Chapter 1

Lecture 1

Advanced Topics in Graph Algorithms

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1.1 Introduction to Graph Theory

1.1.1 Basic Definitions and Notations

A directed graph G consists of a finite set V and an irreflexive binary relation on V. We call the elements of V, *vertices*. The binary relation may be represented either as a collection E of ordered pairs, or as a function from V to its power set: $Adj : V \rightarrow P(V)$.

We call $Adj(v)$ the *adjacency set* of vertex v, and we call the ordered pair $(v, w) \in E$ an *edge*. Clearly : $(v, w) \in E \leftrightarrow w \in Adj(v)$. In this case we say that w is *adjacent* to v, and v and w are *endpoints* of the edge (v, w). The assumption of irreflexivity implies that $(v, v) \notin E$ $(v \in V)$, or equivalently, $v \notin Adj(v)$ $(v \in V)$. $Adj(v)$ is sometimes denoted by $N(v)$ and is called the *open neighborhood* of v. The *closed neighborhood* of v is defined :

$$N[v] = v + Adj(v)$$

We will usually drop the parentheses and the comma when denoting an edge. Thus $xy \in E$ and $(x, y) \in E$ will have the same meaning.

In an undirected graph every edge appears in both directions.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are called *isomorphic*, denoted $G \cong G'$, if there is a bijection $f : V \rightarrow V'$ satisfying for all $x, y \in V$,

$$(x, y) \in E \leftrightarrow (f(x), f(y)) \in E'$$

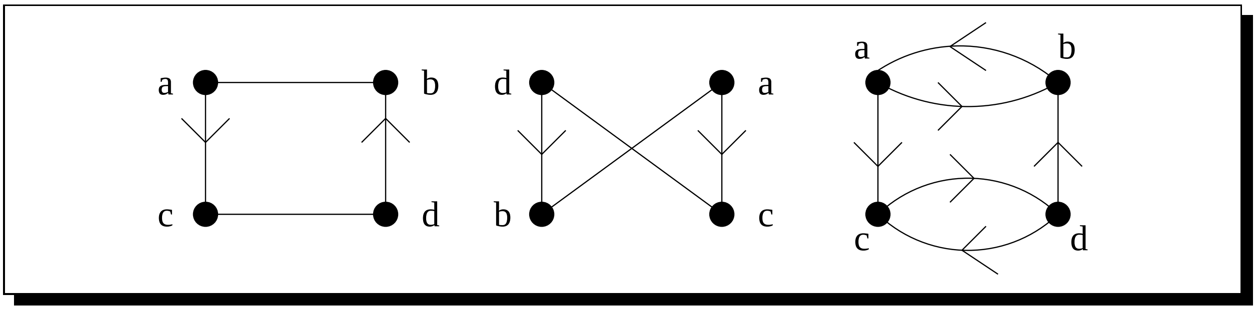

Figure 1.1: Three pictures of the same graph.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The graph $G^{-1} = (V, E^{-1})$ is said to be the *reversal* of G, where $E^{-1} = \{(x, y) \mid (y, x) \in E\}$, that is,

$$xy \in E^{-1} \leftrightarrow yx \in E \ (x, y \in V).$$

We define *symmetric closure* of G to be the graph $\hat{G} = (V, \hat{E})$, where $\hat{E} = E \cup E^{-1}$.

A graph $G = (V, E)$ is called *undirected* if its adjacency relation is symmetric i.e., if $E = E^{-1}$ or $E = \hat{E}$.

A graph is called an *oriented graph* if $E \cap E^{-1} = \emptyset$.

Orientation of an undirected graph is a choice of a direction on every edge in it.

For example : see Figure 1.2.

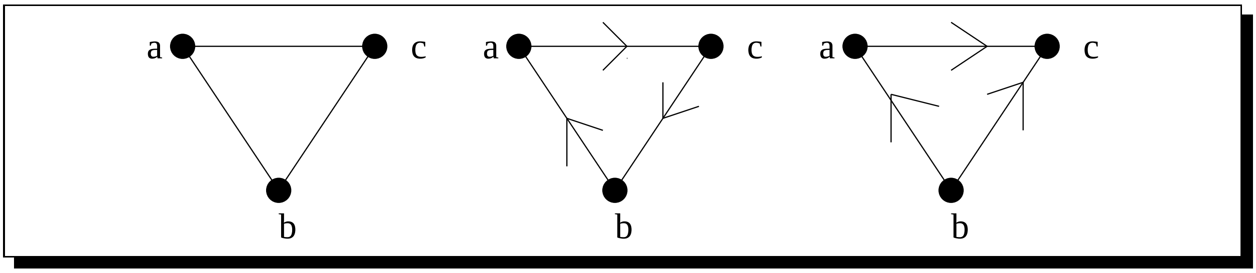

Figure 1.2: Some orientations of a triangle.

