Timer formulas and decidable metric temporal logic

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Abstract

We define a quantitative temporal logic that is based on a simple modality within the framework of monadic predicate logic. Its canonical model is the real line (and not an ω-sequence of some type). It can be interpreted either by behaviors with finite variability or by unrestricted behaviors. For finite variability models it is as expressive as any logic suggested in the literature. For unrestricted behaviors our treatment is new. In both cases we prove decidability and complexity bounds using general theorems from logic (and not from automata theory). The technical proof uses a sublanguage of the metric monadic logic of order, the language of timer normal form formulas. Metric formulas are reduced to timer normal form and timer normal form formulas allow elimination of the metric.

1 Introduction

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Temporal Logic (TL) is a convenient framework for reasoning about the evolving of a system in time. This made TL a popular subject in the Computer Science community and it enjoyed extensive research during the last 20 years. A celebrated theorem by Kamp [Kamp68] proves that the temporal logic based on the two modalities "Since" and "Until" has the same expressive power as the Monadic first order predicate logic, whether time is taken as the set of natural numbers *(discrete time)* or as the set of non negative real numbers (*continuous time*). For discrete time there is also a tight connection between second order predicate logic and automata theory [Büc60], associating with every formula an automaton that accepts exactly the runs that satisfy the formula. In particular this associates also an automaton with every first order logical formula. In classical logic, constructing the automaton that corresponds to a first order formula is neither simpler nor more direct than for second order formulae. In contrast, for temporal logic the construction of the automaton is more straightforward. This may have contributed to the popularity of temporal logic for the treatment of discrete time.

When it came to continuous time models, most of the researchers decided to abandon the natural model of the real line in favor of discrete models in which the elements of the sequence were decorated by some information concerning the real line. This may have been an attempt to pursue the connection with automata theory, as automata were traditionally associated with sequences. We believe that this decision had far reaching consequences that complicated the subsequent research enormously. Firstly the theory could only deal with systems that satisfied a finite variability condition and it did not apply to the behavior of general systems in time (of course, once a general formalism exists it may be worthwhile to modify it to ease the treatment of a favorite family). In some of the approaches it is not even clear which finite variability model was intended. Secondly, the equivalence between first order classical logic and temporal logic that held in the real line by Kamp's theorem was lost. It followed that the choice of a particular temporal logic became an arbitrary choice. It could not be proved robust by the usual argument that it has the same expressive power as classical logic.

Things got worse when the time came to introduce metric into the logic. The pair of timed sequences together with temporal logic is a poor setting for it. When dealing with the real line in predicate logic the addition of the $+$ function is just a trivial change in the signature of the logic. The resulting logic is rich and it is the natural place to look for sublogics that express different properties of the metric. This obvious approach has no counterpart when the model is a sequence and the logic is a temporal logic. Thus a host of different formalisms was introduced which, combined with the different sequential models produced an astonishing variety of "model-logic" pairs. In spite of the efforts invested over the years the research did not converge toward an accepted sequential model and an appropriate metric temporal logic.

In this paper we adopt the classical approach and assume that the underlining model is the ordered real line itself with unary predicates corresponding to properties that may or may not hold at each point. In section 2 we describe the classical background; we define the predicate monadic logic of order and we cite the main decidability results for this logic: for first and for second order logic, for finite variability models and for unrestricted models. All the later decidability results for the metric case will be achieved by reduction to these non metric results. In section 3 we enrich the language, adding to it the $+1$ function. The resulting logic is undecidable and the rest of the paper concerns itself with identifying a good sublanguage and checking its properties. First however we introduce the main technical notion that we use, *the timer*. A predicate X is a timer for the predicate Y if X holds exactly at points where Y has been holding continuously for the last unit of time. Formulas in timer normal form use the $+1$ only to say that one predicate is a timer for another one. In section 4 we prove the main technical "metric elimination" result: There is a simple algorithm (simple to apply, not simple to prove correct) which associates with any timer normal form formula a formula which does not mention the metric, such that one formula is satisfiable if and only if the other is. This reduces the questions on decidability for timer normal form onto the corresponding question in the non metric case. In section 5 we tend to temporal logic and recall its main properties, in particular Kamp's theorem about the equivalence between temporal logic and monadic first order logic of order. In section 6 we define the temporal logic QTL- quantitative temporal logic. In addition to the "Until" and "Since" modalities it has two very simple metric constructs. These are the constructs "φ will happen within one unit of time", and "φ has happened within the last unit of time". These constructs are easily defined in the monadic predicate logic with the $+1$ function. This also gives rise to the natural language $QMLO$- quantitative monadic logic of order. We show that with the added expressive power the temporal logic and the classical logic are still equivalent in expressive power.

In section 7 we show that these simple metric logics are quite expressive. We show that they can express seemingly stronger alternative connectives, in particular all the connectives that we found in the literature. In section 8 we give an algorithm to reduce $QMLO$ and QTL formulas to timer normal form formulas. We utilize the decidability theorem for timer formulas to prove decidability for QTL and $QMLO$ in the class of general behaviors, as well as in the class of finite variability behaviors (remember that the general behaviors could not even be represented using the method of timed sequences, let alone proved decidable).

Finally in section 9 we determine the complexity of our algorithms. We apply automata free proofs of the complexity of pure temporal logic to show that the satisfiability is $PSPACE$ whether we want the answer in general behaviors or in finite variability behaviors. For finite variability behaviors this result was known ([AFH96]). We stress the fact that our proof does not rely on automata theory, so that there is no evidence that automata can give tighter estimates, even in the special cases when they apply.

The main results of this paper were announced in [HR99a]. A survey and detailed comparison with other approaches, models and logics was provided without proofs in [HR2004].

2 Monadic Logic of Order

The most natural language to discuss systems that evolve in time is classical predicate logic. The language has a name for the (discrete or continuous) order relation of the time line, and a supply of unary predicates each to denote a property which the system may or may not have at any point in time. This leads to the following definition of the syntax and the semantics of the pure (non metric) monadic predicate logic:

The syntax of the monadic predicate logic of order - MLO has in its vocabulary *individual* (first order) variables $t_0, t_1, \dots$, monadic *predicate* names $X_0, X_1, \dots$, and one binary relation $<$ (the order). **Atomic formulas** are of the form $X(t), t_1 < t_2$ and $t_1 = t_2$. **Well formed formulas** of the monadic logic MLO are obtained from atomic formulas using Boolean connectives $\neg, \vee, \wedge, \rightarrow$ and the (first order) quantifiers $\exists t$ and $\forall t$. **Second order** Monadic Logic of Order formulas are obtained using also second order quantifiers $\exists X$ and $\forall X$.

Warning on Terminology: In this work we are mainly interested in first-order monadic logic of order, and unless second order is explicitly mentioned the name MLO is reserved for first order logic.

A structure for MLO is a pair $M = \langle A, < \rangle$ where $<$ is a linear order over A. To be exact a structure $M = \langle A, <, P_1, \cdots, P_n \rangle$ may have some predetermined fixed one-place predicates (sets) $P_1, \cdots, P_n$, but we shall use the simple notation $\langle A, < \rangle$ when the particular special predicates are not essential to the discussion.

The important examples for us are $\langle R^+, < \rangle$ and $\langle N, < \rangle$; the non-negative real line and the non-negative integers.

We shall not repeat the standard inductive definition saying when is a formula satisfied. We recall the notation:

$$M, \tau_1, \dots \tau_k; P_1, \dots, \models_{MLO} \varphi(t_1, \dots, t_k; X_1, \dots, X_m)$$

which we also abbreviate to $M \models \varphi[\tau_1, \dots, \tau_k; P_1, \dots, P_m]$ or even to $M \models \varphi[\overline{\tau}, \overline{P}]$ where M is a structure, $\tau_1, \dots, \tau_k$ are elements of M, $P_1 \cdots, P_m$ are unary predicates (i.e sets) in M, φ is a formula and $t_1, \cdots t_k; X_1, \dots, X_m$ include all the variables of φ.

The non negative real line $\langle R^+, < \rangle$ and the non negative integers $\langle N, < \rangle$ respectively are called the **continuous canonical model** and the **discrete canonical model**.

Finite Variability: A function η from a bounded interval I (of reals or rationals) into a set Σ has *finite variability* if there exists a partition of I into a finite sequence $I_1, \dots, I_k$ of subintervals such that η is constant on every subinterval I_j $(j = 1, \dots, k)$. A function η from reals or rationals has finite variability if the restriction of η on every bounded interval has finite variability.

From the above definition it follows that a function η from a bounded subinterval $(a,\ b)$ of the reals into a set Σ has *finite variability* if there exists a finite increasing sequence $a = a_0 < a_1 \cdots < a_n = b$ such that η is constant on every interval (a_i, a_{i+1}). A function η from the non negative real line R^+ into a set Σ has *finite variability* if there exists an increasing unbounded ω-sequence $0 = a_0 < a_1 \cdots < a_n < \dots$ such that η is constant on every interval (a_i, a_{i+1}).

In the computer science literature it is often assumed that a system has finite variability; i.e, that every monadic predicate involved has finite variability (we identify predicates with their characteristic functions).

The (first or second order) *Finite Variability Monadic Logic of Order* is syntactically the logic introduced before with the additional assumption concerning the semantics that all the predicate names and second-order variables range over finite variability predicates.

Remark 1 (About Finite Variability) 1. *The distinction between the class of models with finite variability and the class of unrestricted models is significant also for first order monadic logic. It is easy to express in first order language the fact that any non-singular interval has a point where the predicate X holds and a point where $\neg X$ holds. This formula is satisfiable in the class of unrestricted models but not in the class of finite variability models.*

2. *In the real line model finite variability can be defined by a* first order *formula: The predicate X has finite variability iff it satisfies the formula*

$$\forall t \exists t_1[(t < t_1) \wedge \forall t_2(t < t_2 < t_1) \rightarrow (X(t_2) \leftrightarrow X(t_1)]$$
$$\bigwedge$$
$$\forall t > 0 \exists t_1[(t_1 < t) \wedge \forall t_2(t_1 < t_2 < t) \rightarrow (X(t_2) \leftrightarrow X(t_1)]$$

($t > 0$ can be written as $\exists s(s < t)$).

3. *In contrast, over the rational line finite variability can be defined by a second-order monadic formula but not by a first order one.*

Here are the main decidability results concerning the monadic logic of order:

Theorem 2 *1. The satisfiability of monadic first order logic of order over the reals is decidable [BG85].*

2. The satisfiability of monadic second-order logic of order over the reals is undecidable [She75]

3. The satisfiability of monadic second order logic over the reals with finite variability interpretation is decidable [Rab98].

4. The satisfiability of monadic second order logic over the rationals is decidable [Ra69].

3 Metric Monadic Logic and Timer Formulas

The logic MLO is not suitable to deal with quantitative statements like "X will occur within one unit of time", which are needed for the specifications of systems. A natural strengthening of the language can be obtained by including in it a name for the two place addition function $+$. A more cautious approach is to include the unary function $t+1$ as a function or as a binary relation. It turns out that even this is too generous and it yields an undecidable logic. Nevertheless we shall use this metric logic as a framework in which we look for useful decidable fragments.

Definition 3 *MLO^+, the monadic logic of order with a $+1$ function, is the first order logic built from the binary relation $<$, monadic predicate names and one unary function symbol which we denote either by $S(t)$ or by $t+1$. The standard (canonical) model for this logic is the non-negative real line with the usual $+1$ function and the usual order relation.*

We introduce the following suggestive notations:

$$(\exists v)^{<t}_{>s}\varphi \equiv \exists v((s<v<t)\wedge\varphi) \qquad \textit{and} \qquad (\forall v)^{<t}_{>s}\varphi \equiv \forall v((s<v<t)\rightarrow\varphi)$$

where t and s are terms in the language. We do not have substraction in the language and we define:

$$(\exists v)^{<t}_{>t-1}\varphi \equiv \exists v(t<v+1<t+1\wedge\varphi)$$

Note that for points t smaller than 1 it just means that there is some previous v satisfying φ. We define similarly $(\forall v)^{<t}_{>t-1}\varphi$ (note that $(\forall v)^{<t}_{>t-1}\varphi$ will always be satisfied at $t=0$). We shall also extend these notations to include weak inequalities instead of sharp ones.

