) = \tau_1$ and $\tau_0 + \tau_1 \otimes \alpha \models \varphi$, by (c_2). Hence,

$$(\alpha_0, <, \bar{P}_0) + (\alpha_1, <, \bar{P}_1) \otimes \alpha \models \varphi,$$

and all is well. $\dashv$

An immediate corollary is:

Corollary 4.3. *For $\alpha \in \{\omega^\omega, \omega_1\}$, there is an algorithm that, given a formula $\varphi(\bar{Y})$, decides whether φ has a selector ψ in $(\alpha, <)$, and if so, constructs such a ψ.*

Proof. By the Code Theorem, we can decide whether φ is satisfied in $(\alpha, <)$. If φ is not satisfiable, then $Y_0 \neq Y_0$ (say) selects φ in $(\alpha, <)$. Assume henceforth φ is satisfied in $(\alpha, <)$. Let $n := \mathrm{qd}(\varphi)$, $l := \lg(\bar{Y})$ and denote by $\mathrm{Sat}^{<\omega^\omega}_{n,l}$ the set of (n, l)-Hintikka formulas satisfiable in some countable ordinal. By the Satisfiability Lemma, $\mathrm{Sat}^{<\omega^\omega}_{n,l}$ is computable. Let

$$\mathrm{Sel}^\alpha_\varphi := \{(\tau_0, \tau_1) \in \mathrm{Sat}^{<\omega^\omega}_{n,l} \times \mathrm{Sat}^{<\omega^\omega}_{n,l} \mid \tau_0 + \tau_1 \otimes \alpha \models \varphi\}.$$

It is immediate from the "$(a) \Leftrightarrow (c)$" equivalence of the previous proposition that φ is selectable in $(\alpha, <)$ iff $\mathrm{Sel}^\alpha_\varphi \neq \varnothing$. But, $\mathrm{Sel}^\alpha_\varphi$ is computable. Indeed, fix $\tau_0, \tau_1 \in \mathrm{Sat}^{<\omega^\omega}_{n,l}$. By the Multiplication Theorem, we can compute $\tau_1 \otimes \alpha$. By the Addition Lemma, we can then compute $\tau_0 + \tau_1 \otimes \alpha$. Finally, by the Hintikka Lemma, we can decide whether $\tau_0 + \tau_1 \otimes \alpha \models \varphi$. It follows that we can decide whether φ has a selector in $(\alpha, <)$.

Now, if $\mathrm{Sel}^\alpha_\varphi \neq \varnothing$, pick any $(\tau_0, \tau_1) \in \mathrm{Sel}^\alpha_\varphi$. By the previous proposition, we can compute from (τ_0, τ_1) a selector for φ in $(\alpha, <)$. $\dashv$

As in the previous section, there remains the simple task of reducing the selection problem in $\alpha \in (\omega^\omega, \omega_1)$ to the selection problem in $(\omega^\omega, <)$.

THEOREM 4.4 (Solvability of selection). *There exists an algorithm that, given $\alpha \in [\omega^\omega, \omega_1]$ and a formula $\varphi(\bar{Y})$, decides whether φ has a selector in $(\alpha, <)$, and if so, constructs one for it.*

PROOF. If α is a multiple of ω^ω, then either $\alpha = \omega_1$ or α is monadically equivalent to ω^ω. In each case, the previous corollary does the job. Otherwise, by the Tail Lemma, we may assume $\alpha = \omega^\omega + \zeta$ for $\zeta \in \omega^\omega \setminus 1$. Again, we can decide if φ is satisfied in $(\alpha, <)$ and may assume this is the case. Set $n := \mathrm{qd}(\varphi)$, $l := \lg(\bar{Y})$. Let $\mathrm{Sel}^{\omega^\omega}_{n,l}$ be the set of (n,l)-Hintikka formulas that are both *satisfiable* and *selectable* in $(\omega^\omega, <)$. By the previous corollary, we can compute $\mathrm{Sel}^{\omega^\omega}_{n,l}$. Let $\mathrm{Sat}^{\zeta}_{n,l}$ be the set of (n,l)-Hintikka formulas satisfied in $(\zeta, <)$. By the Code Theorem, $\mathrm{Sat}^{\zeta}_{n,l}$ is computable. We claim that φ has a selector in $(\alpha, <)$ iff

(Sel): there are $\tau \in \mathrm{Sel}^{\omega^\omega}_{n,l}$ and $\tau' \in \mathrm{Sat}^{\zeta}_{n,l}$ such that $\tau + \tau' \models \varphi$.

Furthermore, given such τ, τ', we can compute a selector for φ in $(\alpha, <)$.

Suppose first $\bar{P} \in \mathcal{D}(\varphi, \alpha)$ is definable in $(\alpha, <)$. If we let

$$\tau := \mathrm{type}^n((\alpha, <, \bar{P})_{\restriction[0,\omega^\omega)}) \text{ and } \tau' := \mathrm{type}^n((\alpha, <, \bar{P})_{\restriction[\omega^\omega,\alpha)}),$$

then $\tau + \tau' \models \varphi$ and $\tau' \in \mathrm{Sat}^{\zeta}_{n,l}$. By the Segment Lemma (3.14), $\bar{P} \cap \omega^\omega$ is definable in $(\omega^\omega, <)$. But, $\bar{P} \cap \omega^\omega$ satisfies τ in $(\omega^\omega, <)$, so $\tau \in \mathrm{Sel}^{\omega^\omega}_{n,l}$.

Assume conversely that (Sel) holds. By the previous corollary, we can compute a Ψ that selects τ in $(\omega^\omega, <)$. By Proposition 2.21, we can compute a Ψ' that selects τ' in $(\zeta, <)$. As in the proof of 3.17, we can define ω^ω in $(\alpha, <)$. Thus, the $\psi(\bar{Y})$ that says: "Ψ holds relativized to $[0, \omega^\omega)$ and Ψ' holds relativized to $[\omega^\omega, \infty)$" selects φ in $(\alpha, <)$.

Finally, by the Addition Lemma and the Hintikka Lemma, it is decidable whether (Sel) holds. ⊣

4.2. Idempotents and selection. In this subsection we reexamine the construction of the selector $\psi_{(\tau_0,\tau_1)}$ in the "$(c) \Rightarrow (a)$" direction of Proposition 4.2, to extract data that will turn out useful in section 7 and is also interesting in itself.

Recall that $\psi_{(\tau_0,\tau_1)}$ says: "for the unique W that satisfies $\theta_M(W)$, τ_0 holds relativized to $[0, \min(W))$, and τ_1 holds relativized to $[x, y)$ whenever x, y are successive elements of W." As noted in the proof of 4.2, the construction of $\psi_{(\tau_0,\tau_1)}$ did not depend on the α given in that proposition. In showing that $\psi_{(\tau_0,\tau_1)}$ selects φ in $(\alpha, <)$, we did use the assumption that we can pick $\alpha_0, \alpha_1 < \min\{\alpha, \omega^\omega\}$ such that:

1. for $i \in 2$, τ_i is satisfiable in $(\alpha_i, <)$.
2. $\alpha - \alpha_0$ is a multiple of α_1, and
3. $\tau_0 + \tau_1 \otimes \frac{\alpha - \alpha_0}{\alpha_1} \models \varphi$.

Note that from assumptions (1) and (2) alone it follows that $\psi_{(\tau_0,\tau_1)}$ *defines* in $(\alpha, <)$ a tuple $\bar{P}$ such that $\mathrm{type}^n(\alpha, <, \bar{P}) = \tau_0 + \tau_1 \frac{\alpha - \alpha_0}{\alpha_1}$. Thus,

LEMMA 4.5. *Let $n, l \in \omega$ and $\tau_0, \tau_1 \in H_{n,l}$ both satisfiable in some countable ordinal. Then we can compute a formula $\psi_{(\tau_0,\tau_1)}$ such that for every non-0 multiple α of $\omega^{\mathfrak{p}(n+l)+1}$, $\psi_{(\tau_0,\tau_1)}$ selects $\tau_0 + \tau_1 \frac{\alpha - \alpha_0}{\alpha_1}$ in $(\alpha, <)$.*

Proof. By the Satisfiability Lemma we can pick $\alpha_0, \alpha_1 < \omega^{\mathfrak{p}(n+l)+1}$ such that for $i \in 2$, τ_i is satisfied in $(\alpha_i, <)$. Now, let $\psi_{(\tau_0,\tau_1)}$ be constructed as in the proof of 4.2. Fix $\beta \in \mathtt{On} \setminus 1$ and let $\alpha := \omega^{\mathfrak{p}(n+l)+1}\beta$. Then $\alpha \geq \omega^{\mathfrak{p}(n+l)} \cdot 2$ implies $\alpha_0, \alpha_1 < \alpha$. Since α is a multiple of $\omega^{\mathfrak{p}(n+l)+1}$, so is $\alpha - \alpha_0 = \alpha$. Thus, $\alpha - \alpha_0$ is divisible by α_1. ⊣

We needed assumption (3) to show that the $\bar{P}$ defined by $\psi_{(\tau_0,\tau_1)}$ satisfies φ in $(\alpha, <)$. We now show that we can pick τ_1 to ensure that $\psi_{(\tau_0,\tau_1)}$ will select φ in every *countable* multiple of $\omega^{\mathfrak{p}(n+l)+1}$.

Definition 4.6. Let $n, l \in \omega$, $\tau \in H_{n,l}$ and $\alpha \in \mathtt{On} \setminus 1$. We call τ an *α-idempotent* iff $\tau \otimes \alpha = \tau$.

Lemma 4.7. *An ω-idempotent is an α-idempotent for every $\alpha \in \omega_1 \setminus 1$.*

Proof. Let $n, l \in \omega$ and $\tau \in H_{n,l}$ an ω-idempotent. We proceed by induction on $\alpha \in \omega_1 \setminus 1$. For $\alpha = 1$, there is nothing to prove. Assume $\alpha \in \omega_1 \setminus 2$ and our claim holds for all $\alpha' < \alpha$. If $\alpha = \alpha' + 1$ is a successor, then $\Phi \otimes (\alpha' + 1) = \Phi \otimes \alpha' + \Phi \overset{\text{induction}}{=} \Phi + \Phi \overset{\text{assumption}}{=} \Phi + \Phi \otimes \omega = \Phi \otimes (1 + \omega) = \Phi \otimes \omega = \Phi$. Assume α is limit. Pick a strictly increasing $\langle \beta_i \mid i \in \omega \rangle \subseteq \alpha$ with $\sup\{\beta_i \mid i \in \omega\} = \alpha$ and $\beta_0 := 0$. For $i \in \omega$, let $\alpha_i := \beta_{i+1} - \beta_i$. Then $\alpha = \sum_{i\in\omega} \alpha_i$, so $\Phi \otimes \alpha = \sum_{i\in\omega} \Phi \otimes \alpha_i \overset{\text{induction}}{=} \sum_{i\in\omega} \Phi = \Phi \otimes \omega = \Phi$. ⊣

It turns out that when φ is selectable, we can pick an ω-idempotent τ_1.

Proposition 4.8. *Let $n, l \in \omega$, $\varphi \in \mathfrak{Form}_{n,l}$ and α a non-0* countable *multiple of ω^ω. Assume φ is satisfied in $(\alpha, <)$. Then the following are equivalent:*

(*a*) *φ is selectable in $(\alpha, <)$.*
(*b*) *There exist $\tau_0, \tau_1 \in H_{n,l}$ such that:*
  (b_1) *τ_0, τ_1 are satisfiable in a countable ordinal,*
  (b_2) *$\tau_0 + \tau_1 \models \varphi$, and*
  (b_3) *τ_1 is an ω-idempotent.*

Furthermore, when this is the case, we can compute τ_0, τ_1 as in (*b*).

Proof. We show that (*b*) here holds iff clause (*c*) of Proposition 4.2 holds. Assume (*b*) here holds. Then (c_1) is (b_1). By the previous lemma, $\tau_0 + \tau_1 = \tau_0 + \tau_1 \otimes \alpha$ and (c_2) follows. Assume (*c*) there holds, i.e. there are $\tau_0', \tau_1' \in H_{n,l}$ both satisfiable in a countable ordinal such that $\tau_0' + \tau_1' \otimes \alpha \models \varphi$. Let $\tau_0 := \tau_0'$ and $\tau_1 := \tau_1' \otimes \alpha$. Then ($b_2$) is immediate. Also, if $\alpha_1' < \omega^\omega$ and τ_1' is satisfied in $(\alpha_1', <)$, then τ_1 is satisfied in $(\alpha_1'\alpha, <)$. Hence, (b_1) holds. Finally, α and $\alpha\omega$ have the same monadic theory. By Theorem 2.15, this means $\tau_1 = \tau_1' \otimes \alpha = \tau_1' \otimes (\alpha\omega) = (\tau_1' \otimes \alpha) \otimes \omega = \tau_1 \otimes \omega$.