Let $G = (V, E)$ be an undirected graph. We define the *complement* of G to be the graph $\overline{G} = (V, \overline{E})$, where

$$\overline{E} = \{(x, y) \in V \times V \mid x \neq y \ and \ (x, y) \notin E\}$$

A graph is *complete* if every two distinct vertices are adjacent. The complete graph on n vertices is usually denoted by K_n, and is called a *clique* of size n.

A (partial) *subgraph* of a graph $G = (V, E)$ is defined to be any graph $H = (V', E')$ satisfying $V' \subseteq V$ and $E' \subseteq E$.

Given a subset $A \subseteq V$ of the vertices, we define the *subgraph induced by A* to be $G_A = (A, E_A)$, where $E_A = \{xy \in E \mid x \in A \ and \ y \in A\}$.

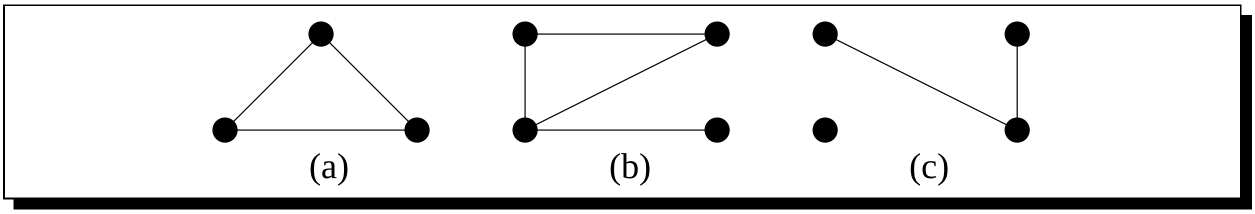

Figure 1.3: (a) The graph K_3.(b) is the complement of (c) and vice versa.

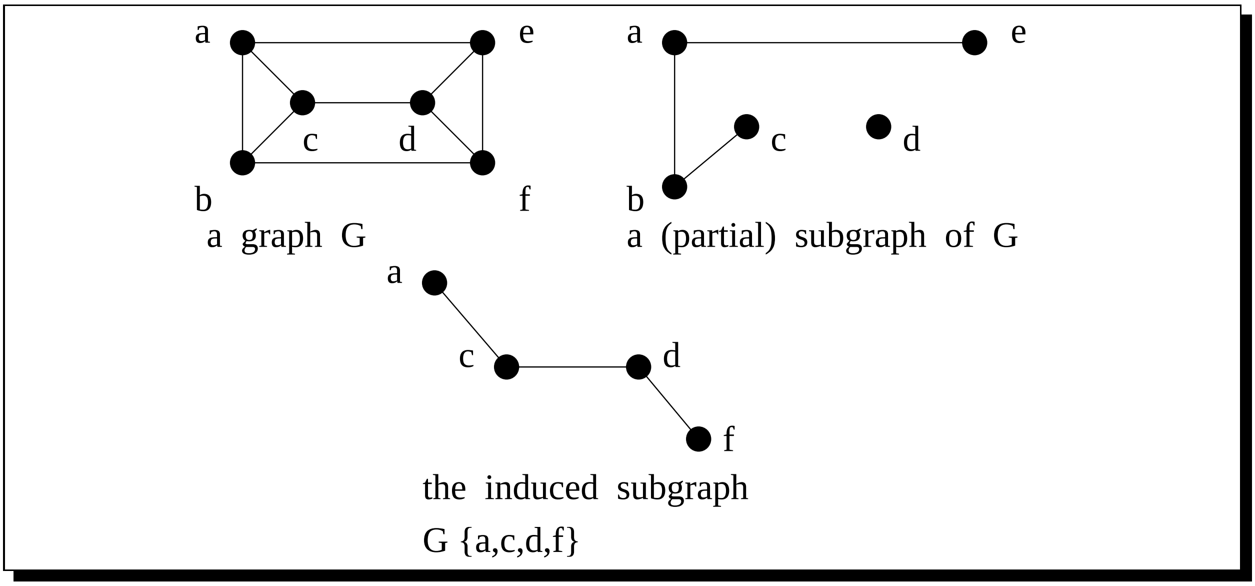

Figure 1.4: Examples of subgraphs.

Let $G = (V, E)$ be an undirected graph. Consider the following definitions :

A subset $A \subseteq V$ of r vertices is an *r-clique@r-clique* if it induces a complete subgraph, i.e., if $G_A \cong K_r$. A single vertex is a 1-clique.