Our strategy is the following: we identify a basic metrical connection between pairs of predicates that we call "Q is a *timer* for P": Q holds whenever P has been true for the entire last unit of time. For any metric logic (first or second order, and later temporal logic) we identify the sublanguage of *formulas in timer normal form.* These are formulas in which the metric enters only as a means to claim that one predicate is a timer for another, and all the rest is non metrical. Our interest in this seemingly unnatural language is due to the following features:

- The timer relation can be approximated by a non metrical formula in such a way that every formula in timer normal form gives rise to a non metrical formula such that the two formulas are both valid, or both invalid.
- Thus decidability for formulas in timer normal form follows from the decidability of the corresponding non metrical logic.
- In fact even the complexity of the problem can be reduced to the non metrical logic.

Therefore the search for decidable metrical logic may be replaced by a search for logics for which every formula is effectively equivalent to a formula in timer normal form. One more feature of the language of formulas in timer normal form is:

- There are very interesting logics which allow an effective translation to timer normal form.

We define the notion of *Timer* by a formula of MLO^+:

Definition 4 (Timer normal form) *1. We call the following MLO^+ formula* a simple Timer Formula

$$\text{Timer}(X, Y) \equiv \forall t(Y(t) \longleftrightarrow (\forall t_1)^{<t}_{>t-1} X(t_1))$$

(X is different from Y). We say that Y is a timer for X

2. For every n we define the formula:

$$\text{Timer}_n(X_1, \ldots X_n, Y_1, \ldots Y_n) \equiv \bigwedge_i Timer(X_i, Y_i)$$

(All $2n$ variables are different from each other.)

3. A formula is said to be in first order (second order) **timer normal form** *if it has the form*

$$\exists \overline{W}(\text{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \wedge \varphi),$$

where $\overline{W}$ is a list of monadic variables and φ is (non metrical) first order (or second order) monadic, respectively.

Note that whether or not X has finite variability a timer Y for X always has. Note also that by its definition a timer Y holds on a (topologically) closed set. By its definition, if Y does not holds at t and will hold in the future then there is a first point after t where Y holds, and it is preceded by an interval of length one where Y does not hold. Since a finite interval contains only finitely many disjoint interval of length one Y can change its value only finitely often in a finite interval.

The set of formulas in timer normal form is (semantically) closed under conjunctions, disjunctions and the prefixing of existential predicate quantifiers. The transformations to timer normal form may involve change of variables. For example, $\text{Timer}(X, Y) \wedge \text{Timer}(Y, Z)$ is equivalent to $\exists W \text{Timer}(X, Y) \wedge \text{Timer}(W, Z) \wedge \forall t(Y(t) \leftrightarrow W(t))$. Formulas in timer normal form are also closed (up to equivalence) under the substitution of a metric free formula in place of a free predicate variable: Instead of substituting φ for X we may simply add the conjunct $\forall t(X(t) \leftrightarrow \varphi(t))$ to the matrix and add the prefix $\exists X$ to the whole formula. It is easy to see that up to simple syntactic transformation the resulting formula is in timer normal form.

Remark 5 (Other "timers") *In the definition of "timer" above Y is a "past persistency timer" for X as it measures that X persisted at least for one unit period in the past. This is the timer that we chose as our basic timer. Three other timers are the "past occurrence timer" which is active if X was true at least at one point in the past unit interval, and the future persistence and future occurrence timers which are defined similarly. These three timers can be expressed by timer normal formulas.*

1. *The past occurrence timer is defined as follows: $\forall t(Y(t) \longleftrightarrow (\exists t_1)^{<t}_{>t-1}X(t_1))$. This formula is equivalent to the following formula: $\exists Z(Timer(\neg X, Z) \wedge \forall t(Y(t) \longleftrightarrow \neg Z(t))$*

2. *The future occurrence timer is $\forall t(Y(t) \longleftrightarrow (\exists s)^{<t+1}_{>t}X(s)))$. To use the convenience of the temporal logic notations we delay the proof that the future occurrence timer is expressible by a timer normal form formula to Section 8 Theorem 27.*

3. *The future persistence timer is defined from the future occurrence timer analogously to (1).*

4 Metric elimination for timer formulas

4.1 Non metrical approximations of timer formulas

Our main theorem associates with every timer normal form formula φ a metric free formula $\overline{\varphi}$ such that φ is satisfiable (or valid) in R^+ with its natural metric iff $\overline{\varphi}$ is satisfiable (respectively, valid) in R with the metric ignored. We start the discussion associating with any timer formula $\text{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ a (metric free) first order *MLO* formula $\overline{\text{Timer}}_n(X_1, \ldots, X_n, Y_1, \ldots Y_n)$ which summarizes the main first order property of the timer. The main theorem will then prove that any model of $\overline{\text{Timer}}_n(X_1, \ldots, X_n, Y_1, \ldots Y_n)$ can be enriched by a non standard $t+1$ function for which $\text{Timer}_n(X_1, \ldots X_n, Y_1, \ldots, Y_n)$ holds. This non standard metric line is isomorphic to the standard metric line and from this the main theorem will follow by the general principle of preservation of truth under isomorphism.

Let $\text{Timer}_n(X_1, \ldots X_n, Y_1, \ldots, Y_n)$ be a given timer formula. The pure monadic first order formula $\overline{\text{Timer}}_n(X_1, \ldots, X_n, Y_1, \ldots Y_n)$ associated with it is the conjunction of nine formulas: Three formulas T1-T3 listing the non metrical requirements enabling Y_i to be a timer, four formulas P1-P4 listing the non metrical properties that enable Y_i to be a timer for X_i and two formulas C1-C2 listing the non metrical properties by which the different timers conform with each other.

1. T1) For every i, Y_i holds at zero:

$$\forall t[\forall s(t \leq s) \longrightarrow Y_i(t)]$$

Zero has no name in our language, but "$t = 0$" is defined by $\forall s(t \leq s)$.

2. T2) Y_i has finite variability:

$$\Big\{\forall t(\exists t_1)_{>t}(\forall t_2)^{<t_1}_{>t}(X_i(t_2) \leftrightarrow X_i(t_1))\Big\} \bigwedge \Big\{(\forall t)_{>0}(\exists t_1)^{<t}(\forall t_2)^{<t}_{>t_1}(X_i(t_2) \leftrightarrow X_i(t_1))\Big\}$$

3. T3) The set of points where Y_i holds is a closed set:

$$\forall t\Big(\neg Y_i(t) \rightarrow \big\{\exists s_1 \exists s_2(s_1 < t < s_2 \wedge (\forall v)^{<s_2}_{>s_1} \neg Y_i(v))\big\}\Big)$$

(because of axiom T1 we don't have to consider the case where $t = 0$)

4. P1) If Y_i holds at a point other than zero then X_i has been true for a while just before that point:

$$\forall t\Big[\big(t \neq 0 \wedge Y_i(t)\big) \rightarrow (\exists s)^{<t}(\forall v)^{<t}_{>s} X_i(v)\Big]$$

5. P2) If X_i does not hold at a point then Y_i does not hold for a while:

$$\forall t\Big(\neg X_i(t) \rightarrow \exists s\big[t < s \wedge (\forall r)^{<s}_{>t} \neg Y_i(r))\big]\Big)$$

6. P3) If X_i is true from some point c forever then Y_i will be true from some point forever.

$$\big(\exists t(\forall s)_{>t} X_i(s))\big) \rightarrow \big(\exists t(\forall s)_{>t} Y_i(s)\big)$$

7. P4) If X_i and Y_i are true at a point b then Y_i stays true as long as X_i does

$$\forall t\Big\{\big(Y_i(t) \wedge X_i(t)\big) \longrightarrow (\forall s)_{>t}\Big(\big[(\forall v)^{<s}_{>t} X_i(v)\big] \longleftrightarrow \big[(\forall v)^{<s}_{>t} Y_i(v)\big]\Big)\Big\}$$

8. C1) If at a point Y_i holds and Y_j does not hold then X_j holds for a shorter interval in the past than X_i

$$\forall t[Y_i(t) \wedge \neg Y_j(t) \longrightarrow \exists s < t((\forall v)^{<t}_{>s} X_i(v) \wedge (\exists v)^{<t}_{>s} \neg X_j(v))]$$

(note that axiom P1 gets close to be a special case of C1).

9. C2) Conversely: If X_i holds for a longer interval in the past than X_j then Y_i will be triggered before Y_j:

$$\forall t(\forall s)_{>t}\Big[(\forall v)^{<t}_{>s}(X_i(v) \wedge \neg Y_i(v)) \wedge (\exists v)^{<t}_{>s} \neg X_j(v)) \longrightarrow \neg Y_j(t)\Big]$$

Definition 6 *1. For every n, $\overline{\text{Timer}}_n$ is the pure first-order MLO formula which is the conjunction of the formulas 1-9 above, for $i, j \leq n$.*

2. *For every formula ψ in timer normal form, $\psi \equiv \exists \overline{W}(\mathrm{Timer}_n(X_1,\ldots,X_n,Y_1,\ldots,Y_n) \wedge \varphi)$ we denote by $\overline{\psi}$ the non metrical formula $\exists \overline{W}(\overline{\mathrm{Timer}}_n(X_1,\ldots,X_n,Y_1,\ldots,Y_n) \wedge \varphi)$*

Clearly if ψ is in timer normal form then every model of ψ is also a model of the non metrical formula $\overline{\psi}$. Our main theorem proves that as far as satisfiability is concerned they are equivalent.

Theorem 7 (Metric elimination theorem) *Let ψ be in timer normal form, $\psi \equiv \exists \overline{W}(\mathrm{Timer}_n(X_1,\ldots,X_n,Y_1,\ldots,Y_n) \wedge \varphi)$, and let $\overline{\psi}$ be its non metrical approximation: $\overline{\psi} \equiv \exists \overline{W}(\overline{\mathrm{Timer}}_n(X_1,\ldots,X_n,Y_1,\ldots,Y_n) \wedge \varphi)$*

Then ψ is satisfiable iff $\overline{\psi}$ is satisfiable.

The theorem will be proved in the next subsection. Note that since $\overline{\psi}$ is obtained from ψ effectively we get an algorithm to check satisfiability for formulas in timer normal form, provided satisfiability is decidable for the non metrical formulas $\overline{\psi}$ which are obtained.

4.2 The proof of the metric elimination theorem

Let R^+ be a model of $\overline{\psi} \equiv \exists \overline{W}(\overline{\mathrm{Timer}}_n(X_1,\ldots,X_n,Y_1,\ldots,Y_n) \wedge \varphi)$. Thus there are sets $P_1,\ldots,P_n,Q_1,\ldots,Q_n$ such that $R^+ \models \overline{\mathrm{Timer}}_n(P_1,\ldots,P_n,Q_1,\ldots,Q_n) \wedge \varphi'$ (we start to use the more convenient notation of the fact that the sets $P_1,\ldots,P_n,Q_1,\ldots,Q_n$ satisfy $\overline{\psi}$. Also, writing φ' we suppress the names of the sets that occur in φ'). We will construct an order preserving bijection $T : R^+ \longrightarrow R^+$ such that $R^+ \models \mathrm{Timer}_n(T(P_1),\ldots,T(P_n),T(Q_1),\ldots,T(Q_n)) \wedge \varphi''$, where φ'' is obtained from φ' by replacing each parameter set A by its image $T(A)$. In fact all that we must show is that $R^+ \models \mathrm{Timer}_n(T(P_1),\ldots,T(P_n),T(Q_1),\ldots,T(Q_n))$. The fact that φ'' also holds is a well known, easily checkable fact about monotone bijections (isomorphisms in *MLO*). So we assume that $R^+ \models \overline{\mathrm{Timer}}_n(P_1,\ldots,P_n,Q_1,\ldots,Q_n)$. We will construct unbounded increasing sequences of points $b_1,b_2,b_3,\cdots$, and $d_1,d_2,d_3,\cdots$, and an increasing sequence of monotone bijections $T_1,T_2,T_3,\cdots$ with $T_i : [0,b_i] \longrightarrow [0,d_i]$, and define T to be the union of these functions. The bounded approximations T_k of T will be defined in a way that will assure that:

$$R^+ \models \mathrm{Timer}_n(T(P_1),\ldots,T(P_n),T(Q_1),\ldots,T(Q_n)).$$

Assume that Y has the timer properties with respect to X, as specified in the list above. Assume that Y does not hold at a point t, but that it holds sometimes in the future. The largest lower bound for the set of points in the future that satisfy Y is itself a point that satisfies Y by the closedness property (T3). Hence there is a first point in the future where Y holds. Such a point b will be called a *triggering point* of Q. By the main property (P1) in the list, X holds for a while before b. The maximal interval where X holds until it triggers Y will be called a *triggering interval* for Q. Clearly these are the intervals that must have length 1 in our metrization. On the other hand we must make sure that every *non triggering* interval, where X holds without causing Y to hold, must have length less than 1. This is all that we are after. Now any point belongs to only finitely many triggering and non triggering intervals, for different pairs of X and Y, and the different triggering and non triggering intervals do not conflict with each other by the axioms (C1) and (C2). This will enable us to deal with one triggering interval at a time, declare its length 1, and define the length of all nearby non triggering intervals to have shorter length.