By the proof of Corollary 4.3, when (*c*) of 4.2 holds, we can compute τ_0', τ_1' as above. Since $\tau_1' \otimes \alpha$ is computable from τ_1', the "furthermore" holds. ⊣

Corollary 4.9. *There is an algorithm that, given a formula $\varphi(\bar{Y})$ selectable in $(\omega^\omega, <)$, constructs a ψ which selects φ in $\omega^{\mathfrak{p}(\mathrm{qd}(\exists \bar{Y}\varphi))+1}\beta$ for* every *$\beta \in \omega_1 \setminus 1$.*

Proof. Let $n := \mathrm{qd}(\varphi)$ and $l := \mathrm{lg}(\bar{Y})$. Compute $\tau_0, \tau_1 \in H_{n,l}$ as in (*b*) of the last proposition. The desired formula is the $\psi_{(\tau_0,\tau_1)}$ of Lemma 4.5. ⊣

§5. Comparing difficulty of selection in a countable ordinal.

5.1. Selection degrees. We know that the formula $\theta_{\omega\,\mathrm{ub}}(Y)$ saying "Y is an unbounded ω-sequence" has no selector in $(\omega^\omega, <)$. Now, let us look at the formula $\theta_{\omega^2\,\mathrm{ub}}$ saying "Y is unbounded and of order type ω^2". It is immediate from Lemma 3.9, that $\theta_{\omega^2\,\mathrm{ub}}$ has no selector in $(\omega^\omega, <)$. But are there any other interesting relations between these two formulas? Can we say, for instance, that $\theta_{\omega^2\,\mathrm{ub}}$ is even "harder" to select than $\theta_{\omega\,\mathrm{ub}}$ (whatever that might mean)? Or, perhaps the other way round? Do we feel that the example of $\theta_{\omega^2\,\mathrm{ub}}$ "contains a new idea" when it comes to our discussion of selection?

To turn these admittedly vague questions into mathematical ones, we require a notion of comparing formulas and perhaps an equivalence relation on them. But, as our example shows, semantical equivalence seems *not* to be the right notion. Note, however, the following. For any unbounded ω^2-sequence $S_2 \subseteq \omega^\omega$, the set of limit points of S_2 (i.e., those $\alpha < \omega^\omega$ such that $\sup(S_2 \cap \alpha) = \alpha$) is an unbounded ω-sequence. Also, this set is *definable* from S_2. On the other hand, given an unbounded ω-sequence $S_1 \subseteq \omega^\omega$, the set $\{\alpha + n \mid \alpha \in S_1, n \in \omega\}$ is an unbounded ω^2-sequence. And, again the latter set is definable from S_1. The example suggests the following definition:

Definition 5.1 (Reduction). Let $\varphi_0(\bar{Y})$, $\varphi_1(\bar{X})$ be formulas and $\mathcal{M}$ a structure. We say that φ_0 is *easier to select* than φ_1 *in* $\mathcal{M}$ (in symbols: $\varphi_0 \preceq_{\mathcal{M}} \varphi_1$) iff *either*:

1. φ_0 is selectable, *or*
2. φ_1 is satisfied in $\mathcal{M}$ and there exists a formula $\psi(\bar{X}, \bar{Y})$ such that for all $\bar{P} \in \mathcal{D}(\varphi_1, \mathcal{M})$,
$$\psi(\bar{P}, \bar{Y}) \text{ selects } \varphi_0 \text{ in } \mathcal{M}.$$

We call ψ a *reduction* of φ_0 to φ_1 in $\mathcal{M}$.

φ_0, φ_1 are *equally easy to select over* $\mathcal{M}$ (in symbols: $\varphi_0 \sim_{\mathcal{M}} \varphi_1$) iff both $\varphi_0 \preceq_{\mathcal{M}} \varphi_1$ and $\varphi_1 \preceq_{\mathcal{M}} \varphi_0$.

We leave the proof of the following easy lemma to the reader.

Lemma 5.2. *Let $\mathcal{M}$ be a structure. Then:*

(*a*) *$\preceq_{\mathcal{M}}$ is a reflexive, transitive relation on the formulas.*
(*b*) *$\sim_{\mathcal{M}}$ is an equivalence relation on the formulas.*

Definition 5.3 (Degrees). Let $\mathcal{M}$ be a structure. We call every $\sim_{\mathcal{M}}$-equivalence class an *$\mathcal{M}$-degree.*

The main result of this section is:

Theorem 5.4. *For every $\alpha \in [\omega^\omega, \omega_1)$, there are two degrees in $(\alpha, <)$: one consisting of all formulas selectable in $(\alpha, <)$ and the other consisting of all non-selectable formulas.*

Note first that,

Lemma 5.5. *All formulas selectable in a structure $\mathcal{M}$ form one $\mathcal{M}$-degree.*

Proof. It is immediate from (1) of Definition 5.1 that all selectable formulas share a degree. It remains to be seen that any formula easier than a selectable formula is itself selectable. Suppose then that $\psi(\bar{X},\bar{Y})$ reduces some formula $\varphi_0(\bar{Y})$ to $\varphi_1(\bar{X})$, where φ_1 is selectable in $\mathcal{M}$. Let $\Psi(\bar{Y})$ be any selector for φ_1 in $\mathcal{M}$. Then $\exists\bar{X}(\Psi(\bar{X})\wedge\psi(\bar{X},\bar{Y}))$ selects φ_0 in $\mathcal{M}$. ⊣

It remains to be seen that all non-selectable formulas are equally easy to select in $(\alpha,<)$. To prove this, we show in subsection 5.3, that the formula $\theta^{\alpha}_{\omega\,\mathrm{ub}}(Y)$ saying "Y is an unbounded ω-sequence in the ω^ω-part of α" is easier to select than every *non*-selectable formula. Then, in subsection 5.4, we show that every formula is easier than $\theta^{\alpha}_{\omega\,\mathrm{ub}}$.

Note that $\mathrm{MTh}(\alpha,<)$ holds all information concerning the degrees in $(\alpha,<)$.

LEMMA 5.6. *Let $\varphi_0(\bar{Y}),\varphi_1(\bar{X})$ be formula, and $\mathcal{M},\mathcal{M}'$ monadically equivalent structures. Then φ_0 is easier than φ_1 to select in $\mathcal{M}$ iff it is easier than φ_1 to select in $\mathcal{M}'$. When this is the case, the same formulas are reductions of φ_0 to φ_1 in $\mathcal{M}$ as are in $\mathcal{M}'$.*

PROOF. Because the monadic theory of a structure determines which formulas are selectable in it, φ_0 is selectable in $\mathcal{M}$ iff it is selectable in $\mathcal{M}'$. Similarly, φ_1 is satisfiable in $\mathcal{M}$ iff it is satisfiable in $\mathcal{M}'$. Finally, for any formula $\psi(\bar{X},\bar{Y})$,

$$\mathcal{M}\models\forall\bar{X}(\varphi_1(\bar{X})\rightarrow\text{sel-ax}(\psi(\bar{X},\bar{Y}),\varphi_0(\bar{Y})))$$

iff $\mathcal{M}'$ satisfies this sentence. Looking again at Definition 5.1, it is clear that our claim is proved. ⊣

Thus, it suffices we handle α of the form $\omega^\omega+\zeta$ for $\zeta<\omega^\omega$. In practice, we shall handle the case of ω^ω and leave the easy generalization to other $\omega^\omega+\zeta$ to the reader.

Note finally that in proving Theorem 5.4, we shall be solving a restricted case of the uniformization problem in a countable ordinal (recall Definition 2.22). Indeed, if $\varphi_0(\bar{Y})$ and $\varphi_1(\bar{X})$ are both satisfiable in a structure $\mathcal{M}$, then φ_0 is easier than φ_1 to select in $\mathcal{M}$ iff $\varphi_1(\bar{X})\wedge\varphi_0(\bar{Y})$ has a uniformizer in $\mathcal{M}$.

We have seen that tuples definable in $(\omega^\omega,<)$ are "combinatorially simple" - they are periodic. Tuples satisfying non-selectable formulas, however, are never periodic (by Proposition 4.2). To handle such tuples, we will use a somewhat more refined tool of "combinatorial analysis", namely, Büchi's notion of an automaton acting on ordinal words and his translation of formulas to automata. We begin, therefore, by recalling the relevant definitions.

5.2. Automata on ordinal words.

DEFINITION 5.7. Let $\alpha\in\mathtt{On}\setminus 1$, $\rho:\alpha\rightarrow Q$ with Q any set. The *limit* of ρ is

$$\lim(\rho):=\{q\in Q\mid\rho^{-1}(q)\text{ is unbounded in }\alpha\}.$$

We call the elements of $\lim(\rho)$ *limit states* of ρ.

DEFINITION 5.8 (Automata). 1. An *automaton* is a tuple $\mathfrak{A}:=\langle Q,\Sigma,\delta_{\mathrm{suc}},\delta_{\mathrm{lim}}\rangle$ where:
- Q and Σ are *finite* non-empty sets,
- $\delta_{\mathrm{suc}}:Q\times\Sigma\rightarrow\mathcal{P}(Q)\setminus 1$, and

- $\delta_{\lim} : \mathcal{P}(Q) \to \mathcal{P}(Q) \setminus 1$.[10]

2. We call Q the *states* of $\mathfrak{A}$, Σ its *input alphabet*, δ_{suc} the *successor transition function* and $\delta_{\lim}$ the *limit transition function*.
3. We call $\mathfrak{A}$ *deterministic* iff for all $q \in Q$ and $\sigma \in \Sigma$, $|\delta_{\text{suc}}(\mathfrak{q}, \sigma)| = 1$ and for all $K \subseteq Q$, $|\delta_{\lim}(K)| = 1$. In that case, we assume that $\delta_{\text{suc}} : Q \times \Sigma \to Q$ and $\delta_{\lim} : \mathcal{P}(Q) \to Q$.
4. An *initialized automaton* is a pair $\mathcal{A} := \langle \mathfrak{A}; q_I \rangle$ with $\mathfrak{A}$ as above and $q_I \in Q$.

Definition 5.9 (Runs). Let $\mathcal{A} := \langle Q, \Sigma, \delta_{\text{suc}}, \delta_{\lim}; q_I \rangle$ be an initialized automaton. Let $\alpha \in \mathtt{On}$, $\Im : \alpha \to \Sigma$.

1. A *run* of $\mathcal{A}$ *on* $\Im$ is a function ρ such that:
   - $\text{rng}(\rho) \subseteq Q$,
   - if α is zero or successor, $\text{dom}(\rho) = \alpha + 1$,
   - if α is limit, $\text{dom}(\rho) \in \{\alpha, \alpha + 1\}$,

   and, for each $\beta \in \text{dom}(\rho)$:
   - $\rho(0) = q_I$,
   - if $\beta + 1 \in \text{dom}(\rho)$, $\rho(\beta + 1) \in \delta_{\text{suc}}(\rho(\beta), \Im(\beta))$,[11] and
   - if β is limit, $\rho(\beta) \in \delta_{\lim}(\lim(\rho|_\beta))$.[12]
2. We call ρ a *strict* run in all cases *but* when α is limit and $\text{dom}(\rho) = \alpha + 1$.
3. If $\mathcal{A}$ is deterministic there exists a unique strict run of $\mathcal{A}$ on $\Im$, denoted by $\text{run}_{\mathcal{A}}(\Im)$.
4. We call ρ a $\text{dom}(\rho)$-*run* to indicate its length.

So far we have automata that can run on ordinal words. To allow them to define classes of such words, we must provide them with acceptance conditions.

Definition 5.10. (Muller automata)

1. A *Muller automaton* is a tuple $\mathfrak{A} := \langle Q, \Sigma, \delta_{\text{suc}}, \delta_{\lim}; \mathcal{F} \rangle$ where $\langle Q, \Sigma, \delta_{\text{suc}}, \delta_{\lim} \rangle$ is an automaton and $\mathcal{F} \subseteq \mathcal{P}(Q)$.
2. We call $\mathcal{F}$ the *acceptance condition* of $\mathfrak{A}$.
3. An *initialized Muller automaton* is a pair $\mathcal{A} := \langle \mathfrak{A}; q_I \rangle$ where $\mathfrak{A}$ is a Muller automaton and $q_I \in Q$.
4. Let $\alpha \in \mathtt{On}$, $\Im : \alpha \to \Sigma$. We say that $\Im$ is *accepted* by $\mathcal{A}$ iff there exists a *strict* run ρ of $\mathcal{A}$ on $\Im$ with $\lim(\rho) \in \mathcal{F}$. We call such a run *accepting*.