A clique K in G is *maximal* if there is no clique of G which properly contains K as a subset.

A clique is *maximum* if there is no clique of G of larger cardinality.

$\omega(G)$ is the number of vertices in a maximum clique of G, and is called the *clique number* of G.

A *clique cover* of size k is a partition of the vertices $V = A_1 + A_2 + ... + A_k$ such that each A_i is a clique.

$k(G)$ is the size of a smallest possible clique cover of G. It is called the *clique cover number* of G.

A *stable set* or *independent set* is a subset X of vertices no two of which are adjacent.

$\alpha(G)$ is the number of vertices in an independent set of maximum cardinality. It is called the *stability number* of G.

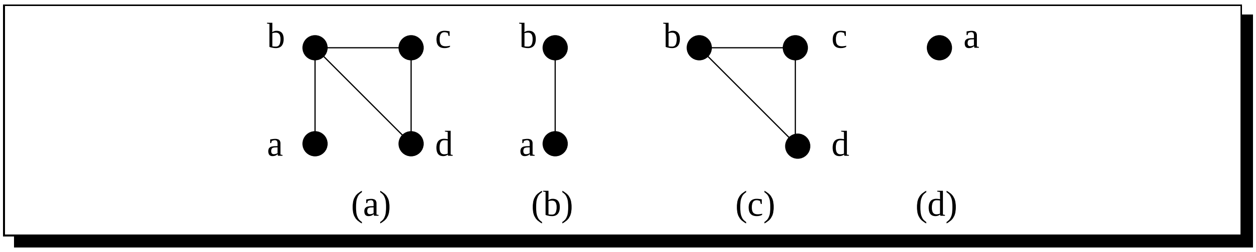

Figure 1.5: (a) A graph G.(b) A maximal clique in G.(c) A maximum clique of G.(d) A 1-clique.

A proper *ccoloring@c-coloring* is a partition of the vertices $V = X_1 + X_2 + ... + X_c$ such that each X_i is an independent set. In such a case, the vertices of X_i are "painted" with the color i and adjacent vertices will receive different colors. If G has a proper c-coloring, we say that G is *ccolorable@c-colorable.*

$\chi(G)$ is the smallest possible c for which there exist *a* proper c-coloring of G, it is called the *chromatic number* of G.

In the graph G of figure 1.5 (a) : $\{b, c, d\}$ and $\{a\}$ are a clique cover of size 2. $\{a, c\}$ is an independent set in G.

Hence, $\omega(G) = 3$; $k(G) = 2$; $\alpha(G) = 2$; $\chi(G) = 3$

It is always true that:

$$\omega(G) \leq \chi(G) \text{ and } \alpha(G) \leq k(G)$$

Since every vertex of a maximum clique must be contained in a different color class in any minimum proper coloring, and every vertex of a maximum stable set must be contained in a different clique in any minimum clique cover.

The following relations are sometimes called *duality*:

$$\omega(G) = \alpha(\overline{G}) \text{ and } \chi(G) = k(\overline{G}).$$

Let $G = (V, E)$ be an arbitrary graph, consider the following definitions :

Chain: A sequence of vertices $[v_0, v_1, v_2, ...v_l]$ is a chain of length l in G if

$$v_{i-1}v_i \in E \text{ or } v_iv_{i-1} \in E \text{ for } i = 1...l.$$

Path: A sequence of vertices $[v_0, v_1, v_2, ...v_l]$ is a path from v_0 *to* v_l in G provided that $v_{i-1}v_i \in E$ for $i = 1..l$. A path or a chain in G is called *simple* if no vertex occurs more than once in the sequence. It is called trivial if $l = 0$. It is called *chordless* if the graph induced by $\{v_0, ...v_l\}$is a path of length l. We denote the chordless path on n vertices by P_n.

Connected graph: A graph G is connected if between any two vertices there exists a chain in G joining them.

Strongly connected graph : A graph G is strongly connected if for any two vertices x *and* y there exists a path in G from x *to* y.

Cycle : A sequence of vertices $[v_0, v_1, v_2, ...v_l, v_0]$ is called a cycle of length $l+1$ (or closed path) if $v_{i-1}v_i \in E$ *for* $i = 1...l$ *and* $v_lv_0 \in E$.

Simple cycle: A cycle $[v_0, v_1, v_2, ...v_l, v_0]$ is a simple cycle if $v_i \neq v_j$ *for* $i \neq j$.