First the definition:

Definition 8 *Let* $R^+ \models \overline{\text{Timer}}_n(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$.

1. *A triggering interval for* Q_i *is a non empty interval (a,b) with the following properties:*
   - $P_i \wedge \neg Q_i$ *holds for any point in the (open) interval (a,b)*
   - $Q_i(b)$
   - *a is the smallest point with this property, i.e,* $\forall t_1 < a (\exists t)^{\leq a}_{>t_1} \neg P_i(t)$

   The point b is a triggering point for Q_i *and the point a is an initializing point for* Q_i.
2. *A* non triggering interval for Q_i *is an interval (a,b) with the following properties:*
   - P_i *holds continuously in the interval (a,b) but not in any larger open interval that contains (a,b).*
   - Q_i *does not hold at b.*

The main properties of these intervals are summarized in the following lemma:

Lemma 9 (The main properties of triggering intervals)
1. *Every open interval that satisfies continuously* $P_i \wedge \neg Q_i$ *is contained in a unique triggering interval for* Q_i *or in a unique non triggering interval (triggering, if* Q_i *is activated before, or just as,* P_i *ceases to hold).*
2. *Every bounded interval contains at most finitely many triggering and initializing points.*
3. *The initializing points form a finite sequence or an infinite sequence that diverges to infinity.*
4. *For any point a there are at most n (the number of timers) non triggering intervals that contain a.*

Proof

1. Let (a, b) be such an interval and let (a', b') be a maximal open interval containing (a, b) where $P_i \wedge \neg Q_i$ holds (i.e, the union of such open intervals containing (a,b)). By axiom P3 b' is not ∞. If Q_i holds at b' this is a triggering interval. Else it is a non triggering interval.
2. In a given interval there are only finitely many triggering points for Q_i because of finite variability (axiom T2), since every triggering point is a point where Q_i changes its value. Every initializing point corresponds to a unique triggering point by axioms C1 and C2. Therefore there are only finitely many initializing points.

3. This follows immediately from the fact that every bounded interval contains only finitely many elements of the sequence.

4. Two triggering intervals for a single P_i cannot overlap since their union would testify that they are not maximal.

$\square$

We are now ready for the proof of the main lemma:

Lemma 10 (Isomorphism lemma) *Let* $R^+ \models \overline{\text{Timer}}_n(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$.

There is a monotone bijection T from R^+ onto R^+ such that:

- *If (a,b) is a triggering interval for some Q_i then*
  $T(b) = T(a) + 1$

- *If (a,b) is a non triggering interval for some Q_i then $T(b) < T(a) + 1$*

Proof

Let $a_1, a_2, \cdots$ be the increasing sequence of initializing points, and for every n let $b_n > a_n$ be the triggering point that corresponds to a_n. We assume first that this sequence is infinite and by lemma 8(3) it diverges to ∞ (later we shall deal with the case that there are only finitely many triggering intervals). We define inductively a bijection $T_n : [0, b_n] \rightarrow [0, d_n]$ with the following properties:

1. $\{d_n : n \in Nat\}$ is an unbounded increasing sequence and T_{n+1} extends T_n.

2. T_n is monotone.

3. T_n respects triggering intervals: If (a,b) is a triggering interval with $b \leq b_n$ (so that $a = a_i$ and $b = b_i$ for some $i \leq n$) then $T_n(b) = T_n(a) + 1$

4. T_n respects non triggering intervals: If (a,b) is a non triggering interval with $b \leq b_n$ then $T_n(b) < T_n(a) + 1$

We start with $[0, b_1]$. There are two cases to consider: $a_1 = 0$ and $a_1 \neq 0$.

Case 1, $a_1 = 0$: This means that some P_i holds on $(0, b_1)$ and b_1 is the first point, apart from 0, where Q_i holds (note that by axiom T1 and P4 P_i does not hold at 0). We choose d_1 to be 1 and map $[0, b_1]$ by an arbitrary monotone function T_1 onto $[0, 1]$. This T_1 has all the properties required: Every triggering interval which is contained in $[0, b_1]$ is equal to $[0, b_1]$ so that the third property is satisfied vacuously. On the other hand every proper subinterval is transformed into a proper subinterval of $[0, 1]$, so that the forth property holds.

Case 2, $0 < a_1$: We choose some $c < 1$ and map $[0, a_1]$ monotonically onto $[0, c]$. By lemma 8 there are at most n non triggering intervals that contain the point a_1 and by axioms C1 and C2 they all terminate before b_1. Let $e < b_1$ be larger than all the right endpoints of non triggering intervals that contain a_1. We map $T_1(e) = 1$ and extend the map monotonously from $[a_1, e]$ onto $[c, 1]$ and then also from $[e, b_1]$ onto $[1, 1 + c]$. By this choice non triggering intervals that start before a_1 are mapped into intervals shorter than 1 by the choice of e whether they terminate before a_1 or contain a_1. Non triggering intervals that start after a_1 and terminate before b_1 are shorter than $[a_1, b_1]$ which is exactly 1. On the other hand $[a_1, b_1]$ is the only triggering interval contained in $[0, b_1]$ and it is mapped onto an interval of length one.

We turn now to the inductive step. We assume that T_n is defined from $[0, b_n]$ onto $[0, d_n]$,and that $T_n(a_n) = d' = d_n - 1$. Remember that we assume that there is a next triggering point a_{n+1}. There are again two cases to consider: $a_{n+1} \leq b_n$ and $a_{n+1} > b_n$

Case 1, $a_{n+1} \leq b_n$: Since $a_n < a_{n+1} \leq b_n$, we have $T_n(a_{n+1}) = d' + c$ with $c \leq 1$. We define $T_{n+1}(b_{n+1}) = d_n + c = d_{n+1}$. It remains to map $[b_n, b_{n+1}]$ monotonously onto $[d_n, d_n + c]$ in a way that will preserve condition 4 for non triggering intervals that end before b_{n+1} and after b_n (intervals ending before b_n are taken care of by the induction assumption). If such an interval begins after a_{n+1} then monotonicity will assure that its image is smaller than $(d_n + c) - (d' + c) = 1$. Thus we are interested only in the finitely many non triggering intervals that contain a_{n+1} and b_n. We may choose points e_1 and e_2 such that $a_n < e_1 < a_{n+1} < b_n < e_2 < b_{n+1}$ so that all the non triggering intervals that contain a_{n+1} and b_n are contained in, and are shorter than, $[e_1, e_2]$. Now $T_n(e_1) = d' + c'$ with $c' < c$. We define next $T_{n+1}(e_2) = d_n + c'$. We conclude by extending T_{n+1} monotonously from $[b_n, e_2]$ onto $[d_n, d_n + c']$ and from $[e_2, b_{n+1}]$ onto $[d_n + c', d_n + c]$. It follows from the choice of e_1 and e_2 that condition 4 remains true for T_{n+1}.

Case 2, $b_n < a_{n+1}$: Let $e_1 < e_2$ be two points between a_n and b_n such that all the non triggering intervals that contain b_n start beyond e_2. $T_n(e_1) = T_n(a_n) + c_1$ and $T_n(e_2) = T_n(a_n) + c_2$. We define $T_{n+1}(a_{n+1}) = d_n + c_1$ and extend T monotonously from $[b_n, a_{n+1}]$ onto $[d_n, d_n + c_1]$. Non triggering intervals that terminate before a_{n+1} remain small. Next we choose a point e_3 between a_{n+1} and b_{n+1} which lies beyond the non triggering intervals that contain a_{n+1} and we define $T_{n+1}(e_3) = d_n + c_2$. It remains to extend monotonously T_{n+1} from $[a_{n+1}, e_3]$ to $[d_n + c_1, d_n + c_2]$ and from $[e_3, b_{n+1}]$ to $[d_n + c_2, d_n + c_1 + 1]$.

This completes the inductive proof and in the limit we obtain a mapping T that satisfies requirements 2-4. To show that the sequence d_n is unbounded we pick a subsequence a_{i_n} of the initializing points with the property that $b_{i_n} < a_{i_{n+1}}$. It follows by monotonicity that
$d_{i_{n+1}} - d_{i_n} > T(b_{i_{n+1}}) - T(a_{i_{n+1}}) = 1$. This completes the proof when there are infinitely many triggering intervals.

It remains to deal with the case where there are only finitely many triggering intervals so that we reach in the construction a state where $T_n : [0, b_n] \to [0, d_n]$ is defined and there are no initializing points beyond a_n. We choose points e_1 and e_2 such that $a_n < e_1 < b_n < e_2$ and such that every non triggering interval that contains b_n starts after e_1 and ends before e_2. $T_n(e_1) = T_n(a_n) + c$ with $c < 1$ and we define $T_{n+1}(e_2) = d_n + c$. We extend T_{n+1} to map $[b_n, e_2]$ onto $[d_n, d_n + c]$. Next we choose b_{n+1} such that $e_2 < b_{n+1}$ and $b_n + 1 < b_{n+1}$. We map $T_{n+1}(b_{n+1}) = d_n + 1 = d_{n+1}$

and extend the mapping from $[e_2, b_{n+1}]$ to $[d_n + c, d_{n+1}]$. The mapping satisfies the requirements and it extends the domain and the range at least by one unit. This construction can be repeated by induction (with the pairs b_{m-1},b_m replacing a_n and b_n of the previous case). This will yield a mapping T that is from R^+ onto R^+. This completes the proof of the lemma. $\square$

Next we show that with this isomorphism construction each Q_i becomes a proper metric timer for P_i:

Lemma 11 *Each $T(Q_i)$ is a timer for $T(P_i)$.*

Proof

We must show:

a) If $t \in T(P_i)$ for all $c < t < c+1$ then $c+1 \in T(Q_i)$

b) If $d \in T(Q_i)$ then $t \in T(P_i)$ for all$d-1 < t < d$

We must also check the case where the point under discussion is less than 1,

c) If $t \in T(P_i)$ for all $t < c+1$ then $c+1 \in T(Q_i)$

d) If $d \in T(Q_i)$ and $d < 1$ then $t \in T(P_i)$ for $0 \leq t < d$.

a) Assume that $T(P_i)$ holds at an interval (c,d) with $d = c+1$, $c = T(a)$ and $d = T(b)$. Then P_i holds at every point in the interval (a,b). If Q_i holds at some point in the interval (a,b) then it holds at b by axiom P4 and T3, and $T(Q_i)$ holds at d. So it remains to check the case that $T(Q_i)$ does not hold at any point in (c,d), and Q_i does not hold at any point in (a,b). Now (a,b) cannot be contained in a non triggering interval for Q_i, as our construction assures that non triggering intervals are mapped onto intervals of length less than 1. By lemma 8 (a,b) is therefore contained in a triggering interval (a',b'). Therefore $(T(a'),T(b'))$ has length 1 and contains (c,d) and Q_i holds at b' and $T(Q_i)$ holds at $T(b')$. By monotonicity $(T(a'),T(b'))$ contains $(T(a),T(b))$ and both have length 1, so that they coincide. Therefore $d = T(b')$ and $T(Q_i)$ holds at d. Hence $T(Q_i)$ holds whenever $T(P_i)$ has been holding for one unit of time.

b+d) Assume that $T(Q_i)$ holds at a point $d = T(b)$. By axiom T3 either b is a triggering point for Q_i or there is a triggering point a before b such that Q_i holds at $[a,b]$. By axioms P2, P_i holds on $[a,b)$ and $T(P_i)$ holds on $[T(a),T(b))$. By our construction $a = 0$ or P_i holds at $(a-1,a)$ (or on $[0,a)$) and $T(P_i)$ holds prior to d for at least one unit of time (or since the beginning of time).

c) Assume that $T(P_i)$ holds on $[0,d)$ and $d = T(b)$. Then P_i holds on $[0,b)$ and by axiom T1, T3 and P4, Q_i holds on b and $T(Q_i)$ holds on $T(b)$.