Definition 5.11. Let φ be a formula with free variables $Y_0, \dots, Y_{l-1}$, $\mathcal{A}$ an initialized Muller automaton with input alphabet ${}^l 2$ and $\mathcal{C}$ a class of ordinals. We say that $\mathcal{A}$ and φ are *equivalent over* $\mathcal{C}$ iff for every $\alpha \in \mathcal{C}$ and $\bar{P} \in {}^l\mathcal{P}(\alpha)$,

$$\bar{P} \in \mathcal{D}(\varphi, \alpha) \text{ iff } \Im^{\alpha}_{\bar{P}} \text{ is accepted by } \mathcal{A}.$$

The following is proved in [BS73], section 4. The proof is far from trivial.

Theorem 5.12 (Büchi, 1973). *There exists an algorithm that, given a formula φ with free-variables $Y_0, \dots, Y_{l-1}$, returns an initialized* deterministic *Muller automaton $\mathcal{A}$ with input ${}^l 2$ which is equivalent to φ over the countable ordinals.*

[10] Our definition is a bit more restrictive than the definition of non-deterministic automata often found in the literature, in that we require that for all $q \in Q$ and $\sigma \in \Sigma$, $\delta_{\text{suc}}(q, \sigma) \neq \varnothing$, and for all $K \subseteq Q$, $\delta_{\lim}(K) \neq \varnothing$.

[11] Or, when $\mathfrak{A}$ is deterministic, $\rho(\beta + 1) = \delta_{\text{suc}}(\rho(\beta), \Im(\beta))$.

[12] Or, when $\mathfrak{A}$ is deterministic, $\rho(\beta) = \delta_{\lim}(\lim(\rho|_\beta))$.

REMARK 5.13. The fact that $\mathcal{A}$ is deterministic will be important when we prove that $\theta_{\omega\,\mathrm{ub}}$ represents the *minimal* non-selectable degree in $(\omega^\omega, <)$. It will not be used when proving that it represents the *maximal* degree.

Note also that the equivalence between MLO formulas and Muller automata, as defined here, breaks in $(\omega_1, <)$. Consult section 6 of [BS73] for more details.[13]

Let $\alpha \in \mathtt{On}\setminus 1$ and $\mathcal{A}$ an initialized automaton. The following lemma essentially says that $\mathrm{MTh}(\alpha, <)$ knows "everything there is to know" concerning the runs of $\mathcal{A}$ on inputs of length α: are there runs taking the automaton into a given state? are there runs with a particular limit? are there runs in which certain states appear while others do not?, etc.

While we only state the existence of a particular formula, it is an easy exercise in formalization to use it (and variants thereof) to "answer" every such question.

LEMMA 5.14. *Let $l \in \omega$, $\mathcal{A}$ an initialized automaton with input ${}^l 2$ and states $Q := \{q_0, \dots, q_{k-1}\}$. Let $q \in Q$. Then there is a formula*

$$\theta^{\text{run-on}}_{\mathcal{A}\to q}(Y_0, \dots, Y_{l-1}, W_0, \dots, W_{k-1},)$$

such that for $\alpha \in \mathtt{On} \setminus 1$, $\bar{P} \in {}^l\mathcal{P}(\alpha)$ and $\bar{Q} \in {}^k\mathcal{P}(\alpha)$:

$(\alpha, <) \models \theta^{\text{run-on}}_{\mathcal{A}\to q}(\bar{P}, \bar{Q})$ iff there exists an $(\alpha+1)$-run ρ of $\mathcal{A}$ on $\Im^{\alpha}_{\bar{P}}$ such that

$\forall i < k . Q_i = \rho^{-1}(q_i) \cap \alpha$ and $\mathrm{last}(\rho) = q$.

Thus, $\theta^{\text{run-on}}_{\mathcal{A}\to q}$ defines (a natural encoding of) the set of $(\alpha+1)$-runs of $\mathcal{A}$ on $\Im^{\alpha}_{\bar{P}}$ that take it into state q. If $\mathcal{A}$ is deterministic, there exists a *unique* $\bar{Q} \in \mathcal{D}(\theta^{\text{run-on}}_{\mathcal{A}\to q}(\bar{P}, \bar{W}), \alpha)$.

REMARK 5.15. In section 7 it will be important that $\theta^{\text{run-on}}_{\mathcal{A}\to q}$ can be taken a *first-order* formula (in the sense of Definition 2.3).

5.3. An unbounded ω-sequence is easier than any non-selectable formula. Suppose $\varphi(\bar{Y})$ is a formula not selectable in $(\omega^\omega, <)$ and $\mathcal{A}$ an automaton equivalent to it over the countable ordinals. Let ρ be an *accepting* ω^ω-run of $\mathcal{A}$ with limit F. Define an ω-sequence $\beta_0, \beta_1, \dots$ as follows: β_0 is the first ordinal after which only limit states of ρ appear in ρ, β_1 is the first β above β_0 such that ρ visits all states of F between β_0 and β, β_2 is the first β' after β_1 such that ρ visits all states of F between β_1 and β', etc. If we let $\beta_\omega := \sup\{\beta_n \mid n \in \omega\}$, then $\lim(\rho|_{\beta_\omega}) = F$. Using the fact that φ is not selectable, we will argue that $\beta_\omega = \omega^\omega$, i.e. that $\beta_0, \beta_1, \dots$ is an *unbounded* ω-sequence. Since the sequence $\beta_0, \beta_1, \dots$ is definable from ρ, this will show that given any tuple satisfying φ, we can define an unbounded ω-sequence, as desired. Below are the details.

We know that φ is selectable in $(\omega^\omega, <)$ iff it is satisfied by a periodic tuple. Let us therefore begin by relating the existence of such a tuple with the existence of periodic accepting runs of $\mathcal{A}$.

NOTATION 5.16. Let S be a set of ordinals and f a function with $S \subseteq \mathrm{dom}(f)$. Then $f \restriction S := f \circ \eta_S$ where $\eta_S : \mathrm{otp}(S) \to S$ is the unique order-preserving isomorphism.

[13] Neeman ([Ne]) defines a generalized notion of finite automaton for which the equivalence holds in *every* ordinal.

LEMMA 5.17. *Let $\varphi(X_0, \dots, X_{l-1})$ be a formula, $\mathcal{A} := \langle \mathfrak{A}; q_I \rangle$ an initialized Muller automaton equivalent to φ over the countable ordinals and α a countable multiple of ω^ω.*

> *If $\mathcal{A}$ has an accepting ω^ω-periodic α-run, then there is an ω^ω-periodic $\bar{P}$ satisfying φ in $(\alpha, <)$.*[14]

PROOF. Let ρ be an accepting ω^ω-periodic α-run of $\mathcal{A}$. Suppose ρ is a run on $\bar{P}'$ for some $\bar{P}' \in {}^l\mathcal{P}(\alpha)$.[15] Let $\alpha_0, \alpha_1 < \omega^\omega$ be a phase and period of ρ, respectively. Let $\bar{P}_0 := \bar{P}' \check{\cap} \alpha_0$, $\bar{P}_1 := \bar{P}' \check{\cap} [\alpha_0, \alpha_0 + \alpha_1)$ and $\bar{P} := \bar{P}_0 + \bar{P}_1 \otimes \alpha$.[16] Then ρ is a run of $\mathcal{A}$ on $\bar{P}$. Since ρ is accepting, $\bar{P} \in \mathcal{D}(\varphi, \alpha)$. ⊣

Our second step is to present a simple "automata-theoretic" criterion for the existence of periodic runs (Lemma 5.23). To do this we need two definitions and two lemmas.

DEFINITION 5.18 (Pumpability). Let $\mathcal{A}$ be an initialized automaton with states Q and limit transition function $\delta_{\lim}$. Let $K \subseteq Q$, $\alpha \in \mathtt{On}$. We say that K is:

1. *stable* in $\mathcal{A}$ iff $K \cap \delta_{\lim}(K) \neq \varnothing$;
2. an *α-limit* in $\mathcal{A}$ iff there exist some limit $\beta < \alpha$ and a β-run ρ of $\mathcal{A}$ such that $\lim(\rho) = K$;
3. *α-pumpable* in $\mathcal{A}$ iff it is both stable and an α-limit in $\mathcal{A}$.

The following easy lemma explains why stable sets are good for us.

LEMMA 5.19. *Let $\mathcal{A}$ be an automaton with states Q and limit transition function $\delta_{\lim}$. Let $K \subseteq Q$. Suppose $\alpha_1 \in \mathtt{Lim}$ and ρ_1 is an α_1-run of $\mathcal{A}$ with $\rho_1(0) \in K \cap \delta_{\lim}(K)$ and $\operatorname{rng}(\rho_1) = \lim(\rho_1) = K$. Then for every $\beta \in \mathtt{On} \setminus 1$, $\rho_1 \otimes \beta$ is a run of $\mathcal{A}$ with $\operatorname{rng}(\rho_1 \otimes \beta) = \lim(\rho_1 \otimes \beta) = K$.*

PROOF. We proceed by induction on β. For $\beta = 1$, there is nothing to show. Let $\beta = \beta' + 1$ be a successor. Then $\rho_1 \otimes \beta = \rho_1 \otimes \beta' + \rho_1$. By the inductive assumption, $\lim(\rho_1 \otimes \beta') = K$. Since $\rho_1(0) \in \delta_{\lim}(K)$, $\rho_1 \otimes \beta' + \rho_1$ is a run of $\mathcal{A}$. Also, $\lim(\rho_1 \otimes \beta) = \lim(\rho_1)$ and $\operatorname{rng}(\rho_1 \otimes \beta) = \operatorname{rng}(\rho_1 \otimes \beta') \cup \operatorname{rng}(\rho_1) = K$. Assume β is limit. It is easy to see that the union of an increasing (by inclusion) sequence of runs of $\mathcal{A}$ is also a run. Thus, $\rho_1 \otimes \beta = \bigcup_{\gamma \in \beta} \rho_1 \otimes \gamma$ is a run. Also, $\operatorname{rng}(\rho_1 \otimes \beta) = \bigcup_{\gamma \in \beta} \operatorname{rng}(\rho_1 \otimes \gamma) = K$. Finally, $\lim(\rho_1 \otimes \beta) = \operatorname{rng}(\rho_1) = K$. ⊣

DEFINITION 5.20 (Fiber). Let $\alpha \in \mathtt{Lim}$ and $\rho : \alpha \to Q$ with $|Q| < \omega$. By induction on $n \in \omega$ define:

> $\beta_0(\rho) := \min\{\beta < \alpha \mid \rho[[\beta, \alpha)] = \lim(\rho)\}$. That is, $\beta_0(\rho)$ is the least ordinal after which all states appearing in ρ are limit states of ρ.
>
> $\beta_{n+1}(\rho) := \min\{\beta \in (\beta_n(\rho), \alpha) \mid \rho[[\beta_n(\rho), \beta)] = \lim(\rho)\}$. That is, $\beta_{n+1}(\rho)$ is the first β after $\beta_n(\rho)$ such that ρ visits all states of $\lim(\rho)$ in the interval $[\beta_n(\rho), \beta)$.

---

[14]We also have that if $\mathcal{A}$ is *deterministic* and $\bar{P} \in {}^l\mathcal{P}(\alpha)$ is ω^ω-periodic, then $\operatorname{run}_{\mathcal{A}}(\Im^{\alpha}_{\bar{P}})$ is ω^ω-periodic. However, there exists a non-deterministic automaton $\mathcal{A}$ with input alphabet 2 such that the only $\Im : \omega^\omega \to 2$ accepted by $\mathcal{A}$ is $\lambda\alpha \in \omega^\omega.0$ (say), and yet $\mathcal{A}$ has no periodic accepting ω^ω-runs.

[15]We shall speak of runs on tuples of subsets of an ordinal, always meaning runs on their characteristic functions.

[16]By which we mean $\bar{P}$ is the unique such that $(\alpha, <, \bar{P}) \cong (\alpha_0, <, \bar{P}_0) + (\alpha_1, <, \bar{P}_1) \otimes \alpha$. We shall continue to use such abbreviations when no confusion is likely to occur.

The *fiber* of ρ is $\mathrm{fib}(\rho) := \{\beta_n \mid n \in \omega\}$.

REMARK 5.21. $\mathrm{fib}(\rho)$ is an ω-seqeunce and $\lim(\rho|_{\sup(\mathrm{fib}(\rho))}) = \lim(\rho)$.