Chordless cycle: A simple cycle $[v_0, v_1, v_2, ...v_l, v_0]$ is chordless if $v_iv_j \notin E$

for every i, j such that $\{(i - j) > 1(mod\ l)\}$. A chordless cycle on n vertices is denoted by C_n.

Bipartite graph: An undirected graph $G = (V, E)$ is bipartite if its vertices can be partitioned into two disjoint independent sets $V = S_1 + S_2$ i.e., every edge has one endpoint in S_1 and one in S_2. We usually denote such graphs by (S_1, S_2, E) to distinguish the two parts.

Complete bipartite graph: A bipartite graph $G = (S_1, S_2, E)$ is complete if for every $x \in S_1$ *and* $y \in S_2$ we have $xy \in E$. We denote by $K_{m,n}$ the complete bipartite graph (S_1, S_2, E) with $\mid S_1 \mid= m$ and $\mid S_2 \mid= n$.

Examples for using the above notation : $K_3 = C_3$ is called a *triangle* ; $\overline{C}_4 = 2K_2$; $K_{n,n} = \overline{2K_n}$; $C_5 = \overline{C}_5$.

1.1.2 Intersection Graphs

Let F be a family of nonempty sets. The *intersection graph* of F is obtained by representing each set in F by a vertex and connecting two vertices by an edge iff their corresponding sets intersect. Examples :

1. Interval graph : The intersection graph of a family of intervals on a linearly ordered set (like the real line) is called an *interval graph*. If these intervals have a unit length, then we call the graph a *unit interval graph*.

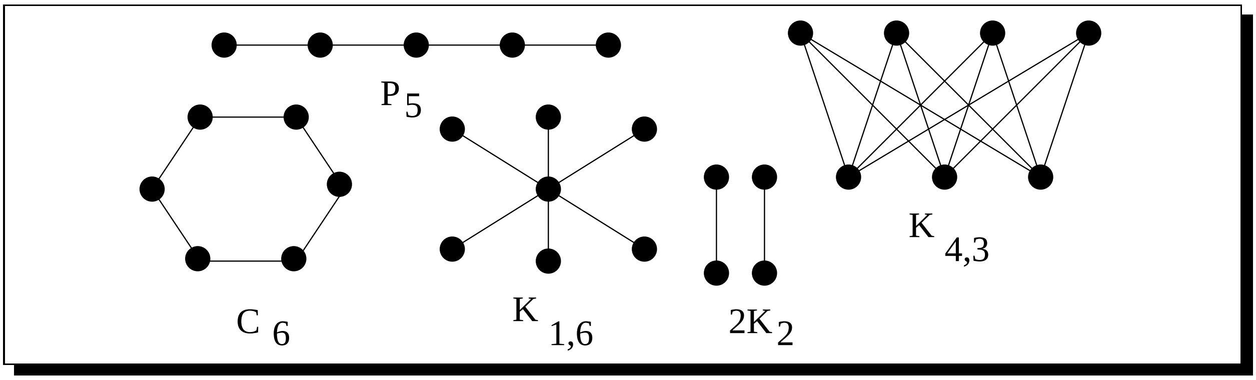

Figure 1.6: Graph examples.

2. Proper interval graph : A proper interval graph is constructed from a family of intervals on a line such that no interval properly contains another.

In 1969 Roberts showed that a graph is a unit interval graph iff it is a proper interval graph.

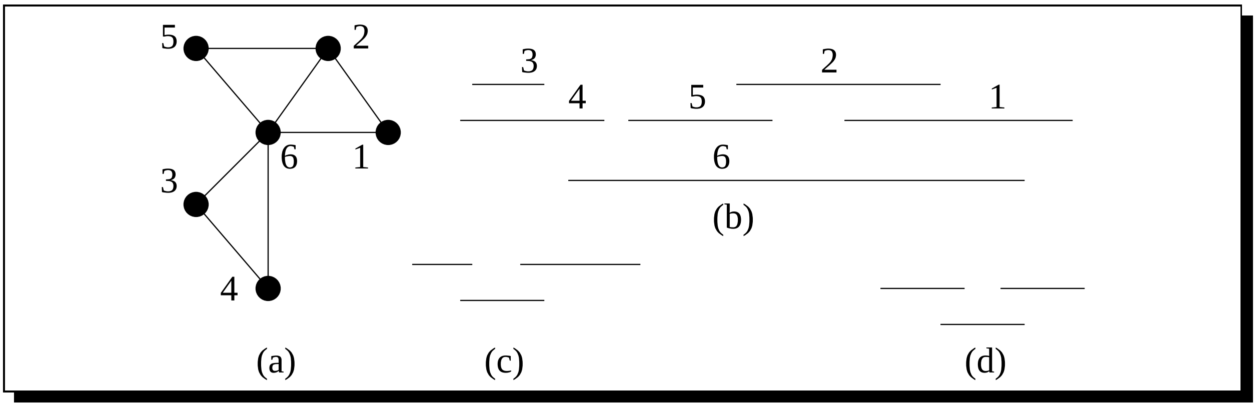

Figure 1.7: (a) A graph G.(b) An interval representation of G.(c) A proper interval representation for P_3.(d) A unit interval representation for P_3.