The proof of the metric elimination theorem: We have now all the ingredients for the proof. Since every model of ψ is a model of $\overline{\psi}$ it is obvious that if ψ is satisfiable then so is also $\overline{\psi}$. Assume on the other hand that $\overline{\psi}$ is satisfied so that there are sets $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ (as well as some others that we don't mention explicitly) such that $R^+ \models \overline{\mathrm{Timer}}_n(P_1, \ldots, P_n, Q_1, \ldots, Q_n) \wedge \varphi'$ (we write φ' to acknowledge the sets that occur in φ' as parameters). Next we use the isomorphism lemma to construct a monotone bijection T for which $R^+ \models \mathrm{Timer}_n(T(P_1), \ldots, T(P_n), T(Q_1), \ldots, T(Q_n))$, by lemma 10. φ on the other hand is a pure *MLO* formula and it is invariant under the order preserving bijection so that $R^+ \models \varphi''$, where each set A in φ' is replaced by $T(A)$. It now follows that $R^+ \models \exists \overline{W} \text{Timer}_n(P_1, \ldots, P_n, Q_1, \ldots, Q_n) \wedge \varphi$. This completes the proof of the metric elimination theorem. $\square$

4.3 Decidability of timer normal form formulas

With the technical work behind us we obtain the decidability results from the corresponding non metric case:

Theorem 12 (Decidability for timer normal form) *1. The satisfiability of a formula in first-order timer normal form in the canonical model is decidable.*

2. The satisfiability of a formula in the second-order timer normal form under the finite variability interpretation is decidable.

Proof Given a formula φ in normal form then its non metric approximation $\overline{\varphi}$ is computed algorithmically. The satisfiability of $\overline{\varphi}$ is decidable by theorem 2 and it is equivalent to the satisfiability of φ by the metric elimination theorem.

$\square$

4.4 Discussion: The mathematical content of the metric elimination theorem

The construction of the bijection in the isomorphism lemma (lemma 10) is the mathematical backbone of the metric elimination procedure. It says that under some assumptions the metric of R^+ can be rescaled (via the bijection T) in a way that all intervals in a chosen set of intervals will have length 1 and all the intervals in another chosen set will have length less than 1. Our two sets of intervals were very simple sets and it is obvious that we just touched upon a general mathematical theorem:

The next theorem gives the most general conditions for the existence of such a rescaling. The conditions are about limits of sequences of intervals and the rescaling defines directly a new "+1" function. We start with the definition of these notions.

Definition 13 *1. A function $S : R^+ \longrightarrow R^+$ is a* **successor function** *if*

- *S is monotone: $x < y \longrightarrow S(x) < S(y)$*
- *$S(x) > x$*
- *S is continuous: if $x_n \longrightarrow x$ then $S(x_n) \longrightarrow S(x)$*

2. If S with these properties is defined only on a closed subset of R^+ it will be called a **partial successor function**.

3. *An interval $[a, b]$ is the* **limit** *of a sequence of intervals $[a_n, b_n]_{n \in N}$ and the sequence of intervals* **converges** *to $[a, b]$ if the sequence $\langle a_n \rangle_{n \in N}$ converges to a and the sequence $\langle b_n \rangle_{n \in N}$ converges to b.*

The next Proposition justifies our terminology

Proposition 14 *A function $S : R^+ \rightarrow R^+$ is a successor function if and only if there exists an order preserving bijection $\rho : R^+ \longrightarrow R^+$ such that $\forall a \in R^+.\ \rho(S(a)) = \rho(a) + 1$, where $+$ is the standard addition of reals.*

Proof Assume that $\rho : R^+ \longrightarrow R^+$ is an order preserving bijection and that $\forall a \in R^+.\ \rho(S(a)) = \rho(a) + 1$. We will show that S is a successor function. Note that $S = \lambda a \in R^+.\rho^{-1}(\rho(a) + 1)$. First observe that S is monotone since ρ preserves order. Second, $\rho(S(a)) = \rho(a) + 1 > \rho(a)$ and by order preservation we derive $S(a) > a$. Finally, every order preserving bijection from the non-negative reals to the non-negative reals is continuous (by Basic Set Theory or Calculus), and the $+1$ function is continuous. Therefore S which is the composition of the order preserving bijection ρ^{-1} with $+1$ is continuous.

Assume for the opposite direction that S has the three properties. We construct an order preserving bijection τ for which $\tau(a + 1) = S(\tau(a))$. ρ is then the inverse of τ. By Basic Set Theory (or calculus) there is an order preserving bijection τ from $[0, 1]$ onto $[0, S(0)]$. We extend τ to a bijection from $[0, 2]$ onto $[0, S(S(0))]$, defining for every t between 0 and 1, $\tau(t + 1) = S(\tau(t))$. Similarly, we extend τ to a bijection from $[0, 3]$ onto $[0, S(S(S(0)))]$, defining for every t between 1 and 2, $\tau(t + 1) = S(\tau(t))$. Continuing in this manner and taking the union of the functions thus constructed we obtain an order preserving bijection from the non-negative real line onto an open prefix of the non-negative real line which satisfies $\tau(a + 1) = S(\tau(a))$. It remains to show that this prefix is all of R^+. Else the range τ has a least upper bound b. Since $b > S(0)$ there is a point c such that $b = S(c)$. c is not an upper bound for the range so that for some d, $\tau(d) > c$. But then by the definition of τ, $\tau(d + 1) = S(\tau(d)) > S(c) = b$. This contradicts the choice of b as an upper bound for the range. $\square$

Theorem 15 (Rescaling theorem) *Let $\mathbb{U}$ and $\mathbb{S}$ be sets of intervals. We call the intervals in $\mathbb{U}$* **unit intervals** *and the intervals in $\mathbb{S}$* **small intervals**. *The following conditions together form a necessary and sufficient condition for the existence of a successor function S for which $S(b) - S(a) = 1$ if $[a, b] \in \mathbb{U}$ and $S(b) - S(a) < 1$ if $[a, b] \in \mathbb{S}$:*

1. *(Downward compatibility) The limit of a sequence of unit intervals is neither a single point, nor is it contained in a small interval (in particular no unit interval is contained in a small interval).*

2. *(upward compatibility) The limit of intervals in $\mathbb{U} \cup \mathbb{S}$ does not properly contain a limit of unit intervals. In particular no unit interval properly contains another unit interval.*

3. *Boundedness Let* $[a_n, b_n]_{n\in N}$ *be a sequence of intervals in* $\mathbb{U} \cup \mathbb{S}$*. If the sequence* $\langle a_n\rangle_{n\in N}$ *is bounded then the sequence* $\langle b_n\rangle_{n\in N}$ *is bounded.*

These conditions are clearly necessary: (1) and (2) are obvious size compatibility conditions, (3) is necessary if R^+ is to have infinite length (note that we do not exclude the possibility that a unit interval will be the limit of small intervals). The proof that these conditions are also sufficient is given in the appendix.

Lemma 10 is a special case of Theorem 15.

4.5 The rational time line

When trying to apply the metric elimination procedure in the case of the rational time line Q^+ we face two major differences. On the one hand even the axioms 1-9 are no longer expressible in first order MLO. In fact an interval of rational numbers that has non rational endpoints cannot be described in first order MLO. On the other hand even second order MLO is decidable over the rational time line (theorem 2). This is very lucky since the axioms 1-9 can be easily expressed in second order MLO. Thus $\forall x, y, z(x < y < z \wedge x \in X \wedge z \in X \longrightarrow y \in X)$ says that X is an interval, and this is used expressing the axioms. With this modification we obtain via a similar proof an even stronger decidability theorem for the rational time line:

Theorem 16 *The satisfiability of a formula of second order timer normal form is decidable over the rational line* Q^+*.*

5 Temporal Logics

Temporal logics evolved in philosophical logic and were enthusiastically embraced by a large body of computer scientists. Temporal logics use logical constructs called "modalities" to create a language that is free from variables and quantifiers. Every modality that was seriously considered corresponds to a formula in relational logic, and it is obvious that temporal logic can be thought of as syntactical sugar for a fragment of classical logic. In its purest form the use of sugaring is harmless since Kamp proved that the expressive power of the temporal logic based on the modalities "until" and "since" equals that of the monadic logic of order (concerning formulas with at most one free variable).

Here is the general logical framework to define temporal logics:

The syntax of the Temporal Logic $TL(O_1^{(k_1)} \cdots O_n^{(k_n)})$ has in its vocabulary *monadic predicate variables* $X_1, X_2, \ldots$ and a sequence of *modality names* with prescribed arity, $O_1^{(k_1)} \cdots O_n^{(k_n)}$ (the arity notation is usually omitted). The formulas of this temporal logic are given by the grammar:

$$\varphi \ ::= \ X \mid \neg\varphi \mid \varphi \wedge \varphi \mid O^{(k)}(\varphi_1, \cdots, \varphi_k)$$

When the particular modality names are unimportant or they are clear from the context we omit them and write TL instead of $TL(O_1^{(k_1)}, \ldots, O_n^{(k_n)})$.

Structures for TL are again linear orders with monadic predicates $M = \langle A, <, P_1, P_2, \dots, P_n, \dots \rangle$, where predicate P_i is assigned to a variable X_i. Every modality $O^{(k)}$ is interpreted in every structure M as an operator $O_M^{(k)} : [\mathbb{P}(A)]^k \to \mathbb{P}(A)$ which assigns "the set of points where $O^{(k)}[S_1, \dots, S_k]$ holds" to the k-tuple $\langle S_1, \dots, S_k \rangle \in \mathbb{P}(A)^k$. (Here $\mathbb{P}$ is the power set notation, and $\mathbb{P}(A)$ denotes the set of all subsets of A.) Once every modality corresponds to an operator the semantics is defined by structural induction:

- for atomic formulas

$$\langle M, \tau \rangle \models_{T} X \quad \text{iff} \quad \tau \in P, \text{ where the monadic predicate } P \text{ is assigned to } X.$$

- for Boolean combinations the definition is the usual one.

- for $O^{(k)}(\varphi_1, \cdots, \varphi_k)$

$$\langle M, \tau \rangle \models_{T} O^{(k)}(\varphi_1, \cdots, \varphi_k) \quad \text{iff} \quad \tau \in O_M^{(k)}(A_{\varphi_1}, \cdots, A_{\varphi_k})$$

  where $A_\varphi = \{ \tau : \langle M, \tau \rangle \models_{T} \varphi \}$ (we suppressed predicate parameters that may occur in the formulas).

Actually we are interested in a more restricted case; for the modality to be of interest the operator $O^{(k)}$ should reflect some intended connection between the sets A_{φ_i} of points satisfying φ_i and the set of points $O[A_{\varphi_1}, \dots, A_{\varphi_k}]$. The intended meaning is usually given by a formula in an appropriate predicate logic:

Truth Tables: A formula $\overline{O}(t_0, X_1, \dots X_k)$ in the predicate logic L is a *Truth Table* for the modality $O^{(k)}$ if for every structure M

$$O_M(P_1, \dots, P_k) = \{ \tau : M \models_{MLO} \overline{O}[\tau, P_1, \dots, P_k] \} .$$

The following are examples of modalities introduced through truth tables. The modalities *until* and *since* which are introduced here are the most commonly chosen modalities in temporal logic for computer science.