LEMMA 5.22. *Let $\mathcal{A}$ be an initialized automaton with states Q and $K \subseteq Q$. Suppose $\alpha \in \mathtt{On}$ and K is an α-limit in $\mathcal{A}$. Then K is also an ω^ω-limit in $\mathcal{A}$.*

PROOF. We only need the case $\alpha = \omega_1$. Using Lemma 5.14, write a formula $\theta_K^{\lim}(\bar{W})$ saying: "The domain is a limit ordinal and $\bar{W}$ encodes a run of $\mathcal{A}$ (of length the domain) whose limit is K." If K is an ω_1-limit in $\mathcal{A}$, then by (2) of Definition 5.18, there is some *countable* limit β where $\theta_K^{\lim}$ is satisfied. By the Satisfiability Lemma (4.1), this means that there is also some limit $\beta' < \omega^\omega$ where $\theta_K^{\lim}$ is satisfied. Thus, K is an ω^ω-limit in $\mathcal{A}$, as claimed.

For the general case, note that $\theta_K^{\lim}$ can be taken first-order and apply Theorem 7.7. ⊣

LEMMA 5.23 (Fiber Lemma). *Let $\mathcal{A}$ be an initialized automaton with states Q, $K \subseteq Q$ and α a multiple of ω^ω. Then the following are equivalent:*

(a) *K is α-pumpable in $\mathcal{A}$.*
(b) *There exists an ω^ω-periodic α-run of $\mathcal{A}$ whose limit is K.*
(c) *There exists an α-run ρ of $\mathcal{A}$ with* bounded *fiber and with limit K.*

In fact, the fiber of every *ω^ω-periodic run is bounded in ω^ω.*

PROOF. (a) $\Rightarrow$ (b): Since K is an α-limit in $\mathcal{A}$, there are $\beta \in \mathtt{Lim} \cap \alpha$ and a β-run ϱ of $\mathcal{A}$ with $\lim(\varrho) = K$. By the previous lemma, we may assume $\beta < \omega^\omega$. Since K is stable, we can pick a $q \in K \cap \delta_{\lim}(K)$ (where $\delta_{\lim}$ is the limit transition function of $\mathcal{A}$). Then there is a $\gamma_0 \in [\beta_0(\varrho), \beta)$ such that $\varrho(\gamma_0) = q$ (in fact, there are cofinaly many such γ_0 below β). Let $\rho_0 := \varrho|_{\gamma_0}$, $\rho_1 := \varrho \restriction [\gamma_0, \beta)$ and $\rho := \rho_0 + \rho_1 \otimes \alpha$. Since α is a multiple of ω^ω and $\gamma_0, \beta - \gamma_0 < \omega^\omega$, $\alpha = \gamma_0 + (\beta - \gamma_0)\alpha$. Thus, $\mathrm{dom}(\rho) = \alpha$, indeed. By definition of γ_0, $\rho_1(0) = q$. By choice of ρ_1, $\lim(\rho_1) = K$, which implies $K \subseteq \mathrm{rng}(\rho_1)$. By definition of $\beta_0(\varrho)$, $\varrho[[\beta_0(\varrho), \beta)] = K$, so, in particular, $\mathrm{rng}(\rho_1) = K$. By Lemma 5.19, $\rho_1 \otimes \alpha$ is a run of $\mathcal{A}$ from state q with limit K. Hence, ρ is a run of $\mathcal{A}$ with limit K.

(b) $\Rightarrow$ (c): Let ρ be an ω^ω-periodic α-run of $\mathcal{A}$ with $\lim(\rho) = K$. Let α_0, $\alpha_1 < \omega^\omega$ be a phase and period for ρ respectively, and ρ_0, ρ_1 the corresponding partial runs. Then $K = \lim(\rho) = \mathrm{rng}(\rho_1)$. Therefore, $\beta_0(\rho) \leq \alpha_0$. Since also $\lim(\rho_0 + \rho_1 \otimes \omega) = K$, it follows $\sup(\mathrm{fib}(\rho)) \leq \alpha_0 + \alpha_1\omega < \omega^\omega$.

(c) $\Rightarrow$ (a): Suppose ρ is an accepting run of $\mathcal{A}$ with limit K and $\mathrm{fib}(\rho)$ bounded in α. Let $\beta := \sup(\mathrm{fib}(\rho))$. Then $\beta \in \mathtt{Lim} \cap \alpha$ and $\lim(\rho|_\beta) = \lim(\rho) = K$, which shows that K is an α-limit in $\mathcal{A}$. Furthermore, since $\beta > \beta_0(\rho)$, $\rho(\beta) \in K$. But, $\rho(\beta) \in \delta_{\lim}(\lim(\rho|_\beta))$, so $\rho(\beta) \in K \cap \delta_{\lim}(K)$. ⊣

Now, if we add to the last result the simple observation that the fiber of a run is definable from the run, we obtain the desired result:

PROPOSITION 5.24. *There is an algorithm that, given $\alpha \in [\omega^\omega, \omega_1)$ and a formula $\varphi(\bar{X})$, constructs a formula $\psi(\bar{X}, Y)$ such that the following are equivalent:*

(a) *φ is* not *selectable in $(\alpha, <)$.*
(b) *ψ is a reduction in $(\alpha, <)$ of $\theta^\alpha_{\omega\,\mathrm{ub}}$ to φ,*

where $\theta^{\alpha}_{\omega\,\mathrm{ub}}(Y)$ says: "Y is an unbounded ω-sequence in the ω^ω-part of α."

In particular, for any formula φ not selectable in $(\alpha, <)$, $\theta^{\alpha}_{\omega\,\mathrm{ub}}$ is easier to select in $(\alpha, <)$ than φ.

PROOF. We handle the case $\alpha = \omega^\omega$ (equivalently, a countable multiple of ω^ω) and leave the general case to the reader. By Theorem 5.12, we can compute an initialized *deterministic* Muller automaton $\mathcal{A}$ equivalent to φ over the countable ordinals. Let $Q = \{q_0, \dots, q_{k-1}\}$ be its states. It is easy to write a formula $\theta_{\mathrm{fib}}(W_0, \dots, W_{k-1}, Y)$ as follows. For every $\rho : \omega^\omega \to Q$ and $\bar{Q} \in {}^k\mathcal{P}(\omega^\omega)$, if $\bar{Q}$ "encodes" ρ, i.e. for all $i < k$, $Q_i := \rho^{-1}(q_i)$, then $\theta_{\mathrm{fib}}(\bar{Q}, Y)$ defines $\mathrm{fib}(\rho)$ in $(\omega^\omega, <)$. Let $\theta^{\text{run-on}}_{\mathcal{A}}(\bar{X}, \bar{Y})$ be as in Lemma 5.14.[17] Set

$$\psi(\bar{X}, Y) := \exists \bar{W}(\theta^{\text{run-on}}_{\mathcal{A}}(\bar{X}, \bar{W}) \wedge \theta_{\mathrm{fib}}(\bar{W}, Y)),$$

Let $\bar{P} \in {}^l\mathcal{P}(\omega^\omega)$. Since $\mathcal{A}$ is deterministic, there is a unique (strict) run of $\mathcal{A}$ on $\Im^{\alpha}_{\bar{P}}$, and the unique $\bar{Q}$ that satisfies $\theta^{\mathrm{run}}_{\mathcal{A}}(\bar{P}, \bar{Q})$ encodes this run. Then $\psi(\bar{P}, Y)$ defines in $(\alpha, <)$ the fiber of this run. Let us show this ψ is as claimed.

If φ is selectable in $(\omega^\omega, <)$, then by Lemma 5.5, there can be no reduction of the non-selectable $\theta_{\omega\,\mathrm{ub}}$ to φ in $(\omega^\omega, <)$. In particular, ψ is not such a reduction.

Assume φ is not selectable in $(\omega^\omega, <)$. By Proposition 4.2, there exists no ω^ω-periodic $\bar{P} \in \mathcal{D}(\varphi, \omega^\omega)$. Thus, by Lemma 5.17, there exists no accepting ω^ω-periodic ω^ω-run of $\mathcal{A}$. The Fiber Lemma now tells us that there is no accepting ω^ω-run of $\mathcal{A}$ with bounded fiber. Since, given any $\bar{P} \in \mathcal{D}(\varphi, \omega^\omega)$, $\psi(\bar{P}, Y)$ defines the fiber of $\mathrm{run}_{\mathcal{A}}(\mathcal{I}^{\omega^\omega}_{\bar{P}})$, it follows that for any $\bar{P} \in \mathcal{D}(\varphi, \omega^\omega)$, $\psi(\bar{P}, Y)$ defines an unbounded ω-sequence in ω^ω, as was to be shown. ⊣

The next corollary extracts from the proof data that are needed in section 7.

COROLLARY 5.25. *Let $\varphi(\bar{X})$ be a formula not selectable in $(\omega^\omega, <)$. Then there is a* first-order *formula $\phi_{\preceq}(\bar{X}, \bar{W}, Y)$ such that:*

(*a*) $(\omega^\omega, <) \models \forall \bar{X} \forall \bar{W} \forall Y \big((\varphi(\bar{X}) \wedge \phi_{\preceq}(\bar{X}, \bar{W}, Y)) \to \theta_{\omega\,\mathrm{ub}}(Y)\big)$, *and*

(*b*) *for every $\alpha \in \mathtt{Lim}$, $(\alpha, <) \models \forall \bar{X} \exists \bar{W} \exists Y (\phi_{\preceq}(X, \bar{W}, Y))$.*

PROOF. Using the notation of the above proof, let $\phi_{\preceq}(\bar{X}, \bar{W}, Y) := \theta^{\text{run-on}}_{\mathcal{A}}(\bar{X}, \bar{W}) \wedge \theta_{\mathrm{fib}}(\bar{W}, Y)$. It is easy to see that θ_{fib} is first-order and, by Remark 5.15, so is $\theta^{\text{run-on}}_{\mathcal{A}}$. By the proof above, (*a*) holds, while (*b*) simply says that $\mathcal{A}$ has a run on any input of whatever limit length and that this run has a fiber. ⊣

5.4. Any formula is easier than an unbounded ω-sequence. Let again $\mathcal{A}$ be an automaton equivalent to a non-selectable φ. Denote by F the limit of an accepting ω^ω-run of $\mathcal{A}$. By the Fiber Lemma, F is not ω^ω-pumpable. We shall prove, however, that there is a $K \subsetneq F$ which *is* ω^ω-pumpable (Proposition 5.29). Suppose $q \in K \cap \delta_{\lim}(K)$. Then, given an unbounded ω-sequence S, we will be able to define an ω^ω-run of $\mathcal{A}$ with limit F roughly as follows (Proposition 5.31):

> Use some run ρ_0 of length $< \omega^\omega$ to get $\mathcal{A}$ into state q. From q there is a run ρ_1 of limit length $< \omega^\omega$ whose limit is K and which mentions only states in K. Start pumping ρ_1 (that is, repeat it over and over again). This can be done since $q \in K \cap \delta_{\lim}(K)$. Whenever you encounter an element of S, stop pumping ρ_1 and use (once) a run ρ_2 of length $< \omega^\omega$ that mentions *all*

[17] Only here we do not care what state the run takes $\mathcal{A}$ into.

states of F and *only* them. Then go back to pumping ρ_1, until you again encounter an element of S.

Having defined this run, we will be able to define a tuple $\bar{P}$ on which it is a run. Since F is an accepting limit, $\bar{P}$ will satisfy φ. Below are the details.

NOTATION 5.26. Let $\mathfrak{A} := \langle Q, \Sigma, \delta_{\mathrm{suc}}, \delta_{\mathrm{lim}} \rangle$ be an automaton and $F \subseteq Q$. The *restriction of* $\mathfrak{A}$ *to* F is the automaton

$$\mathfrak{A}|_F := \langle F, \Sigma, \delta_{\mathrm{suc}}|_{F\times\Sigma}, \delta_{\mathrm{lim}}|_{\mathcal{P}(F)} \rangle.$$

That is, $\mathfrak{A}|_F$ is the automaton gotten from $\mathfrak{A}$ by removing all states not in F.

DEFINITION 5.27. Let $\mathcal{A} := \langle \mathfrak{A}; q_I \rangle$ be an initialized Muller automaton with states Q, limit transition function δ_{lim} and acceptance condition $\mathcal{F}$. Let $K, F \subseteq Q$, $q \in Q$ and $\alpha \in \mathtt{On}$. Call (q, K, F) an *α-pumping triple* of $\mathcal{A}$ iff:

1. K is an α-limit in $\mathcal{A}$,
2. $q \in \delta_{\mathrm{lim}}(K) \cap K$,[18]
3. $K \subseteq F$,
4. there exist a *successor* $\beta < \alpha$ and a β-run ρ of $\langle \mathfrak{A}|_F; q \rangle$ such that $\mathrm{last}(\rho) = q$ and $\mathrm{rng}(\rho) = F$, and
5. $F \in \mathcal{F}$.