Example 1 (Tables for Modalities)

- *The modality $\Diamond X$, "eventually X", is defined by*

$$\varphi(t_0, X) \equiv \exists t > t_0 \ X(t).$$

- *The modality $\overleftarrow{\Diamond} X$, "$X$ has happened before", is defined by $\varphi(t_0, X) \equiv \exists t < t_0 \ X(t)$.*

- *The modality X **U** Y, "X until Y", is defined by*

$$\varphi(t_0, X, Y) \equiv \exists t_1 (t_0 < t_1 \wedge Y(t_1) \wedge \forall t (t_0 < t < t_1 \to X(t))).$$

- *The modality* $X\ \mathbf{S}\ Y$, *"*X *since* Y*", is defined by*

$$\varphi(t_0, X, Y) \equiv \exists t_1 (t_0 > t_1 \wedge Y(t_1) \wedge \forall t (t_1 < t < t_0 \rightarrow X(t))).$$

- $Always(X)$ *is the modality that states that* X *always holds. Its truth table is:*

$$\varphi(t_0, X) \equiv (t_0 = t_0) \wedge \forall t.\ X(t).$$

The following observation justifies our view that temporal logic can be thought of as an alternative approach (sugaring) to fragments of the predicate logic in which they are defined:

Proposition 17 *If every modality in a temporal logic TL has a truth table in the logic MLO then for every formula* $\varphi(X_1, \ldots, X_n)$ *of TL there corresponds effectively (and naturally) a formula* $\overline{\varphi}(t_0, X_1, \ldots, X_n)$ *of MLO such that for every* M*,* $\tau \in M$ *and predicates* $P_1, \ldots, P_n$

$$\langle M, \tau, P_1, \ldots, P_n \rangle \models_{T} \varphi \quad \text{iff} \quad \langle M, \tau, P_1, \ldots, P_n \rangle \models_{MLO} \overline{\varphi}\ .$$

In particular by example 1, the temporal logic $TL(\ \mathbf{U},\ \mathbf{S})$ with the modalities "until" and "since" describes a fragment of (first-order) MLO.

5.1 Expressive Completeness of $TL(until, since)$

The discussion above brings up the question as to which simple collection of truth tables and modalities suffices to represent all of the monadic logic. Formally:

Definition 18 (Expressive Completeness) *Let* L *be a fragment of first-order logic and let* C *be a class of structures. A temporal logic* TL' *is expressively complete for* L *over* C *if for every formula* $\psi(t_0, X_1, ..., X_n) \in L$ *with a single free first order variable* t_0*, there is a formula* $\varphi \in TL'$ *which is equivalent to* ψ *over* C*, and for every formula* $\varphi \in TL'$*, there is a formula* $\psi(t_0, X_1, ..., X_n) \in L$ *with a single free first order variable* x_0 *which is equivalent to* φ *over* C*.*

We could of course take all the formulas with one free variable of MLO as truth tables for modalities. This would yield an expressively complete, extremely uninteresting temporal logic. A much smaller collection of modalities suffices, as can be seen from the following beautiful theorem:

Theorem 19

1. *([Kamp68], reproved in [GPSS80]): The temporal logic* $TL(until, since)$ *is expressively complete for (first-order) MLO over the two canonical structures: the real line and the natural numbers.*

2. *[GHR94] There is a pair of modalities* $X until_s Y$ and $X since_s Y$ *("Stavi's modalities") which together with the since and until is expressively complete for (first-order) MLO over the class of all linear orders.*

Hence from now on when we say simply "temporal logic" or "pure temporal logic" we mean $TL(until, since)$.

6 Quantitative Temporal Logic (QTL) and Quantitative Monadic Logic of Order ($QMLO$)

We already indicated in our discussion of the Timer formulas what seems to us the most economical of metric notions that should be introduced into the discussion: there is the notion of distance in the background and we want to be able to say that within one unit of time in the past (or in the future) such and such happened (or will happen). This leads to the predicate logic $QMLO$ - quantitative monadic logic of order, and QTL - quantitative temporal logic of order.

Definition 20 (Quantitative Temporal Logic - QTL**)** *For any set of temporal modalities* Γ *which is expressively complete for the first order monadic logic of order we denote by* $QTL(\Gamma)$ *the temporal logic constructed from* Γ *and two new modalities* $\Diamond_1 X$ *and* $\overleftarrow{\Diamond}_1 X$ *defined by the tables with* t_0 *free:*

$$\Diamond_1 X : \qquad \exists t((t_0 < t < t_0 + 1) \wedge X(t)) \tag{3}$$

$$\overleftarrow{\Diamond}_1 X : \qquad \exists t((t < t_0 < t + 1) \wedge X(t)) \ . \tag{4}$$

We shall denote by QTL the quantitative logic $QTL(\ \mathbf{U},\ \mathbf{S})$.

Remark 21 *The reader may suspect that since* $TL(\ \mathbf{U},\ \mathbf{S})$ *is expressively complete it would suffice to add the new modalities only to this basic temporal logic. While this is true in view of the next theorem, it cannot be assumed a priori; the expressive power of two equivalent logics may differ after adding the same construct to both.*

We also introduce the derived modalities:

$$\Box_1 X = \neg\Diamond_1\neg X \qquad\qquad \overleftarrow{\Box}_1 X = \neg\overleftarrow{\Diamond}_1\neg X$$

Next we intend to identify the fragment of MLO^+ that corresponds to QTL. This fragment will use the function $t+1$ only in a very restricted form as indicated in (3) and (4). We recall our notation of bounded quantifiers: $(\exists t)^{<t_0+1}_{>t_0}$ and $(\exists t)^{<t_0}_{>t_0-1}$ are shorthand for the following: if φ is a formula of MLO^+ then

$$(\exists t)^{<t_0+1}_{>t_0}\varphi \equiv \exists t(t_0 < t < t_0 + 1 \wedge \varphi(t)) \tag{5}$$

$$(\exists t)^{<t_0}_{>t_0-1}\varphi \equiv \exists t((t < t_0 < t + 1) \wedge \varphi(t)) \tag{6}$$

Definition 22 (Quantitative Monadic Logic of Order - $QMLO$**)** $QMLO$ *is the fragment of* MLO^+ *which is built from the atomic formulas* $t_1 < t_2, t_1 = t_2, X(t)$ *(*t, t_1, t_2 *variables) using Boolean connectives, first order quantifiers and the following rule: if* $\varphi(t)$ *is a formula of QMLO with* t *its only first order free variable then* $(\exists t)^{<t_0+1}_{>t_0}\varphi$ *and* $(\exists t)^{<t_0}_{>t_0-1}\varphi$ *are formulas of* $QMLO$*.*

Remark 23 *The reader should not take lightly the restricting feature that t is to be the only free variable in φ. Relaxing this requirement yields non decidable logics and our subsequent quite non trivial research deals with careful modifications that give rise to stronger yet decidable logics [HR2004].*

Since the new modalities of QTL have truth tables in $QMLO$ it is clear that QTL corresponds to a fragment of $QMLO$. The following observation shows that it is complete:

Theorem 24 (Expressively completeness of QTL in $QMLO$) *Let Γ be any expressive complete set of modalities (over the reals) for first-order MLO.*

For every formula p over the modalities $\{\Gamma, \diamondsuit_1, \overleftarrow{\diamondsuit}_1\}$ there is a formula $\varphi(t_0)$ of QMLO effectively computable from p such that $\varphi(t_0)$ is equivalent (over the reals) to p.

For every formula $\varphi(t_0)$ of QMLO (with no variable except t_0 free) there is a formula p over the modalities $\{\Gamma, \diamondsuit_1, \overleftarrow{\diamondsuit}_1\}$ effectively computable from $\varphi(t_0)$ such that $\varphi(t_0)$ is equivalent (over the reals) to p. In particular QTL is expressibly complete for $QMLO$.

Proof The first claim is similar to Proposition 17. Once the modalities have truth tables in some predicate logic the translation of every formula follows by an easy induction.

The second claim is proved by induction on the total number of bounded quantifiers in the $QMLO$ formula $\varphi(t_0)$. If there are none then $\varphi(t_0)$ is equivalent to some $TL(\Gamma)$ formula since Γ is expressively complete. Otherwise we eliminate some innermost occurrence of a bounded quantifier in φ: Assume $(\exists t)^{<t_1+1}_{>t_1}\theta(t)$ is a subformula of φ with θ in pure monadic logic. Note that t_1 maybe t_0 and that t is the only variable free in θ. Therefore, there is some formula $\psi(t_0, X)$ that has one more predicate variable X and one bounded quantifier fewer than $\varphi(t_0)$ such that φ is obtained from $\psi(t_0, X)$ by substituting $(\exists t)^{<t_1+1}_{>t_1}\theta(t)$ for $X(t_1)$. By assumption there is a $TL(\Gamma)$ formula θ_1 which describes in every model the same predicate as $\theta(t)$ and therefore $\diamondsuit_1\theta_1$ describes the same predicate as $(\exists t)^{<t_1+1}_{>t_1}\theta(t)$. By the induction assumption there is a QTL formula $\psi_1(X)$ which is equivalent to $\psi(t_0, X)$. It is now clear that $\psi_1(\diamondsuit_1\theta_1)$ describes in every model the same set as $\varphi(t_0)$.

$\square$

Note that the theorem proves that the addition of the new metric modalities to *any* complete set Γ of modalities yields the fragment of $QMLO$. Hence there is no loss of generality by choosing from now on $\Gamma = \{\mathbf{U}, \mathbf{S}\}$.

7 The Expressive Power of $QMLO$ (and QTL)

The two modest modalities $\overleftarrow{\diamondsuit}_1$ and $\diamondsuit_1$ when added to the *until* and the *since* connectives are quite strong in expressive power. They are capable of expressing modalities concerning intervals other than the open unit interval and they may be used to express all of the operators which we found in the literature for real time logic. In view of the expressive completeness theorem we can phrase the expressibility results in terms of the temporal logic QTL or in terms of the predicate logic $QMLO$.

7.1 General Bounded Quantifiers

The modalities $\overleftarrow{\Diamond}_1$ and $\Diamond_1$ refer to the existence of a witness in the next or in the previous (open) unit of time. More general modalities can be defined from these two. In fact any open, closed or half open, bounded or unbounded interval can be the domain of a definable operator.

Notations like $(\exists t)^{<t_0+n+m}_{>t_0+n}$, $(\exists t)^{\leq t_0+n+m}_{\geq t_0+n}$, and $(\exists t)^{<\infty}_{>t_0+n}$ are self explanatory. Here are formulas of *QMLO* that define more general bounded quantifiers from the basic bounded quantifier $(\exists t)^{<t_0+1}_{>t_0}$:

(a) $(\exists t)^{<t_0+1}_{\geq t_0} X(t) \equiv X(t_0) \vee (\exists t)^{<t_0+1}_{>t_0} X(t)$

(b) $(\exists t)^{\leq t_0+1}_{>t_0} X(t) \equiv (\exists t)^{<t_0+1}_{>t_0} X(t) \vee [\text{First }(t_0, X) \wedge (\forall t_1)^{<t_0+1}_{>t_0} (\exists t)^{<t_1+1}_{>t_1} X(t)]$

where First (t_0, X) says that there is a first point past t_0 for which $X(t)$ holds:

(7) $$\text{First }(t_0, X) \equiv \exists t_1[t_0 < t_1 \wedge X(t_1) \wedge \forall t(t_0 < t < t_1 \rightarrow \neg X(t))]$$

(Clearly when $X(t)$ has a first point in the future then the last conjunct in (b) is equivalent to $X(t_0 + 1)$).

(c) $(\exists t)^{<\infty}_{>t_0+1} X(t) \equiv (\forall t)^{\leq t_0+1}_{>t_0} \exists t_1 . t_1 > t \wedge X(t_1)$

(d) $(\forall t)^{<t_0+n+1}_{>t_0+n} X(t) \equiv (\forall t)^{<t_0+n}_{>t_0+n-1} (\exists t_2)^{<t+1}_{>t} (\forall t_1)^{<t_2+1}_{>t_2} X(t_1)$ for $n > 0$

This can be summarized:

Theorem 25 *The extension L of QMLO by the following rules is expressively equivalent to QMLO over the canonical model.*

if $\varphi(t)$ is an L formula with the only free variable t then the following are L formulae:

1. $(\exists t)^{<t_0+n+m}_{>t_0+n}\varphi(t)$, *where n is an integer and m a positive natural number.*
2. $(\exists t > t_0+n)\varphi(t)$ *(denoted also by $(\exists t)^{<\infty}_{>t_0+n}\varphi(t)$) and $(\exists t < t_0+n)\varphi(t)$ (denoted by $(\exists t)^{<t_0+n}_{>-\infty}\varphi(t)$), where n is an integer.*
3. *Similar to (1) or (2) above with weak inequality replacing one or both occurrences of the strong inequality.*

Henceforth, we will freely use these generalized quantifiers which are definable in *QMLO* and also the corresponding modalities like $\diamond_{[n,m]}X$ which is defined by $(\exists t)^{\leq t_0+m}_{\geq t_0+n} X(t)$, and similarly $\Box_{[n,m]}X$. In this new notation $\Diamond_1 X = \diamond_{(0,1)} X$ and $\overleftarrow{\Diamond}_1 X = \diamond_{(-1,0)} X$

7.2 Modalities in Real Time Logics

We define in *QMLO* the main metric modalities that were previously proposed:

1. In [MP93] the main metric modalities used are of the form $\Gamma(X) > n$ where n is a natural number and similar modalities with the inequality reversed and with the inequalities replaced by weak inequalities. The meaning of $\Gamma(X) > n$ is that X was true for the last n units of time. $\Gamma(X) = n$ means that X started exactly n units ago and was true from then until now. In *QMLO* this translates to:

$$[\Gamma(X) > n] \equiv (\forall t)_{>t_0-n}^{<t_0} X(t)$$

$$[\Gamma(X) = n] \equiv (\forall t)_{>t_0-n}^{<t_0} X(t) \wedge ((\forall t)_{>t_0-n}^{<t_0-n+1})(\exists s)_{>t-1}^{<t} \neg X(s)$$

   Thus the modalities of Manna, Pnueli are definable in *QMLO* and *QTL*.