Denote the set of α-pumping triples by $\mathfrak{P}^{\alpha}_{\mathcal{A}}$.

We leave the proof of the following combinatorial fact to the reader.

LEMMA 5.28 (Finite Partition Lemma). *Let $\alpha \in \mathtt{On}$, $\Delta : \omega^{\alpha} \to D$ where D is finite. Then there is $d \in D$ with* $\mathrm{otp}(\Delta^{-1}(d)) = \omega^{\alpha}$.

LEMMA 5.29 (Existence of pumping triples). *Let $\mathcal{A}$ be an initialized Muller automaton. Suppose $\mathcal{A}$ has an accepting ω^{ω}-run. Then the set $\mathfrak{P}^{\omega^{\omega}}_{\mathcal{A}}$ of ω^{ω}-pumping triples is not empty.*

PROOF. Let ρ be an accepting ω^{ω}-run of $\mathcal{A}$, $F := \lim(\rho)$, $k := |F|$, and let $C_0 := [\beta_0(\rho), \omega^{\omega})$, where $\beta_0(\rho)$ is the least ordinal above which only limit states of ρ appear.

For any $B \subseteq \omega^{\omega}$, let ∂B denote the set of limit points of B (in ω^{ω}). Let δ_{lim} be the limit transition function of $\mathcal{A}$. We define a sequence $\langle C_i \mid i \leq k+1 \rangle$ such that for all $i \leq k+1$:

(a) $\mathrm{otp}(C_i) = \omega^{\omega}$,

and, if $i \leq k$:

(b) $C_{i+1} \subseteq \partial C_i$,

(c) for $\gamma, \gamma' \in C_{i+1}$, $\lim(\rho|_{\gamma}) = \lim(\rho|_{\gamma'})$ and $\rho(\gamma) = \rho(\gamma')$.

C_0 was defined above. Let $i \leq k$ and assume C_i has already been defined. By (a) of the inductive assumption, $\mathrm{otp}(C_i) = \omega^{\omega}$, so $\mathrm{otp}(\partial C_i) = \omega^{\omega}$, too. For $\gamma \in \partial C_i$, write $\Delta(\gamma) := (\lim(\rho|_{\gamma}), \rho(\gamma))$. Clearly, $\mathrm{rng}(\Delta) \subseteq \mathcal{P}(Q) \times Q$ is finite. We may therefore apply the Finite Partition Lemma to obtain a $C_{i+1} \subseteq \partial C_i$ with $\mathrm{otp}(C_{i+1}) = \omega^{\omega}$ such that $|\Delta[C_{i+1}]| = 1$, which is exactly what is needed in (a)-(c). This completes the construction.

For $i \in [1, k+1]$, let K_i, q_i such that for *any* $\gamma \in C_i$, $\Delta(\gamma) = (K_i, q_i)$. Then,

[18] Note that (1) and (2) imply that K is α-pumpable in $\mathcal{A}$.

(*d*) $K_i \subseteq F$. Indeed, $C_i \subseteq C_0$, so for each $\gamma \in C_i$, $\gamma > \beta_0(\rho)$.
(*e*) $q_i \in \delta_{\lim}(K_i)$. This is immediate from the definition of a run.

Also, if $i \leq k$:

(*f*) $K_i \cup \{q_i\} \subseteq K_{i+1}$. Indeed, let $\gamma \in C_{i+1}$. Then $\gamma \in \partial C_i$, so we can pick an increasing $\langle \gamma_j \mid j \in \omega \rangle \subseteq C_i$ with $\sup\{\gamma_j \mid j \in \omega\} = \gamma$. For each $j \in \omega$, $\lim(\rho|_{\gamma_j}) = K_i$. Therefore, all states of K_i are unbounded in γ. On the other hand, $\rho(\gamma_j) = q_i$, so also $q_i \in \lim(\rho|_\gamma) = K_{i+1}$.

Thus, $\langle K_i \mid i \in [1, k+1] \rangle$ is an increasing sequence of subsets of F. Since $|F| = k$, there is $i \in [1, k]$ with $K_i = K_{i+1}$ (note that $K_1 \neq \varnothing$). We claim that $(q_i, K_i, F) \in \mathfrak{P}^{\omega^\omega}_{\mathcal{A}}$ (numbers are as in Definition 5.27):

1. Every $\gamma \in C_i$ shows that K_i is an α-limit in $\mathcal{A}$.
2. By (*e*) and (*f*), $q_i \in \delta_{\lim}(K_i) \cap K_{i+1}$. But, $K_{i+1} = K_i$.
3. is (*d*).
4. Pick any $\gamma \in C_i$. Since $F = \lim(\rho)$, there exists a $\beta \in (\gamma, \omega^\omega)$ such that $\rho[[\gamma, \beta)] = F$. Pick $\gamma' \in C_i$ with $\gamma' > \beta$. Then $\rho \restriction [\gamma, \gamma']$ is a run as in (4) of the definition.
5. Since $F = \lim(\rho)$ and ρ is an accepting run.

$\dashv$

COMMENT. A similar proof would show that $\mathfrak{P}^{\alpha}_{\mathcal{A}} \neq \varnothing$ under the weaker assumption that α is a multiple of $\omega^{|F|+2}$ (with F the limit of an accepting run).

The next lemma follows easily from Lemma 5.14.

LEMMA 5.30. *For all $\alpha \in \mathtt{On} \setminus 1$ and $\mathcal{A}$ an initialized Muller automaton, the set of α-pumping triples of $\mathcal{A}$ is determined by* $\mathrm{MTh}(\alpha, <)$ *and computable from it.*

Finally, we have:

PROPOSITION 5.31. *There exists an algorithm that, given $\alpha \in [\omega^\omega, \omega_1)$ and a formula $\varphi(\bar{Y})$, builds a formula $\psi(X, \bar{Y})$ which is a reduction in $(\alpha, <)$ of φ to $\theta^{\alpha}_{\omega\,\mathrm{ub}}$.*

In particular, for any formula φ we have that φ is easier to select in $(\alpha, <)$ than $\theta^{\alpha}_{\omega\,\mathrm{ub}}$.

PROOF. We handle the case $\alpha = \omega^\omega$ (equivalently, a multiple of ω^ω). By Theorem 5.12, we can compute an initialized Muller automaton $\mathcal{A} := \langle \mathfrak{A}; q_I \rangle$ which is equivalent to φ over the countable ordinals. By Lemma 5.30, we can compute $\mathfrak{P}^{\omega^\omega}_{\mathcal{A}}$. If $\mathfrak{P}^{\omega^\omega}_{\mathcal{A}} = \varnothing$, then by Lemma 5.29, $\mathcal{A}$ has no accepting ω^ω-runs. By choice of $\mathcal{A}$, this means φ is not satisfied in $(\omega^\omega, <)$ and we can take $\psi(X, \bar{Y}) := X \neq X$ (say). Assume this is not the case and pick any $(q, K, F) \in \mathfrak{P}^{\omega^\omega}_{\mathcal{A}}$. Use Lemma 5.14 to write formulas as follows:

$\varphi_0(\bar{Y})$ says: "There is run of $\mathcal{A}$ on $\bar{Y}$ (of length the successor of the domain) that takes $\mathcal{A}$ into state q".[19]

$\varphi_1(\bar{Y})$ says: "The domain is a limit ordinal and there is a (strict) run of $\langle \mathfrak{A}|_K; q \rangle$ on $\bar{Y}$ whose limit is K".

[19]Thus, in the notation of Lemma 5.14, φ_0 *is* $\exists \bar{W} \theta^{\text{run-on}}_{\mathcal{A} \to q}(\bar{Y}, \bar{W})$.

$\varphi_2(\bar{Y})$ says: "There is a run of $\langle\mathfrak{A}|_F; q\rangle$ on $\bar{Y}$ (of length the successor of the domain) that takes it into q and whose range is F".

We claim that each of these formulas is satisfied in some ordinal below ω^ω. Indeed, by (1) of Definition 5.27, K is an ω^ω-limit in $\mathcal{A}$, i.e. there are $\beta \in \mathtt{Lim}\cap\omega^\omega$ and a β-run ρ of $\mathcal{A}$ such that $\lim(\rho) = \beta$. Let $l := \lg(\bar{Y})$ and pick $\bar{P} \in {}^l\mathcal{P}(\beta)$ such that ρ is a (strict) run of $\mathcal{A}$ on $\bar{P}$. Since $q \in K$ and $\lim(\rho) = K$, there is a $\gamma_0 \in (\beta_0(\rho), \beta)$ such that $\rho(\gamma_0) = q$. Then $\rho \restriction [0, \gamma_0]$ is a run of $\mathcal{A}$ on $\bar{P}\check{\cap}\gamma_0$, so $(\gamma_0, <, \bar{P}\check{\cap}\gamma_0) \models \varphi_0$. Next, $\rho[[\gamma_0, \beta)] \subseteq K$, since $\gamma_0 \geq \beta_0(\rho)$. Thus, $\rho \restriction [\gamma_0, \beta)$ is a run of $\langle\mathfrak{A}|_K; q\rangle$ on $\bar{P}\check{\cap}[\gamma_0, \beta)$ with limit K. In other words, $([\gamma_0, \beta), <, \bar{P}\check{\cap}[\gamma_0, \beta)) \models \varphi_1$. Similarly, satisfiability of φ_2 below ω^ω follows from (4) of Definition 5.27.

Fix $i \in 3$. Use the Satisfiability Lemma to compute an $\alpha_i \in \omega^\omega \setminus 1$ in which φ_i is satisfied. By Proposition 2.21, compute a ψ_i that selects φ_i in $(\alpha_i, <)$. Let $\bar{P}_i$ be the unique element of $\mathcal{D}(\psi_i, \alpha_i)$.

Pick $k \in \omega$ such that $\omega^k > \alpha_0, \alpha_1, \alpha_2$. We define $\mathfrak{f} : \mathcal{D}(\theta_{\omega\,\mathrm{ub}}, \omega^\omega) \to \mathcal{D}(\varphi, \omega^\omega)$ as follows. Fix an unbounded ω-sequence $S \subseteq \omega^\omega$. Define $S_k := \{\alpha + \omega^k \mid \alpha \in S\}$. Then S_k is also an unbounded ω-sequence in ω^ω and every element in S_k is a multiple of ω^k. Note further that S_k is definable from S (because, by the Definability-below-ω^ω Lemma, ω^k is definable). Let $\{\zeta_i \mid i \in \omega\}$ be the order-preserving enumeration of S_k. Then $\mathfrak{f}(S)$ is the unique $\bar{P} \in {}^l\mathcal{P}(\omega^\omega)$ which satisfies:

(*a*) $(\omega^\omega, <, \mathfrak{f}(S))_{\restriction[0,\zeta_0)} \cong (\alpha_0, <, \bar{P}_0) + (\alpha_1, <, \bar{P}_1) \otimes \frac{\zeta_0-\alpha_0}{\alpha_1}$,

and, for every $i \in \omega$,

(*b*) $(\omega^\omega, <, \mathfrak{f}(S))_{\restriction[\zeta_i,\zeta_{i+1})} \cong (\alpha_2, <, \bar{P}_2) + (\alpha_1, <, \bar{P}_1) \otimes \frac{\zeta_{i+1}-(\zeta_i+\alpha_2)}{\alpha_1}$.

Since $\omega^k > \alpha_0, \alpha_1, \alpha_2$ and ζ_0 and $\zeta_{i+1} - \zeta_i$ are multiples of ω^k, $\zeta_0 = \zeta_0 - \alpha_0$ is a multiple of α_1 and so is $\zeta_{i+1} - \zeta_i = \zeta_{i+1} - (\zeta_i + \alpha_2)$. Therefore, both (*a*) and (*b*) make sense.

We must show that $(\omega^\omega, <) \models \varphi(\mathfrak{f}(S))$. To this end we present a run ρ of $\mathcal{A}$ on $\mathfrak{f}(S)$ which is accepting. Let ϱ_0 be an $(\alpha_0 + 1)$-run of $\mathcal{A}$ on $\bar{P}_0$ with $\mathrm{last}(\varrho_0) = q$. Write $\rho_0 := \varrho_0|_{\alpha_0}$. Let ρ_1 be an α_1-run of $\langle\mathfrak{A}|_K; q\rangle$ on $\bar{P}_1$ with limit K. Finally, let ϱ_2 be an (α_2+1)-run of $\langle\mathfrak{A}|_F; q\rangle$ on $\bar{P}_2$ with $\mathrm{last}(\varrho_2) = q$ and $\mathrm{rng}(\varrho_2) = F$. Set $\rho_2 := \varrho_2|_{\alpha_2}$ ($\varrho_0, \rho_1, \varrho_2$ exist since, for $i \in 3$, $(\alpha_i, <) \models \varphi_i(\bar{P}_i)$). Let $\rho : \omega^\omega \to Q$ satisfy:

(*a′*) $\rho|_{\zeta_0} = \rho_0 + \rho_1 \otimes \frac{\zeta_0-\alpha_0}{\alpha_1}$,

and, for every $i \in \omega$,

(*b′*) $\rho \restriction [\zeta_i, \zeta_{i+1}) = \rho_2 + \rho_1 \otimes \frac{\zeta_{i+1}-(\zeta_i+\alpha_2)}{\alpha_1}$.

It is easy to see that ρ is a run of $\mathcal{A}$ on $\mathfrak{f}(S)$. Briefly:

1. ρ begins in q_I, since ρ_0 does.
2. By Lemma 5.19, for any $\beta \in \mathtt{On} \setminus 1$, $\rho_1 \otimes \beta$ is a run of $\langle\mathfrak{A}|_K; q\rangle$ with limit and range K. In particular, this holds for $\beta = \frac{\zeta_0-\alpha_0}{\alpha_1}$ or $\beta = \frac{\zeta_{i+1}-(\zeta_i+\alpha_2)}{\alpha_1}$ for some $i \in \omega$.
3. Since ρ_0 takes us into q, where $\rho_1 \otimes \frac{\zeta_0-\alpha_0}{\alpha_1}$ begins, the initial segment $\rho|_{\zeta_0}$ is a ζ_0-run with limit K.

4. Since both ρ_2 and $\rho_1 \otimes \frac{\zeta_{i+1}-(\zeta_i+\alpha_2)}{\alpha_1}$ begin in q and take us back into q, ρ is an ω^ω-run as desired.

Now, clearly $\lim(\rho) \subseteq \mathrm{rng}(\rho_1) \cup \mathrm{rng}(\rho_2) = K \cup F = F$. But, for every $i \in \omega$, $\rho[[\zeta_i, \zeta_i + \alpha_2)] = \mathrm{rng}(\rho_2) = F$. Since $S_k = \{\zeta_i \mid i \in \omega\}$ is unbounded in ω^ω, $F \subseteq \lim(\rho)$. Thus, ρ is accepting.

We conclude that, indeed, $\mathfrak{f}$ takes unbounded ω-sequences and returns tuples satisfying φ. If we write a $\psi(X, \bar{Y})$ that defines $\mathfrak{f}$ in $(\omega^\omega, <)$, we would be done. This being very much like the proof of "$(c) \Rightarrow (a)$" in Proposition 4.2, we allow ourself to be brief.

First, write $\theta_M(X, W)$ such that for any unbounded ω-sequence S, if $S_k = \{\zeta_i \mid i \in \omega\}$ is as above, then $\theta_M(S, W)$ defines in $(\omega^\omega, <)$ the set

$$\{\alpha_0 + \alpha_1\beta \mid \beta < \frac{\zeta_0 - \alpha_0}{\alpha_1}\} \cup S_k \cup \bigcup_{i\in\omega}\{\zeta_i + \alpha_2 + \alpha_1\beta \mid \beta < \frac{\zeta_{i+1} - (\zeta_i + \alpha_2)}{\alpha_1}\}.$$

Here, all we use is the definability of $\alpha_0, \alpha_1, \alpha_2$ and the definability of S_k from S.

Then, the desired $\psi(X, \bar{Y})$ says: "for the unique W that satisfies $\theta_M(X, W)$:

$\psi_0(\bar{Y})$ holds relativized to $[0, \min(W))$, and if x, y are successive elements of W, then:

if $x \in W$, then $\psi_2(\bar{Y})$ holds relativized to $[x, y)$, and

if $x \notin W$, then $\psi_1(\bar{Y})$ holds relativized to $[x, y)$." $\dashv$

A nice corollary of the last proof is:

Corollary 5.32. *Let $\alpha \in \omega_1 \setminus 1$ and $\varphi(\bar{Y})$ a formula not selectable in $(\alpha, <)$. Then $|\mathcal{D}(\varphi, \alpha)| = 2^{\aleph_0}$.*

Proof. We handle the case $\alpha = \omega^\omega$. Let $\mathcal{A}$ be an initialized *deterministic* Muller automaton equivalent to φ over the countable ordinals.[20] Since φ is not selectable in $(\omega^\omega, <)$, φ is satisfied in $(\omega^\omega, <)$. Thus, $\mathcal{A}$ has an accepting ω^ω-run, so there are ω^ω-pumping triples of $\mathcal{A}$. Let (q, K, F) be one. Let α_1, α_2, $\bar{P}_1$, $\bar{P}_2$, $k \in \omega$ and $\mathfrak{f} : \mathcal{D}(\theta_{\omega\,\mathrm{ub}}, \omega^\omega) \to \mathcal{D}(\varphi, \omega^\omega)$ be as in the proof of the previous proposition. Let Seq_k denote the class of all unbounded ω-sequences in ω^ω all of whose members are multiples of ω^k. Clearly, $|\,\mathrm{Seq}_k\,| = 2^{\aleph_0}$. We show that $\mathfrak{f}|_{\mathrm{Seq}_k}$ is 1-1, which will finish the proof.

Let $S, S' \in \mathrm{Seq}_k$, $S \nsubseteq S'$. Let S_k, S'_k be obtained from S, S' as in the previous proof. Pick $\xi \in S \setminus S'$ and write $\zeta := \xi + \omega^k$. Then $\zeta \in S_k \setminus S'_k$ (use the fact that all elements of S, S' are multiples of ω^k). Let ρ, ρ' be the unique runs of $\mathcal{A}$ on $\mathfrak{f}(S)$, $\mathfrak{f}(S')$, respectively. For $i \in \{1, 2\}$, let ρ_i be the unique run of $\langle \mathfrak{A}; q\rangle$ on $\Im^{\alpha_i}_{\bar{P}_i}$. By definition of $\mathfrak{f}$, $\rho[[\zeta, \zeta{+}\alpha_2)] = \mathrm{rng}(\rho_2) = F$. In particular, $\rho[[\zeta, \zeta{+}\omega^k)] = F$ (recall $\omega^k > \alpha_2$). But, if ζ' is the least member of S' above ζ, then $\rho' \restriction [\zeta, \zeta') = \rho_1 \otimes \frac{\zeta'-\zeta}{\alpha_1}$. Since $\mathrm{rng}(\rho_1) = K$, $\rho[[\zeta, \zeta')] = K$. Since ζ', too, is a multiple of ω^k, we have, in particular, $\rho[[\zeta, \zeta + \omega^k)] = K$. Since φ is not selectable, $K \neq F$ (or else F is an accepting set which is ω^ω-pumpable in $\mathcal{A}$). Since $\mathcal{A}$ is deterministic and $\rho(\zeta) = \rho'(\zeta) = q$, this must mean $\mathfrak{f}(S) \cap [\zeta, \zeta + \omega^k) \neq \mathfrak{f}(S') \cap [\zeta, \zeta + \omega^k)$, i.e. $\mathfrak{f}(S) \neq \mathfrak{f}(S')$, as was to be shown. $\dashv$

[20] Recall that in the previous proof we did *not* need the assumption $\mathcal{A}$ is deterministic.

Note that Proposition 5.31 leaves open an interesting question. It tells us that with an unbounded ω-sequence $S \subseteq \omega^\omega$ as a parameter, we can select all formulas in $(\omega^\omega, <)$. But, the formulas we select using S do not themselves "mention" S. In other words, the proposition does not tell us that $(\omega^\omega, <, S)$ has the selection property. In fact, let $\varphi(X, Y)$ says: "If $x < x'$ are successive elements of X, then $Y \cap [x, x')$ is an ω-sequence unbounded in $[x, x')$" and $S := \{\omega^k \mid k \in \omega\}$. Then it is easy to show φ has no selector in $(\omega^\omega, <, S)$.[21] It is therefore natural to ask:

> Can $(\omega^\omega, <)$ be expanded by finitely many subsets of ω^ω to have the selection property?

In [RS] we provide an affirmative answer to this question:

Proposition 5.33 ([RS]). *There is an unbounded ω-sequence $S \subseteq \omega^\omega$ such that:*

(a) *the monadic theory of $(\omega^\omega, <, S)$ is decidable,*
(b) *$(\omega^\omega, <, S)$ has the selection property, and*
(c) *for any formula $\varphi(X, \bar{Y})$, a selector for φ in $(\omega^\omega, <, S)$ is computable.*

An essential ingredient of the proof given there is Corollary 6.1 below.

§6. Every formula is selectable in some countable ordinal. This short section is devoted to the proof of Proposition 3.19. On our way there, we state a simple combinatorial corollary of (the proof of) Proposition 5.31, which seems of independent interest.

Corollary 6.1. *Let $n, l \in \omega$ and $\varphi \in \mathfrak{Form}_{n,l}$ which is satisfied in $(\omega^\omega, <)$. Then we can compute $\tau_{sel}, \tau_{suf} \in H_{n,l}$ such that:*

(a) *τ_{sel} is satisfiable and* selectable *in $(\omega^\omega, <)$,*
(b) *τ_{suf} is satisfiable in $(\omega^\omega, <)$, and*
(c) *$\tau_{sel} + \tau_{suf} \models \varphi$.*

Furthermore, τ_{sel} and τ_{suf} can be computed from φ.

Proof. Fix $\tau \in H_{n,l}$. By Theorem 2.15, there is $\mathfrak{c} = \mathfrak{c}(n, l) \in \omega$ such that for every chain $\mathcal{I}$, $\tau \otimes \mathcal{I}$ depends only on the $\mathfrak{c}$-type of $\mathcal{I}$. Let $p := \mathfrak{p}(\mathfrak{c})$. Then for $\alpha, \alpha' \in \omega_1 \setminus 1$, $\tau \otimes \omega^p \alpha = \tau \otimes \omega^p \alpha'$. In particular, $\tau \otimes \omega^p = \tau \otimes \omega^{p+1}$, which means $\tau \otimes \omega^p$ is an ω-idempotent (Definition 4.6).

Now, go back to the proof of 5.31 (we use the notation used there). From what was just said, it follows $\mathrm{type}^n((\alpha_1, <, \bar{P})) \otimes \omega^{\mathfrak{c}}$ is an ω-idempotent. Of course, $\mathrm{type}^n((\alpha_1, <, \bar{P})) \otimes \omega^{\mathfrak{c}} = \mathrm{type}^n((\alpha_1, <, \bar{P}_1) \otimes \omega^{\mathfrak{c}})$. But, the l-chain $(\alpha_1, <, \bar{P}_1) \otimes \omega^{\mathfrak{c}}$ satisfies every requirement placed on $(\alpha_1, <, \bar{P}_1)$ in 5.31: its domain $\alpha_1 \omega^{\mathfrak{c}}$ is smaller than ω^ω and $\rho_1 \otimes \omega^{\mathfrak{c}}$ is a run of $\mathcal{A}$ on this l-chain beginning in state q, mentioning only states in K and with limit K (use Lemma 5.19). We could therefore use $(\alpha_1, <, \bar{P}_1) \otimes \omega^{\mathfrak{c}}$ in our construction of $\mathfrak{f}$ in 5.31. In other words, we may assume $\tau_1 := \mathrm{type}^n(\alpha_1, <, \bar{P}_1)$ is an ω-idempotent.

[21] Recall that in our semantics some of the free variables play the role of monadic predicate symbols. When interpreting $\varphi(X, Y)$ in $(\omega^\omega, <, S)$, we intend for X to be interpreted as S, while Y serves as a free variable in the usual sense. So, in selecting $\varphi(X, Y)$ we need to define a subset of ω^ω and not a pair of subsets (cf. Definition 2.2).