2. Wilke introduced a relative distance construct "the first time $X(t)$ occurs after t_0 is in distance smaller than, larger than or equal to n" [Wilke94]. Specifically, Wilke uses a parameterized predicate formula $d(X,t) \sim n$ which can be interpreted as a family of truth tables for modalities. These modalities depend on the choice of the distance n and on the choice between equality or kind of inequality . In *QMLO* these modalities can be defined by:

$$\begin{aligned} d_{<n}(X) &\equiv \mathrm{First}(t_0, X) \wedge (\exists t)_{>t_0}^{<t_0+n} X(t) \\ d_{=n}(X) &\equiv (\exists t)_{>t_0}^{\leq t_0+n} X(t) \wedge (\forall t)_{>t_0}^{<t_0+n} \neg X(t) \end{aligned}$$

3. The popular logic *MITL[AFH96] is based on an infinite set of modalities until$_I$ where I is a non-singular interval with integer endpoints - these modalities are called constrained until modalities. In one of the different variants of this operator $X\,until_I Y$ means that there is t in the interval I such that in t units of time Y will hold and until then X holds (see also [HRS98] for a different meaning). The constrained until modalities are easily expressible in QTL:*

   (a)

$$X\ \mathbf{U}_{(0,m)} Y \textit{ is equivalent to } X\ \mathbf{U} Y \wedge \Diamond_{(0,m)} Y$$

   (b)

$$X\ \mathbf{U}_{[0,m)} Y \textit{ is equivalent to } Y \vee X\ \mathbf{U}_{(0,m)} Y$$

   (c) for $n > 0$

$$X\ \mathbf{U}_{(n,n+m)} Y \textit{ is equivalent to } \Box_1 X \wedge \Diamond_1 \Box_{(0,n)} (X \wedge X\ \mathbf{U} Y) \wedge \Diamond_{(n,n+m)} Y$$

   i.e: X and $X\ \mathbf{U}Y$ hold from the current point t until a point after $t+n$. The last clause assures that the Y that makes $X\ \mathbf{U}Y$ true does not occur after the interval $(t+n, t+n+m)$.

(d) *for* $n > 0$

$$X\ \mathbf{U}_{[n,n+m)}Y \text{ is equivalent to } X\ \mathbf{U}_{(n,n+m)}Y \vee (\Box_{(0,n)}X \wedge \Diamond_{[n,n+1)}(Y \wedge \Box_{(-1,0)}X))$$

i.e: either X *holds up to a point in* $(t+n, t+n+m)$ *where* Y *holds, or* X *holds up to* $t+n$ *and* Y *holds at* $t+n$*. This last case is covered by the existence of a* t_1 *in* $[t+n, t+n+1)$ *where* Y *holds and where* X *has been true for the last unit of time.*

The following modifications take care of the case when the intervals are closed on the right.

(e)

$$X\ \mathbf{U}_{(0,m]}Y \text{ is equivalent to } X\ \mathbf{U}Y \wedge \Diamond_{(0,m]}Y$$

(f)

$$X\ \mathbf{U}_{[0,m]}Y \text{ is equivalent to } Y \vee X\ \mathbf{U}_{(0,m]}Y$$

(g) *for* $n > 0$

$$X\ \mathbf{U}_{(n,n+m]}Y \text{ is equivalent to } \Box_1 X \wedge \Diamond_1 \Box_{(0,n)}(X \wedge X\ \mathbf{U}Y) \wedge \Diamond_{(n,n+m]}Y$$

(h) *for* $n > 0$

$$X\ \mathbf{U}_{[n,n+m]}Y \text{ is equivalent to } X\ \mathbf{U}_{(n,n+m]}Y \vee (\Box_{(0,n)}X \wedge \Diamond_{[n,n+1)}(Y \wedge \Box_{(-1,0)}X))$$

Hence, we have the following proposition.

Proposition 26 *MITL and QTL are expressively equivalent.*

4. *In [HRS98] a modality* $\triangleright$ *is defined as* $\triangleright_{(n,n+m)}X$ *- the first instance of* $X(t)$ *in the future lies in the interval* (t_0+n, t_0+n+m)*. This is defined in QMLO by:*

$$\triangleright_{(n,n+m)}X \equiv\ First(t_0, X) \wedge (\exists t)^{<t_0+n+m}_{>t_0+n} X(t) \wedge \neg(\exists t)^{\leq t_0+n}_{>t_0} X(t)$$

7.3 Modalities with rational interval

Clearly we cannot express "P will happen within the next half unit of time", using only modalities $\mathbf{U}$, $\mathbf{S}$, $\Diamond_1$,and $\overleftarrow{\Diamond}_1$. Therefore the extension of QTL by the infinite set of modalities $\{\Diamond_r\ :\ r \text{ is rational}\} \cup \{\overleftarrow{\Diamond}_r\ :\ r \text{ is rational}\}$ strengthens the expressive power of the language ($\Diamond_{\frac{1}{n}}P$ is interpreted as "P will happen within the next $\frac{1}{n}$ fraction of unit of time").

Not surprisingly however as far as satisfiability is concerned there is a simple reduction of this extension to the QTL as follows: Assume that $r_1, \ldots, r_n$ are all the rationals that are mentioned in a formula ϕ. Let r be a common deviser of $r_1, \ldots, r_n$, so there are integers $d_1, \ldots, d_n$ such that $r_i = d_i \times r$ for $i = 1, \ldots, n$. Therefore all the modalities of the formula are of the form $\Diamond_{d_i \times r}$ or of the form $\overleftarrow{\Diamond}_{d_i \times r}$. Let ϕ' be the formula where each $d_i \times r$ is replaced by d_i. Then every model of ϕ

can be made into a model of ϕ' by a simple rescaling, and vice versa. Hence ϕ and ϕ' are equivalent as far as satisfiability is concerned. Moreover, by the result of subsection 7.1, ϕ' is equivalent to a QTL formula and therefore its satisfiability is decidable (by Corollary 29 below).

In [AFH96] modalities of the form $\mathbf{U}_I$ and $\mathbf{S}_I$ are considered where I is an interval with rational (not necessary integer) endpoints. The transformations similar to those in section 7.2 can show that this logic has the same expressive power as the extension of QTL mentioned above and as $TL(\ \mathbf{U},\ \mathbf{S}, \{\Diamond_{\frac{1}{n}}\ :\ n \in Nat\} \cup \{\overleftarrow{\Diamond}_{\frac{1}{n}}\ :\ n \in Nat\})$.

If we allow a modality $\Diamond_\alpha$, where α is irrational then the decidability question becomes more intriguing from mathematical point of view, but it may loose its appeal to the working computer scientist.

8 Decidability of $QMLO$ and QTL

Theorem 28 will show that every $QMLO$ formula is equivalent to a formula in timer normal form. We shall show first that $\Diamond_1 X$ is expressible by a formula in timer normal form. Note that the formula $\Diamond_1 X$ is not expressible by a formula in $TL(until, since, \overleftarrow{\Diamond}_1)$; the existential quantification over sets is essential in expressing $\Diamond_1 X$ by a formula in timer normal form.

Theorem 27 (Reducing the future to the past) $\Diamond_1 X$ *is equivalent to a formula in timer normal form.*

Proof First observe that if

$$Always(Y \leftrightarrow \Diamond_1 X)$$

then

1. If Y holds at t then Y holds until the next occurrence of X.

2. If X holds at t then Y was true for the previous unit of time and

3. If X was false for a unit interval of time then Y was false at the beginning of the interval (hence somewhere in the interval).

Hence, $Always(Y \leftrightarrow \Diamond_1 X)$ implies the following formula ψ

$$\psi \equiv Always(Y \rightarrow Y until X) \wedge Always(X \rightarrow \overleftarrow{\Box}_1 Y) \wedge Always((\overleftarrow{\Box}_1 \neg X) \rightarrow \overleftarrow{\Diamond}_{\leq 1} \neg Y).$$

Moreover, the first and the third conjuncts in ψ imply that $Y \rightarrow \Diamond_1 X$; the second conjunct implies $\Diamond_1 X \rightarrow Y$.

Therefore, ψ is equivalent to $Always(Y \leftrightarrow \Diamond_1 X)$. Hence,

$$\Diamond_1 X \text{ is equivalent to } \exists Y (Y \wedge \psi).$$

Therefore, it remains to show that ψ is equivalent to a formula in timer normal form.

Note that all metrical modalities in ψ are past modalities. We construct the timer normal form for ψ in two steps.

First we get rid of weak order in $\overleftarrow{\Diamond}_{\leq 1}\neg Y$. Similar to the equation (b) of Sect. 7.1 we have that $\overleftarrow{\Diamond}_{\leq 1}\neg Y$ is equivalent to $\overleftarrow{\Diamond}_1\neg Y \vee (last(\neg Y) \wedge \overleftarrow{\Box}_1\overleftarrow{\Diamond}_1\neg Y)$, where the modality $last(Z)$ holds at t iff there is a last occurrence of Z before t (it can be expressed as $\neg Z since Z$). Hence, ψ is equivalent to

$$Always(Y\rightarrow Y until X)\wedge Always(X\rightarrow\overleftarrow{\Box}_1 Y)\wedge Always((\overleftarrow{\Box}_1\neg X)\rightarrow(\overleftarrow{\Diamond}_1\neg Y\vee(last(\neg Y)\wedge\overleftarrow{\Box}_1\overleftarrow{\Diamond}_1\neg Y))).$$

Using the law $\overleftarrow{\Diamond}_1 Z \leftrightarrow \neg\overleftarrow{\Box}_1\neg Z$ we can rewrite the last formula as

$$Always(Y\rightarrow Y until X)\wedge Always(X\rightarrow\overleftarrow{\Box}_1 Y)\wedge Always((\overleftarrow{\Box}_1\neg X)\rightarrow(\neg\overleftarrow{\Box}_1 Y\vee(last(\neg Y)\wedge\overleftarrow{\Box}_1\neg\overleftarrow{\Box}_1 Y))).$$

Finally, introducing a timer for every subformula that starts with $\overleftarrow{\Box}_1$ we can easily see that ψ is equivalent to the following formula in timer normal form:

$$\begin{aligned}&\exists T_1T_2T_3.Timer(\neg X,T_1)\wedge Timer(Y,T_2)\wedge Timer(\neg T_2,T_3)\wedge\\ &\qquad\wedge Always(Y\rightarrow Y until X)\wedge Always(X\rightarrow T_2)\wedge Always(T_1\rightarrow(\neg T_2\vee(last(\neg Y)\wedge T_3)).\end{aligned}$$

(Here $Timer(\neg Z, W)$ stands for $\exists Z'.Timer(Z', W) \wedge Always(Z'\leftrightarrow\neg Z)$.) □

The decidability of *QMLO* (and hence of *QTL*) is proved by reduction to timer normal form.