Let $\tau_0 := \mathrm{type}^n(\alpha_0, <, \bar{P}_0)$ and $\tau_{sel} := \tau_0 + \tau_1$. Proposition 4.8 tells us τ_{sel} is selectable in $(\omega^\omega, <)$. Pick any unbounded ω-sequence $S \subseteq \omega^\omega$ and let $\mathfrak{f}(S) \in \mathcal{D}(\varphi, \omega^\omega)$ be constructed from it as in 5.31. Then $\tau_{suf} := \mathrm{type}^n((\omega^\omega, <, \mathfrak{f}(S))_{\restriction[\alpha_0+\alpha_1,\omega^\omega)})$ is certainly satisfied in $(\omega^\omega, <)$. However, $\mathrm{type}^n(\omega^\omega, <, \mathfrak{f}(S)) = \mathrm{type}^n((\omega^\omega, <, \mathfrak{f}(S))_{\restriction[0,\alpha_0+\alpha_1)}) + \mathrm{type}^n((\omega^\omega, <, \mathfrak{f}(S))_{\restriction[\alpha_0+\alpha_1),\omega^\omega}) = \tau_{sel} + \tau_{suf}$. Since $\mathfrak{f}(S)$ satisfies φ in $(\omega^\omega, <)$, it follows $\tau_{sel} + \tau_{suf} \models \varphi$, as desired.

We leave it to the reader to show the computability of τ_{sel}, τ_{suf}. ⊣

The following is Proposition 3.19.

PROPOSITION 6.2. *Let $n, l \in \omega$ and $\varphi \in \mathfrak{Form}_{n,l}$. Then for every $\zeta \in [\omega^{\mathfrak{p}(n+l)}, \omega^\omega)$, φ is selectable in $(\omega^\omega + \zeta, <)$.*

PROOF. Write $p := \mathfrak{p}(n+l)$. Pick $\alpha \in [\omega^\omega + \omega^p, \omega^\omega \cdot 2)$. We must show φ has a selector φ in $(\alpha, <)$ and may assume φ is satisfied in $(\alpha, <)$. Denote by α_p the ω^p-part of α. We treat the case $\alpha_p \neq \alpha$, and leave the (simpler) case $\alpha_p = \alpha$ to the reader.

There must exist $\tau_p, \tau' \in H_{n,l}$ with τ_p satisfied in α_p and τ' in $\alpha - \alpha_p$ such that $\tau_p + \tau' \models \varphi$. Since α_p is a multiple of ω^p, it is $(n+l)$-equivalent to ω^ω, so τ_p is satisfied in $(\omega^\omega, <)$. Pick τ_{sel}, τ_{suf} as in the previous corollary, setting $\varphi := \tau_p$. Then τ_{sel} is selectable in $(\omega^\omega, <)$, say by a selector Ψ_{sel}. Note that $\alpha_p - \omega^\omega$ is a non-0 multiple of ω^p (use that $\alpha \geq \omega^\omega + \omega^p$). Thus, τ_{suf} is also satisfied in $(\alpha_p - \omega^\omega, <)$. Since $\alpha_p - \omega^\omega < \omega^\omega$, we can compute a selector Ψ_{suf} for τ_{suf} in $(\alpha_p - \omega^\omega, <)$. Similarly, a selector Ψ' for τ' in $(\alpha - \alpha_p, <)$ exists. Then the ψ which says "Ψ_{sel}, Ψ_{suf} and Ψ' hold relativized to $[0, \omega^\omega)$, $[\omega^\omega, \alpha_p)$ and $[\alpha_p, \alpha)$, respectively" selects φ in $(\alpha, <)$. ⊣

§7. Selection between first-order and second-order logic. In this section we often write "FO" instead of "first-order". Recall that in our definition of a FO-formula $\phi(\bar{Y})$ quantifiers range only over elements of the domain, but the free variables are allowed to range over subsets.[22] Let us call the set of first-order MLO formulas the *first-order fragment* of MLO. We show that all results concerning the selection property and selection problem in *countable* ordinals carry through from MLO to its FO-fragment.[23] On the other hand, $(\omega_1, <)$ has the *first-order selection property*, i.e. every FO formula is selectable in $(\omega_1, <)$ (by a FO formula). To prove the latter statement, we show: (1) every FO formula selectable in $(\omega^\omega, <)$ is also selectable in $(\omega_1, <)$; (2) any FO formula not selectable in $(\omega^\omega, <)$ is not even *satisfied* in $(\omega_1, <)$ (so, trivially, it is selectable). In (2) we use the fact that $\theta_{\omega\,\mathrm{ub}}$ represents the least non-selectable degree in $(\omega^\omega, <)$.

First, we need to slightly refine the Satisfiability Lemma.

DEFINITION 7.1 ($\mathcal{L}$-chain). Let $\mathcal{L}_0$ be the set of one element labeled chains. For $i \in \omega$, let $\mathcal{L}_{i+1}$ be defined as follows: a labeled chain C is in $\mathcal{L}_{i+1}$ iff it is either in $\mathcal{L}_i$ or there are $C_0, C_1 \in \mathcal{L}_i$ such that $C = C_0 + C_1$ or $C = C_1 \otimes \omega$.

[22] This is significant. For instance, there is a first-order $\phi(Y)$ such that the only subset of ω satisfying ϕ in $(\omega, <)$ is the set of even numbers. On the other hand, there is no first-order formula $\phi(y)$, with y an *individual* variable, such that for any $n \in \omega$, $(\omega, <) \models \phi(n)$ iff n is even.

[23] It is easy to do the same for the results of section 5.

A labeled chain C is an *$\mathcal{L}$-chain* iff $C \in \mathcal{L}_i$ for some $i \in \omega$. The least such i is then called the *rank* of C.

LEMMA 7.2. *Any formula $\varphi(\bar{Y})$ satisfiable in a countable ordinal is satisfied by an $\mathcal{L}$-chain. When this is the case, we may compute a* FO *formula χ defining an $\mathcal{L}$-chain satisfying φ.*

PROOF. [Sh75] proves the first claim. Shelah further shows that we can compute a bound $\mathfrak{r}$ on the rank of an $\mathcal{L}$-chain satisfying φ. Now, every $\mathcal{L}$-chain can be described by a finite "list of instructions": we need to know which chains with domain 1 to begin with, and how to apply addition and multiplication by ω. Let $n := \mathrm{qd}(\varphi)$, $l := \lg(\bar{Y})$. Given any instruction list, we may compute the n-type of the chain constructed from the list. Indeed, we first compute the n-types of the chains of length 1 with which the construction begins; we then use the Addition and Multiplication Lemmas to compute n-types for the chains obtained at the various stages of the construction, up until the final one. Finally, it is clear that there are only finitely many possible instruction lists for l-chains of rank at most $\mathfrak{r}$. Thus, we may search through these instruction lists, until we find one yielding an $\mathcal{L}$-chain whose n-type implies φ.

To complete the proof the reader will show (by an easy induction on rank) that, given any instruction list, we may compute a FO formula defining the corresponding $\mathcal{L}$-chain. ⊣

LEMMA 7.3. *If $\alpha \in [1, \omega_1]$ and $\varphi(\bar{Y})$ is an* MLO *formula selectable in $(\alpha, <)$, then there is a* first-order *$\chi(\bar{Y})$ that selects φ in $(\alpha, <)$. Furthermore, χ is computable from α and φ.*

PROOF. If φ is not satisfied in $(\alpha, <)$ take $\chi := Y \neq Y$ (say). Assume $\mathcal{D}(\varphi, \alpha) \neq \varnothing$. If $\alpha \in \omega^\omega \setminus 1$, let θ_α be a FO-*sentence* defining α in the class of ordinals. Thus, if β is an ordinal, then $(\beta, <) \models \theta_\alpha$ iff $\beta = \alpha$. Then take χ to be a FO-formula defining an $\mathcal{L}$-chain satisfying $\theta_\alpha \wedge \varphi$, as in the previous lemma.

For $\alpha = \omega^\omega$ or $\alpha = \omega_1$, we again look at the proof of Proposition 4.2. We use the notation used there. By what was just said, we can take the ψ_0, ψ_1 which select τ_0, τ_1 in α_0, α_1 to be FO formulas. This leaves one place where second-order quantification is used in $\psi_{(\tau_0,\tau_1)}$. Recall that we had a formula $\theta_M(W)$ defining the set $M := \{\alpha_0 + \alpha_1\beta \mid \beta < \alpha\}$. Then $\psi_{(\tau_0,\tau_1)}$ said: "For every (that is, the unique) W which satisfies $\theta_M(W) \ldots$". This, of course, is a second-order quantifier "$\forall W(\theta_M(W) \to \ldots)$".[24]

Now, assume we could take $\alpha_1 = \omega^k > \alpha_0$ for some $k \in \omega$. Then M would consist of α_0 and all multiples of ω^k below α. But, there is a FO formula $\theta_k(x)$ that is satisfied (in any ordinal) only by multiples of ω^k: $\theta_1(x)$ says "x is a limit ordinal", $\theta_2(x)$ says "x is the limit of limit ordinals", etc. By the Definiability-below-ω^ω Lemma, there also exists a FO formula $\theta_{\alpha_0}(x)$, which in every ordinal $> \alpha_0$ is satisfied only by α_0. Using $\theta_{\alpha_0}(x) \vee \theta_k(x)$ instead of θ_M, we eliminate the set quantifier.

So, it remains to be seen we can have $\alpha_1 = \omega^k$. By Proposition 4.8, we may assume $\tau_1 \otimes \omega = \tau_1$. Pick any k such that $\omega^k > \alpha_0, \alpha_1$. Then ω^k is a multiple of α_1. By Lemma 4.7, $\tau_1 = \tau_1 \otimes \frac{\omega^k}{\alpha_1}$ is also satisfied in $(\omega^k, <)$, and we are done.

[24]The fact that θ_M itself is FO does not help us in eliminating this quantifier.

Finally, for $\alpha = \omega^\omega + \zeta$ where $\zeta \in \omega^\omega \setminus 1$, we proceed as in the proof of Proposition 4.4, using the fact that there are FO formulas defining $[0, \omega^\omega)$ and $[\omega^\omega, \alpha)$ in $(\alpha, <)$. ⊣

Though the following corollary is phrased somewhat vaguely, we hope our intention is clear.

COROLLARY 7.4. *Let $\mathcal{L}$ be a logic such that:*

1. *For every first-order ϕ, we can compute a formula Λ of $\mathcal{L}$ equivalent to it.*
2. *For every formula Λ of $\mathcal{L}$, we can compute an* MLO *formula φ equivalent to it.*

Then:

(a) *If $\alpha \in \omega_1 \setminus 1$, then $(\alpha, <)$ has the selection property for $\mathcal{L}$-formulas iff $\alpha < \omega^\omega$, and*

(b) *the selection problem for $\mathcal{L}$-formulas in any $\alpha \in [1, \omega_1]$ is solvable.*

PROOF. We do (b) and leave (a) to the reader. Let Λ be an $\mathcal{L}$-formula. Compute an MLO φ as in (2). Decide whether φ has a selector in $(\alpha, <)$. If no, we are done. If yes, use the previous lemma, to compute a FO selector for φ in $(\alpha, <)$, and using (1), translate it back into $\mathcal{L}$. ⊣

There are interesting logics as in the corollary. A famous example is *weak* MLO, where quantifiers range over *finite* subsets of the domain.

We now show that (a) of the corollary cannot be extended to the case $(\omega_1, <)$ even when $\mathcal{L} = \text{FO}$. First, we must know that, if in the presentation of the composition method in subsection 2.3, we replace everywhere the word "formula" with the words "FO formula", then all results remain valid. Thus, $FO\mathfrak{Form}_{n,l}$ is the set of FO formulas in $\mathfrak{Form}_{n,l}$; inside which we find a finite set $FOH_{n,l}$ of FO Hintikka formulas, exactly one of which is satisfied in every l-structure $\mathcal{M}$; we call it the FO *n-type* of $\mathcal{M}$. The taking of ordered sums preserves FO n-equivalence, so that the sum of FO n-types can be defined. Finally, we have FO analogues of Theorem 2.15 and the Multiplication Theorem. So far, nothing new. What is new is:

THEOREM 7.5 ([GR02]). *Let $k \in \omega$, $\mathcal{I}$ any chain. Then ω^k and $\omega^k + \omega^k \otimes \mathcal{I}$ cannot be distinguished by any* FO *sentence of quantifier-depth k. In particular, any two multiples of ω^ω have the same first-order theory.*

Now, let $n, l \in \omega$ and $\tau \in H_{n,l}$. If τ is satisfiable in some structure, then there is a unique $\phi \in FOH_{n,l}$ with $\tau \models \phi$. We call ϕ the FO *restriction* of τ.