Theorem 28 (Reduction to Timer Normal Form) *There is an algorithm which associates with any formula $\varphi(\bar{t},\overline{Z})$ of QMLO variables $X_1,\dots,X_n,Y_1,\dots,Y_n$ and a formula $\overline{\varphi}(\bar{t},X_1,\dots,X_n,Y_1,\dots,Y_n,\overline{Z})$ of pure MLO such that $\varphi(\bar{t},\overline{Z})$ is equivalent to the formula:*

$$\exists\overline{X}\exists\overline{Y}(\mathrm{Timer}_n(X_1,\dots,X_n,Y_1,\dots,Y_n)\wedge\overline{\varphi}(\bar{t},\overline{X},\overline{Y},\overline{Z}))$$

Proof The proof is by induction on the number of metric quantifiers in φ. If there are none then φ is already in pure *MLO* form. Else we may focus on an innermost metric quantifier and we assume for simplicity that it is of the form $(\forall t)^{<t_1}_{>t_1-1}\theta(t,\overline{Z})$ so that there is some formula $\psi(\bar{t},\overline{Z},Y))$ that has one more predicate variable Y and one bonded quantifier fewer than φ such that φ is obtained from $\psi(\bar{t},\overline{Z},Y)$ by substituting $(\forall t)^{<t_1}_{>t_1-1}\theta(t,\overline{Z})$ for $Y(t_1)$. It is clear that φ is equivalent to

$$\exists YX(\psi\wedge\mathrm{Timer}(X,Y)\wedge\forall t.X(t)\leftrightarrow\theta(t,Z)),$$

where X is a fresh variable. By the inductive assumption ψ has a timer normal form $\bar{\psi}$. I.e, $\psi(\bar{t},\overline{Z},Y)$ is equivalent to the formula:

$$\exists\overline{X}\exists\overline{Y}(\mathrm{Timer}_n(X_1,\dots,X_n,Y_1,\dots,Y_n)\wedge\overline{\psi}(\bar{t},\overline{X},\overline{Y},\overline{Z},Y))$$

Hence φ is equivalent to the following formula in timer normal form

$$\exists\overline{X}\exists\overline{Y}\exists Y\exists X(\mathrm{Timer}_n(X_1,\dots,X_n,Y_1,\dots,Y_n)\wedge\mathrm{Timer}(X,Y)\wedge\overline{\psi}\wedge\forall t.X(t)\leftrightarrow\theta(t)).$$

If the innermost simple metric formula that we chose starts with a future quantifier it is dealt with similarly using Theorem 27. □

Thus from the decidability theorem for timer normal form we get:

Corollary 29 *1. Satisfiability in the canonical model is decidable for QMLO and for QTL.*

2. Satisfiability in the finite variability canonical model is decidable for QMLO and for QTL.

Note that the theorem proves for the first time decidability for the model with unrestricted variability and gives a simple automata-free proof for the finite variability model.

9 The Complexity of QTL

In this section we show that our general approach yields quite straightforwardly the expected complexity bounds for metric temporal logic, without appealing to automata theoretical techniques.

We provide a polynomial time reduction from QTL to $TL(\ \mathbf{U},\ \mathbf{S})$ which preserves satisfiability. We then apply theorem 34 from [Rab98a] and theorem 35 due to Reynolds [Re99] which state that the satisfiability problem of $TL(\ \mathbf{U},\ \mathbf{S})$ is PSPACE complete both for the canonical model and for the finite variability canonical model. These theorems were proved without appealing to automata theory.

We start with the observation that the formula $\overline{Timer}$ of the predicate logic (see section 5) is expressible in Temporal Logic, by a simple formula:

Lemma 30 *There is a QTL formula $timer_n(X_1,\dots X_n,Y_1,\dots Y_n)$ of size $O(n^2)$ over the modalities Until and Since such that $\overline{Timer}_n(X_1,\dots X_n,Y_1,\dots Y_n)$ is equivalent over the canonical model to $timer_n(X_1,\dots X_n,Y_1,\dots Y_n)$.*

Proof

It is not difficult to see that the axioms 1-9 of $\overline{Timer}$ translate into QTL. We want to estimate the complexity of the translation. In each of the axioms 1-7 the formula depends in a trivial way on the parameter i, but it clearly translates by Kamp's theorem into a QTL formula whose length is fixed for all n and all i. Hence the length of the conjunction over the 7 axioms and the n indices is linear in n. Similarly axioms 8 and 9 translate into a formula whose length does not depend on n, i or j, so that the conjunction with i and j ranging over $\{1,\cdots,n\}$ is of size $O(n^2)$. $\square$

Next we want to show how a QTL formula translates into a TL formula that is satisfiable if and only if the original formula is satisfiable. There is an intermediate step similar to that played by timer normal form for predicate logic. The role of the timer formula is taken by the QTL formulas $Always(X \leftrightarrow \Diamond_1 Z)$ and $Always(X \leftrightarrow \overleftarrow{\Diamond}_1 Z)$. Recall that $Always(X)$ is the modality that states that X always holds, with the truth table: $(t_0 = t_0) \wedge \forall t.\ X(t)$.

Lemma 31 *For every QTL formula φ with k metrical quantifiers there is a TL formula ψ and a subset F of $\{1\dots k\}$ such that the size of ψ is linear in the size of φ and φ is satisfiable if and only if the conjunction of ψ with $\bigwedge_{i\in F} Always(X_i \leftrightarrow \Diamond_1 Z_i) \wedge \bigwedge_{i\in\{1,\dots,k\}\setminus F} Always(X_i \leftrightarrow \overleftarrow{\Diamond}_1 Z_i)$ is satisfiable.*

Proof The proof is by induction on k. If there are no metric quantifiers we are done. Assume the claim holds for fewer than k quantifiers and that φ has k quantifiers. Let (say) $\Diamond_1\psi$ be an innermost occurrence of a metric modality so that ψ is non metric (the proof is similar when the metric modality is different). Thus there is a formula $\theta(X)$ with $k-1$ metric modalities such that φ is obtained from it replacing X by $\Diamond_1\psi$. Then φ is satisfiable iff the following formula is satisfiable: $\theta(X) \wedge Always(Z \leftrightarrow \psi) \wedge Always(X \leftrightarrow \Diamond_1 Z)$. The formula $Always(Z \leftrightarrow \psi)$ is non metric, and replacing θ according to the induction assumption and rearranging the formula we obtain the required normal form formula. □

Next we show that the metric formulas that remain in the last lemma are expressible using the Timer formulas:

Lemma 32 *1. The formula $Always(X_i \leftrightarrow \overleftarrow{\Diamond}_1 Z_i)$ is equivalent to $Timer_1(\neg X_i, \neg Z_i)$.*

2. The formula $Always(X_i \leftrightarrow \Diamond_1 Z_i)$ is equivalent to a simple formula in timer normal form.

Proof The first claim is obvious, while the second is just a restatement of Theorem 27. □

The above lemmas and Theorem 7 imply

Theorem 33 *There is a polynomial time algorithm that for every QTL formula φ of size m with k metrical modalities constructs a TL formula ψ of size $O(k^2 \times m)$ such that (1) ψ is satisfiable in the canonical model iff φ is satisfiable in the canonical model. (2) ψ is satisfiable in the finite variability canonical model iff φ is satisfiable in the finite variability canonical model.*

This completes the reduction of QTL to TL. To use this reduction to compute the complexity of QTL over the standard model we use the following non metric results:

Theorem 34 *[Rab98a] Let Γ be any finite set of modalities definable in monadic first-order logic of order. The satisfiability problem for TL formulas over Γ in the finite variability canonical model is in PSPACE.*

Theorem 35 *[Re99] Let Γ be any finite set of modalities definable in monadic first-order logic of order. The satisfiability problem for TL formulas over Γ in the canonical model is in PSPACE.*

We note here that the proofs of the above theorems is automata free and relies on the compositional method [She75].

Corollary 36 *1. The satisfiability problem for QTL in the finite variability canonical model is in PSPACE.*

2. The satisfiability problem for QTL in the canonical model is in PSPACE.

9.1 The complexity of $MITL$

In [AFH96], an algorithm is constructed to check satisfiability for formulas of the language $MITL$, the popular (at the time) dialect of metric temporal logic. By theorem 4.4.1 of [AFH96] the algorithm is in $O(2^{m\times c\times log(m\times c)})$, where m is the largest integer mentioned in the formula and c is the number of propositions, connectives and modalities in the formula (or approximately its length). The tedious construction using timed automata may lead the reader to believe that it would be difficult to find an efficient algorithm without using automata theory. Fortunately this is not the case: The natural transformations from $MITL$ to QTL, when done carefully allow a simple reduction to the QTL satisfiability algorithm.

$MITL$ formulas are defined via the syntax:

$$\varphi \ ::= \ X|\varphi_1 \wedge \varphi_2|\neg\varphi|\varphi_1 \ \mathbf{U}_I\varphi_2|\varphi_1 \ \mathbf{S}_I\varphi_2|$$

where I is a non trivial (i.e. contains at least two points) open, closed or half open interval with non negative integer endpoints. The semantics of these modalities were given in subsection 7.2.

We define a natural size function for *MITL*: $Size(X) = 1$, $Size(\varphi_1 \wedge \varphi_2) = Size(\varphi_1) + Size(\varphi_2)$, $Size(\neg\varphi) = Size(\varphi) + 1$, $Size(\varphi_1 \ \mathbf{U}_{(n,m)}\varphi_2) = Size(\varphi_1) + Size(\varphi_2) + n + m$. Similar definitions are given to closed and half closed intervals and to the "Since" modalities. It is clear that the Size that we defined is not larger than $m \times c$ in [AFH96].

Theorem 37 *1. The satisfiability problem for MITL formulae φ is in $PSPACE(size(\varphi))$.*

2. The satisfiability problem for MITL formulae φ in the finite variability canonical model is in $PSPACE(size(\varphi))$.

The proof follows from the following lemma, and the algorithm for QTL satisfiability, in Corollary 36.

Lemma 38 *There is a linear time algorithm which associates with any formula φ of MITL a formula ψ of QTL such that φ is satisfiable iff ψ is satisfiable.*

Proof

We saw in section 7.2 how to reduce the two place metric modalities to simple formulas that involve only one place metric modalities. For completeness we explicitly transform the unary modalities that we used into QTL modalities (this was already sketched out in section 7.1):

$$\Diamond_{(0,m)}X \text{ is equivalent to } \Diamond_1\Diamond_1\cdots\Diamond_1 X$$

with m diamonds.

$$\Diamond_{(0,m]}X \text{ is equivalent to } \Diamond_{(0,m)}X \vee ((\neg X) \ \mathbf{U}X \wedge \Box_1\Diamond_{(0,m)}X)$$

i.e: either X occurs within m units of time or there is a first occurrence of X ($\neg X\ \mathbf{U}X$), and this first occurrence cannot be strictly farther than m units of time.

$$\Diamond_{(n,n+m)}X \text{ is equivalent to } (\Diamond_1\Box_1)(\Diamond_1\Box_1)\cdots(\Diamond_1\Box_1)\Diamond_m X$$

with n pairs of modalities. The correctness of the translation follows by induction from the observation that $\Diamond_1 Y$ holds at $t+1$ iff $(\Diamond_1\Box_1)\Diamond_1 Y$ holds at t.

$$\Diamond_{[n,n+m)}X \text{ is equivalent to } \Diamond_{(n,n+m)}X \vee \Box_{(n,n+1)}((\neg X)\ \mathbf{S}X \wedge \Box_{(-1,0)}X)$$

i.e: Either X occurs in the open required interval, or there is a last occurrence of X before the open interval, and it is not strictly smaller than n.

$$\Diamond_{(n,n+m]}X \text{ is equivalent to } \Diamond_{(n,n+m)}X \vee \Box_{(n+m-1,n+m)}((\neg X)\ \mathbf{U}X \wedge \Diamond_1 X)$$

i.e: Either X occurs in the open interval or there is a first occurrence past the open interval, and it is not strictly larger than $n+m$. With the last two transformations it is also easy to handle the case where the interval is closed on both sides.

From the above equivalences we obtain that $X\ \mathbf{U}_I Y$ (and $X\ \mathbf{S}_I Y$) is equivalent to a *QTL* formula of size linear in the size of $X\ \mathbf{U}_I Y$ (respectively $X\ \mathbf{S}_I Y$). The following observation is also immediate:

Observation. If in an *MITL* formula ψ the modalities $\mathbf{U}_I$ and $\mathbf{S}_I$ are applied only to variables then ψ is equivalent to a *QTL* formula of length $\mathrm{O}(\mathrm{size}(\psi))$.

We reduce general formulas to formulas where $\mathbf{U}_I$ and $\mathbf{S}_I$ are applied only to variables. This reduction is polynomial and preserves satisfiability. Finally we apply the observation above.

Let φ be a formula. We denote by $SUB(\varphi)$ the set of subformulas of φ and by $Var(\varphi)$ the set of variables that occur in φ. For every $\psi \in SUB(\varphi) \setminus Var(\varphi)$ we introduce a fresh variable Y_ψ. For $X \in Var(\varphi)$ we use Y_X for X.

For $\psi \in SUB(\varphi) \setminus Var(\varphi)$ we define node_ψ as follows:

$$\mathrm{node}_\psi = \begin{cases} Y_\psi \leftrightarrow \neg Y_{\psi_1} & \text{if } \psi \text{ is } \neg\psi_1 \\ Y_\psi \leftrightarrow (Y_{\psi_1} \wedge Y_{\psi_2}) & \text{if } \psi \text{ is } \psi_1 \wedge \psi_2 \\ Y_\psi \leftrightarrow Y_{\psi_1}\ \mathbf{U}Y_{\psi_2} & \text{if } \psi \text{ is } \psi_1\ \mathbf{U}\psi_2 \\ Y_\psi \leftrightarrow Y_{\psi_1}\ \mathbf{S}Y_{\psi_2} & \text{if } \psi \text{ is } \psi_1\ \mathbf{S}\psi_2 \\ Y_\psi \leftrightarrow Y_{\psi_1}\ \mathbf{U}_I Y_{\psi_2} & \text{if } \psi \text{ is } \psi_1\ \mathbf{U}_I\psi_2 \\ Y_\psi \leftrightarrow Y_{\psi_1}\ \mathbf{S}_I Y_{\psi_2} & \text{if } \psi \text{ is } \psi_1\ \mathbf{S}_I\psi_2 \end{cases}$$

We define the translation $\mathrm{Tr}(\varphi)$ as

$$\mathrm{Tr}(\varphi) = Y_\varphi \wedge \bigwedge_{\psi \in SUB(\varphi)\setminus Var(\varphi)} Always(\mathrm{node}_\psi)$$

It is clear that (A) φ is satisfiable iff $\mathrm{Tr}(\varphi)$ is satisfiable. Moreover, (B) the size of $\mathrm{Tr}(\varphi)$ is linear in the size of φ.

In $\mathrm{Tr}(\varphi)$ only variables are arguments of $\mathbf{U}_I$ and $\mathbf{S}_I$. Hence Lemma 38 follows from (A), (B) and the observation above. $\Box$

If we avoid the introduction of new variables we get a QTL formula which is actually equivalent to the $MITL$ formula (and not just with regards to satisfiability). Unfortunately we were unable to modify the procedure to ensure that the translation will still be polynomial. Thus the search for a polynomial translation of $MITL$ to QTL is still open.

However, the proof of Lemma 38 shows that when formulas are represented as dags (directed acyclic graphs) then there is a linear time equivalence transformation between dag representations of $MITL$ and QTL formulas.

10 Concluding remarks

Our aim was to get the theory of real time logic back to mainstream mathematical logic: to use the real line itself as the canonical model and to use classical predicate logic as a framework. Within this framework finite variability predicates and general predicates are treated uniformly, and a natural simple metric temporal logic QTL emerges and is shown to absorb the previously defined metric constructs. We proved that this logic is decidable for finite variability predicates as well as for general predicates by a reduction to the non metric case. The same methods allow for the reduction of the complexity of metric temporal formulas to non metric formulas. The main tool used is the reduction of metric formulas to timer normal form formulas and the reduction of timer normal formulas to non metric formulas. In [HR99] we show how the decidability of second order logic over finite variability models together with reduction to timer normal form give rise to stronger decidable metric logics.

A Appendix - the proof of the rescaling theorem

Let $\mathbb{U}$ and $\mathbb{S}$ be given satisfying the three conditions in the theorem. We outline the proof and afterwards we will fill in the details.

1. We define a closed set $\mathbb{C}$ that contains all the left points of intervals in $\mathbb{U}$. We will define a partial successor function S on $\mathbb{C}$ so that every interval in $\mathbb{U}$ will have length 1. We show also that we may assume that $\mathbb{C}$ is unbounded. We will ensure that the following **main condition** holds:

    - No small interval (in $\mathbb{S}$) has length 1
    - No limit of small intervals strictly contains a unit interval.

2. It remains to extend S to a total monotone function that satisfies the main condition. Since $\mathbb{C}$ is a closed set, its complement is a countable disjoint union of open intervals. The extension of S will be done to one open interval at a time. By induction this extends S to all of R^+.

3. Hence the main step in the proof is the following: Given a closed set $\mathbb{C}$ with a partial successor function S that satisfies the main condition, and given a closed interval $[a, b]$ whose intersection with $\mathbb{C}$ is just the two points a and b, we will extend S to a partial successor function on $\mathbb{C} \cup [a, b]$

that respects the main condition (we may assume that $[a, b]$ is a finite interval since we make sure that $\mathbb{C}$ is unbounded). To this end we concentrate on the closed interval $[a, S(b)]$ that contains only two unit intervals $[a, S(a)]$ and $[b, S(b)]$. This will allow us to enlarge the small intervals of $\mathbb{S}$ that are contained in $[a, S(b)]$ to larger intervals with rational endpoints, without contradicting the main condition.

4. The actual construction now applies the Cantor back and forth argument to define S on the rational points of $[a, b]$ onto the rational points of $[S(a), S(b)]$ one point at a time, retaining the main condition. This map extends to a successor function on the set $\mathbb{C} \cup [a, b]$.

Now for the details. We start by making sure that $\mathbb{U}$ contains intervals with arbitrary large endpoints. Otherwise there is a bound M past which there are no intervals of $\mathbb{U}$. Let K be a point past all the small intervals (in $\mathbb{S}$) with the leftend before M. Such a K exists by the boundedness condition in the theorem. We may add $[M, K+1]$ to $\mathbb{U}$ and repeat this construction by induction.

Next we add to $\mathbb{U}$ all the limits of sequences of intervals in $\mathbb{U}$. The three conditions in the theorem continue to hold as any limit of a new sequence of intervals in $\mathbb{U}$ is also a limit of old intervals. We define now $\mathbb{C}$ to be the set of left endpoints of intervals in $\mathbb{U}$. $\mathbb{C}$ is a closed set: if $a \in \mathbb{C}$ then there is a sequence of intervals in $\mathbb{U}$ whose left endpoints converge to a. By the boundedness condition their right endpoints are bounded and have a converging subsequence. This assures that a is the left endpoint of a limit interval. Moreover, it follows from the conditions in the theorem that every point in $\mathbb{C}$ is the left endpoint of a single interval in $\mathbb{U}$. We define now $S(a)$ to be the right endpoint of the unique interval of $\mathbb{U}$ that starts with a. Note also that $\mathbb{C}$ is unbounded by the modification that we made. It is obvious from the conditions in the theorem that the main condition of (1) holds.

We describe now the main step in the construction of S. We assume that S is defined on the closed set $\mathbb{C}$ and that the only points in $[a, b]$ that are in $\mathbb{C}$ are a and b. We assume also that the main condition is satisfied. For the moment we call *small intervals* intervals in $\mathbb{S}$ that are inside the interval $[a, S(b)]$ and that are not already included in one of the two unit intervals which are included in $[a, S(b)]$, i.e, not subintervals of $[a, S(a)]$ or $[b, S(b)]$. We show first that we can replace the small intervals by larger intervals with rational endpoints, without losing the main condition. Define $c = min\{(a + S(b))/2, S(a)\}$ and $d = max\{(a + S(b))/2, b\}$. By the main condition we may find points a_1 slightly larger than a, and b_1 slightly smaller than $S(b)$ such that all the subintervals $[\alpha, \beta]$ of $[a, S(b)]$ with $\alpha < c$ satisfy $\beta < b_1$ and all the intervals $[\alpha, \beta]$ with $d < \beta$ satisfy $a_1 < \alpha$. We change now each $[\alpha, \beta]$ with $\alpha < c$, that is not a subinterval of $[a, S(a)]$, to $[(a + \alpha)/2, min\{(2\beta - S(a)), d\}]$. We change all the intervals $[\alpha, \beta]$ with $d < \beta$ that are not subintervals of $[b, S(b)]$ into $[max\{(2\alpha - b, d\}, (\beta + S(b))/2]$. Finally we change all the small intervals not yet accounted for to $[c, d]$. The rationale behind this division is that the small intervals of the first kind do not endanger the main condition regarding $[b, S(b)]$ and our measured enlarging assures that if left endpoints converge to a then the right endpoints cannot converge to a point larger than $S(a)$. Similarly small intervals of the second kind does not endanger $[a, S(a)]$ and we took care of

$[b, S(b)]$. Small intervals of the third kind does not endanger any of the two and we could just extend them to one small interval $[c, d]$. To ensure that endpoints are always rational we make sure that c and d are rational and in every case that the enlargement does not have rational endpoints we slightly shrink it back to obtain a proper enlargement with such endpoints.

With these preparations behind us we are ready for the actual definition of the function S on the interval $[a, b]$. We will follow Cantor's back and forth construction, recalling that any monotone function from the rational points in $[a, b]$ onto the rational points in $[S(a), S(b)]$ extends to a continuous map between $[a, b]$ and $[S(a), S(b)]$. We start by enumerating all the rational points in $[a, b]$ including, and starting with, a, b (even if they are not rational). We call it *the source sequence*. We create a second sequence, *the target sequence* that starts with the points $S(a)$, $S(b)$, and enumerates all the rational points in $[S(a), S(b)]$. We define S by induction on n so that at the n-th step S is defined for $2n$ rationals $a_1, \cdots, a_{2n}$ (increasing in the natural order) onto the increasing sequence $b_1, \cdots, b_{2n}$ respectively so that:

- $a_1, \cdots, a_{2n}$ includes at least the first n points in the source sequence.
- $b_1, \cdots, b_{2n}$ includes at least the first n points in the target sequence
- The main condition holds also for the $2n$ unit intervals $[a_i, b_i]$.

We lay the groundwork for the inductive construction with the pairs a, b and $S(a), S(b)$. The induction assumption holds by our assumptions on $\mathbb{C}$, $\mathbb{S}$ and S (and our careful enlargement).

For the inductive step, let e be the first element in the source sequence which is not yet in the domain of S (by assumption it is at least the $(n+1)st$ elements in the sequence). We extend S to include e in its domain. Let i be such that $a_i < e < a_{i+1}$ (there is always such an i as $a_1 = a$ is the left edge of the interval and $a_{2n} = b$ is its right edge). To retain monotonicity we can partner e with any point in the target sequence which lies between b_i and b_{i+1}. To ensure that small intervals will receive small lengths we will choose the target more carefully: Let e' be a point such that $e < e' < a_{i+1}$. All the small intervals with left edge smaller than e' have right edge which is smaller than b_{i+1} as $[a_{i+1}, S(a_{i+1})]$ is not strictly contained in a small interval. Moreover, since it is not strictly contained in the limit of small sets we conclude that there is some $d' < b_{i+1}$ which is larger than all the small intervals that start before e'. We choose some f in the target sequence such that $d' \leq f < b_{i+1}$ and define $f = S(e)$. Also, since $S(a_{i+1}) > a_{i+1}$ we can choose the point f to be larger than a_{i+1} and in particular $f > e$. With this definition S remains monotone on $a_1, \cdots, a_{2n-1}, c$ and the new unit interval $[e, S(e)]$ is not contained in any small interval. In a similar way we extend now the function S to include in its range the first point of the target sequence that was not yet in the range. This concludes the inductive step. We have now a monotone function from all the rationals in $[a, b]$ onto all the rationals in $[S(a), S(b)]$ and it extends to a monotone continuous function from $[a, b]$ onto $[S(a), S(b)]$. This in turn extends S from $\mathbb{C} \cup (a, b)$ onto $S(\mathbb{C}) \cup (S(a), S(b))$. It remains to show that our main condition holds. All the new unit intervals are strictly inside $[a, S(b)]$ and it is clear that only small intervals inside this interval can jeopardize the main condition. But our construction made sure that each of them has length less than 1 (is a proper subset of a unit interval) and by continuity of S their limit cannot be an interval larger than 1. This completes the proof.

$\square$

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