PROPOSITION 7.6. *If a* FO *formula is selectable in $(\omega^\omega, <)$, then it has a* FO *selector over the class of all multiples of ω^ω.*

PROOF. Let $n, l \in \omega$ and let $\chi \in FO\mathfrak{Form}_{n,l}$ be selectable in $(\omega^\omega, <)$. Let $\tau_0, \tau_1 \in H_{n,l}$ both satisfiable below ω^ω such that $\tau_0 + \tau_1 \otimes \omega^\omega \models \chi$. Let ϕ_0, ϕ_1 be the FO restrictions of τ_0, τ_1, respectively. Then $\phi_0 +_{\text{FO}} (\phi_1 \otimes_{\text{FO}} \omega^\omega) \models \chi$. Let $\alpha \in \mathtt{On} \setminus 1$. By the last theorem, ω^ω and $\omega^\omega\alpha$ have the same FO-theory. Since multiplication of FO-types is recursive in this theory, $\phi_0 +_{\text{FO}} (\phi_1 \otimes_{\text{FO}} \omega^\omega\alpha) = \phi_0 +_{\text{FO}} (\phi_1 \otimes_{\text{FO}} \omega^\omega)$. By Lemma 4.5, there is formula $\psi = \psi_{(\tau_0, \tau_1)}$ which selects $\phi_0 +_{\text{FO}} (\phi_1 \otimes_{\text{FO}} \omega^\omega)$ in every multiple of ω^ω. By the proof of Lemma 7.3, we know it can be taken FO. ⊣

Next, we need to know that for FO formulas, the Satisfiability Lemma can be extended to all ordinals. This is due to Läuchli and Leonard ([LL66], [Ro82]).

THEOREM 7.7. *Let $\chi(\bar{Y})$ be a* FO *formula satisfied in some ordinal. Then there is $\alpha < \omega^\omega$ in which χ is satisfied.*

COROLLARY 7.8. *Let α be a multiple of ω^ω and $\chi(\bar{Y})$ a* FO *formula which is satisfied in $(\alpha, <)$. Then χ is also satisfied in $(\omega^\omega, <)$.*

PROOF. Suppose $(\omega^\omega, <) \models \forall\bar{Y}\neg\chi(\bar{Y})$. Let n be the quantifier-depth of this sentence and $p := \mathfrak{p}(n)$ as in the $\mathfrak{p}$-Lemma. Then the MLO sentence $\forall\bar{Y}\neg\chi$ also holds in every countable multiple of ω^p. Now, there is a FO sentence θ_p which is satisfied by an ordinal β iff β is a multiple of ω^p. Then $\chi(\bar{Y}) \wedge \theta_p$ is not satisfied in any countable ordinal, so by the previous theorem, it is not satisfiable in any ordinal whatsoever. This means that $\chi(\bar{Y})$ is not satisfiable in any multiple of ω^p, in particular, in any multiple of ω^ω. ⊣

LEMMA 7.9. *Let $\chi(\bar{X})$ be a first-order formula not selectable in $(\omega^\omega, <)$. Then χ is not satisfied in $(\omega_1, <)$.*

PROOF. Since $\chi(\bar{X})$ is not selectable in $(\omega^\omega, <)$, there is a FO formula $\phi_\preceq(\bar{X}, \bar{W}, Y)$ as in Corollary 5.25. Recall that

1. $(\omega^\omega, <) \models \forall\bar{X}\forall\bar{W}\forall Y\big(\chi(\bar{X}) \wedge \phi_\preceq(\bar{X}, \bar{W}, Y) \rightarrow \theta_{\omega\,\mathrm{ub}}(Y)\big)$.

Since $\theta_{\omega\,\mathrm{ub}}$ and $\chi(\bar{Y})$ are also FO, the formula in brackets here is FO. By the previous Corollary,

2. $(\omega_1, <) \models \forall\bar{X}\forall\bar{W}\forall Y\big(\chi(\bar{X}) \wedge \phi_\preceq(\bar{X}, \bar{W}, Y) \rightarrow \theta_{\omega\,\mathrm{ub}}(Y)\big)$.

Since there is no unbounded ω-sequence in ω_1, we must have

3. $(\omega_1, <) \models \forall\bar{X}\forall\bar{W}\forall Y(\neg\chi(\bar{X}) \vee \neg\phi_\preceq\big(\bar{X}, \bar{W}, Y)\big)$.

But by 5.25,

4. $(\omega_1, <) \models \forall\bar{X}\exists\bar{W}\exists Y\phi_\preceq(\bar{X}, \bar{W}, Y)$.

Therefore, we must have $(\omega_1, <) \models \forall\bar{X}\neg\chi(\bar{X})$, as desired. ⊣

Summing up, we have

COROLLARY 7.10. $(\omega_1, <)$ *has the* FO *order selection property, but not the* MLO *selection property.*

§8. The Church synthesis problem. What is known as the "Church synthesis problem" was first posed by A. Church in [Ch63] for the case of $(\omega, <)$. Church uses the language of automata theory. It was McNaughton (see [Mc66]) who first observed that it can be equivalently phrased in game-theoretic language.

DEFINITION 8.1. For an ordinal α and a formula $\varphi(X, Y)$, the *McNaughton game* $\mathcal{G}^\alpha_\varphi$ is a game of perfect information of length α between two players, X and Y.

At stage $\beta < \alpha$, X either accepts or rejects β; then, Y also decides whether to accept or to reject β.

For a play π, we denote by X_π (resp. Y_π) the set of ordinals $< \alpha$ accepted by X (resp. Y) during the play. Then,

$$Y \text{ wins } \pi \text{ iff } (\alpha, <) \models \varphi(X_\pi, Y_\pi).$$

What we want to know is: Does either one of X and Y have a *winning strategy* in $\mathcal{G}^\alpha_\varphi$? If so, which of them? That is, can X choose his moves so that, whatever way Y responds we have $\neg\varphi(P_\pi, Q_\pi)$? Or can Y respond to X's moves in a way that ensures the opposite?

Since at stage $\beta < \alpha$, X has access only to $Q_\pi \cap [0, \beta)$ and Y has access only to $P_\pi \cap [0, \beta]$, it seems that the following formalizes the notion of a strategy in this game well:

Definition 8.2 (Causal operator). Let α be an ordinal, $f : \mathcal{P}(\alpha) \to \mathcal{P}(\alpha)$. We call f *causal* (resp. *strongly causal*) iff for all $P, P' \subseteq \alpha$ and $\beta < \alpha$, if

$$P \cap [0, \beta] = P' \cap [0, \beta] \text{ (resp. } P \cap [0, \beta) = P' \cap [0, \beta)),$$

then

$$f(P) \cap [0, \beta] = f(P') \cap [0, \beta].$$

That is, if P and P' agree *up to and including* (resp. *up to*) β, then so do $f(P)$ and $f(P')$.

So a winning strategy for Y is a causal $f : \mathcal{P}(\alpha) \to \mathcal{P}(\alpha)$ such that for every $P \subseteq \alpha$, $(\alpha, <) \models \varphi(P, f(P))$. It is easy to come up with examples of formulas where Y has no winning strategy. For example, if $\varphi_X(X, Y)$ says "if $X = \varnothing$, then $Y = \text{All}$; otherwise, $Y = \varnothing$" and $\alpha \geq 2$, then X has a winning strategy in $\mathcal{G}^\alpha_{\varphi_X}$. This leads to

Definition 8.3 (Game version of the Church synthesis problem). Let α be an ordinal. Given a formula $\varphi(X, Y)$, decide whether Y has a winning strategy in $\mathcal{G}^\alpha_\varphi$.

Note the similarities and dissimilarities between this problem and the uniformization problem. In uniformization we are also given a formula $\varphi(X, Y)$ and to every P we try and "respond" with a Q such that $\varphi(P, Q)$ holds. Only we do not restrict ourselves to causal responses. Thus, the formula φ_X described above is its *own* uniformizer in every α. On the other hand, we do restrict ourselves to *definable* (in $(\alpha, <)$) responses. In the definition just given, we did *not* require that the strategy (=causal operator) be definable.

In [BL69], Büchi and Landweber prove the decidability of the Church synthesis problem in $(\omega, <)$. What is even more important, they show that in the case of $(\omega, <)$ we *can* restrict ourselves to definable causal (or strongly causal) operators.

Theorem 8.4 (Büchi-Landweber, 1969). *Let $\varphi(X, Y)$ be a formula. Then:*

- Determinacy*: One of the players has a winning strategy in the game $\mathcal{G}^\omega_\varphi$.*
- Decidability*: It is decidable* which *of the players has a winning strategy.*
- Definable strategy*: The player who has a winning strategy, also has a* definable *winning strategy.*
- Synthesis algorithm*: We can compute a formula $\psi(X, Y)$ that defines (in $(\omega, <)$) a winning strategy for the winning player in $\mathcal{G}^\omega_\varphi$.*[25]

[25] If ψ is to define a strategy for X, then we should take Y as the domain variable, so to speak.

As mentioned in the introduction, it seems that Büchi and Landweber believed their theorem would generalize to all countable ordinals. However, if we let $\varphi(X, Y)$ say "Y is an unbounded ω-sequence" (so, we ignore X), then by Proposition 3.17, Y cannot have a *definable* strategy in $\mathcal{G}_{\varphi}^{\omega^{\omega}}$. Indeed, if $\psi(X, Y)$ defined such a strategy, then $\exists X(X = \varnothing \wedge \psi(X, Y))$ (say) would select an unbounded ω-sequence in ω^{ω}. On the other hand, Y does win this game: she simply plays a fixed unbounded ω-sequence $S \subseteq \omega^{\omega}$, ignoring X's moves.

What we know at present concerning the extension of Büchi and Landweber's result to countable ordinals is summarized in the following theorem.

THEOREM 8.5. *Let α be a* countable *ordinal, $\varphi(X, Y)$ a formula.*

- Determinacy*: One of the players has a winning strategy in the game $\mathcal{G}_{\varphi}^{\alpha}$.*
- Decidability*: It is decidable* which *of the players has a winning strategy.*
- Definable strategy*: If $\alpha < \omega^{\omega}$, then the player who has a winning strategy, also has a* definable *(in $(\alpha, <)$) winning strategy. For every $\alpha \geq \omega^{\omega}$, there is a formula for which this fails.*
- Synthesis algorithm*: If $\alpha < \omega^{\omega}$, we can compute a formula $\psi(X, Y)$ that defines a winning strategy for the winning player in $\mathcal{G}_{\varphi}^{\alpha}$.*

A proof of this theorem can be found in [Ra]. It uses the composition method to reduce games of every countable length to games of length ω.

Note that Theorem 8.5 leaves an interesting question open, namely: can we decide those McNaughton games for which there exists a definable winning strategy? It seems reasonable to expect this is the case.

CONJECTURE 8.6. *There is an algorithm that, given $\alpha \in [\omega^{\omega}, \omega_1)$ and a formula $\varphi(X, Y)$, decides whether there is a* definable *winning strategy in $\mathcal{G}_{\varphi}^{\alpha}$, and if so, returns a ψ defining one.*

Rabinovich ([Ra]) shows that if the conjecture holds for $\alpha = \omega^{\omega}$, then it is true.

Next, we have seen that the only stumbling block for selection in $(\omega^{\omega}, <)$ was selecting an unbounded ω-sequence (recall Proposition 5.31). We believe an analogous statement may be true concerning definability of a winning strategy for games of length ω^{ω}:

CONJECTURE 8.7. *For every formula $\varphi(X, Y)$, there is a formula $\psi(W, X, Y)$ such that for every unbounded ω-sequence $S \subseteq \omega^{\omega}$, $\psi(S, X, Y)$ defines in $(\omega^{\omega}, <)$ a winning strategy for the winner of $\mathcal{G}_{\varphi}^{\omega^{\omega}}$. Moreover, ψ can be computed from φ.*

Finally, we mention that for uncountable ordinals the situation changes radically. Let $\varphi_{\mathrm{spl}}(X, Y)$ say: "X is stationary, $Y \subseteq X$ and both Y and $X \setminus Y$ are stationary" (recall Definition 3.11). Then it follows immediately from [LaS] that each of the following statements is consistent with ZFC:

1. None of the players has a winning strategy in $\mathcal{G}_{\varphi_{\mathrm{spl}}}^{\omega_1}$.
2. Y has a winning strategy in $\mathcal{G}_{\varphi_{\mathrm{spl}}}^{\omega_1}$.
3. X has a winning strategy in $\mathcal{G}_{\varphi_{\mathrm{spl}}}^{\omega_1}$.

In other words, ZFC can hardly tell us anything concerning this game. On the other hand, S. Shelah (private communication) tells us he believes it should be possible to prove:

CONJECTURE 8.8. *It is consistent with ZFC that $\mathcal{G}_{\varphi}^{\omega_1}$ is determined for* every *formula φ.*

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