1.1.3 Circular-arc Graphs

Consider the following generalization of the notion of intervals on a line. If we join the two ends of our line, thus forming a circle, the intervals will become arcs on the circle. Allowing arcs to slip over and include the point of connection, we obtain a class of intersection graphs called the circular-arc graphs, which properly contains the interval graphs. In other words : a graph is a circulation arc-graph if it is the intersection graph of a family of arcs on a circle.

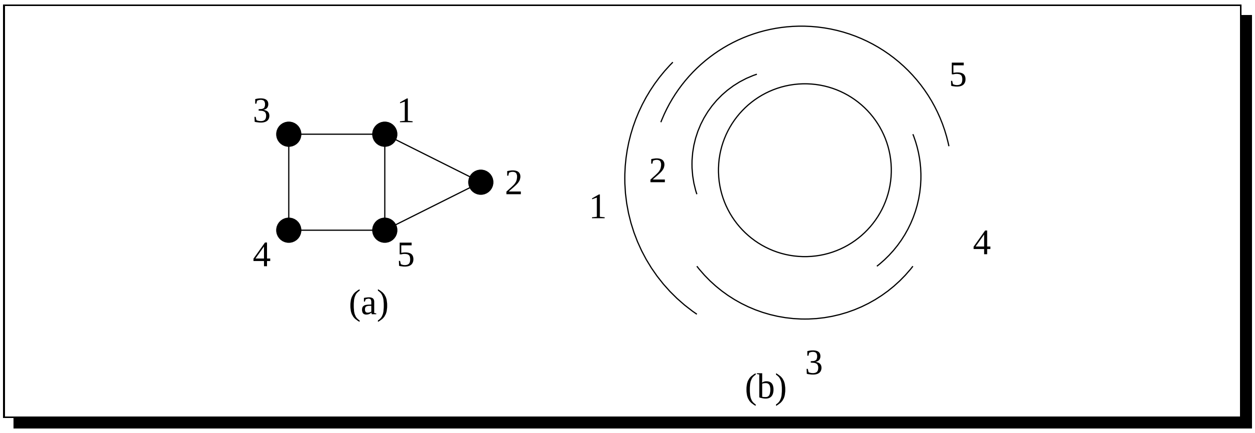

Figure 1.8: (a) A graph G.(b) A circular-arc representation of G.

Example 1.1 *The traffic flow at the corner of Holly, Wood and Vine is pictured in the next page in figure 1.9 (a). Two lanes are called compatible if traffic can flow on both of them simultaneously without collisions. Certain lanes are compatible with one another, such as c and j or d and k, while others are incompatible, such as b and f. In order to avoid collisions, we wish to install a traffic light system to control the flow of vehicles. Each lane will be assigned an arc on a circle representing the time interval during which it has a green light. Incompatible lanes must be assigned disjoint arcs. The circle may be regarded as a clock representing an entire circle which will be continually repeated.*

An arc assignment for our example is given in figure 1.9 (c). In general if G is the intersection graph of the arcs of such an assignment (see the figure 1.9 (b)), and if H is the compatibility relation defined on the pairs of lanes, then clearly G is a (partial) subgraph of H. In our example, the compatible pairs $(d,k),(h,j),$ and (i,j) are in H but are not in G.

A *proper circular arc graph* is the intersection graph of a family of arcs none of which properly contains another. In a different generalization of interval graphs Renz [1970] characterized the intersection graphs of paths in a tree. Walter [1972], Buneman [1974], and Gavril [1974] carried this idea further and showed that the intersection graphs of subtrees of a tree are exactly the triangulated graphs (the graphs with no induced C_k for $k \geq 4$).

A *permutation diagram* consists on n points on each of two parallel lines, and n straight line segments matching the points. The intersection graph of the line segments is called a *permutation graph.*

If the $2n$ points are located randomly around a circle, then the matching segments will be chords of the circle, and the resulting class of intersection graphs properly contains the permutation graphs. This class is called the class of *circle graphs.*

A simple argument shows that every proper circular-arc graph is also a circle graph. We may assume that no pair of arcs together covers the entire circle. For each arc on the circle

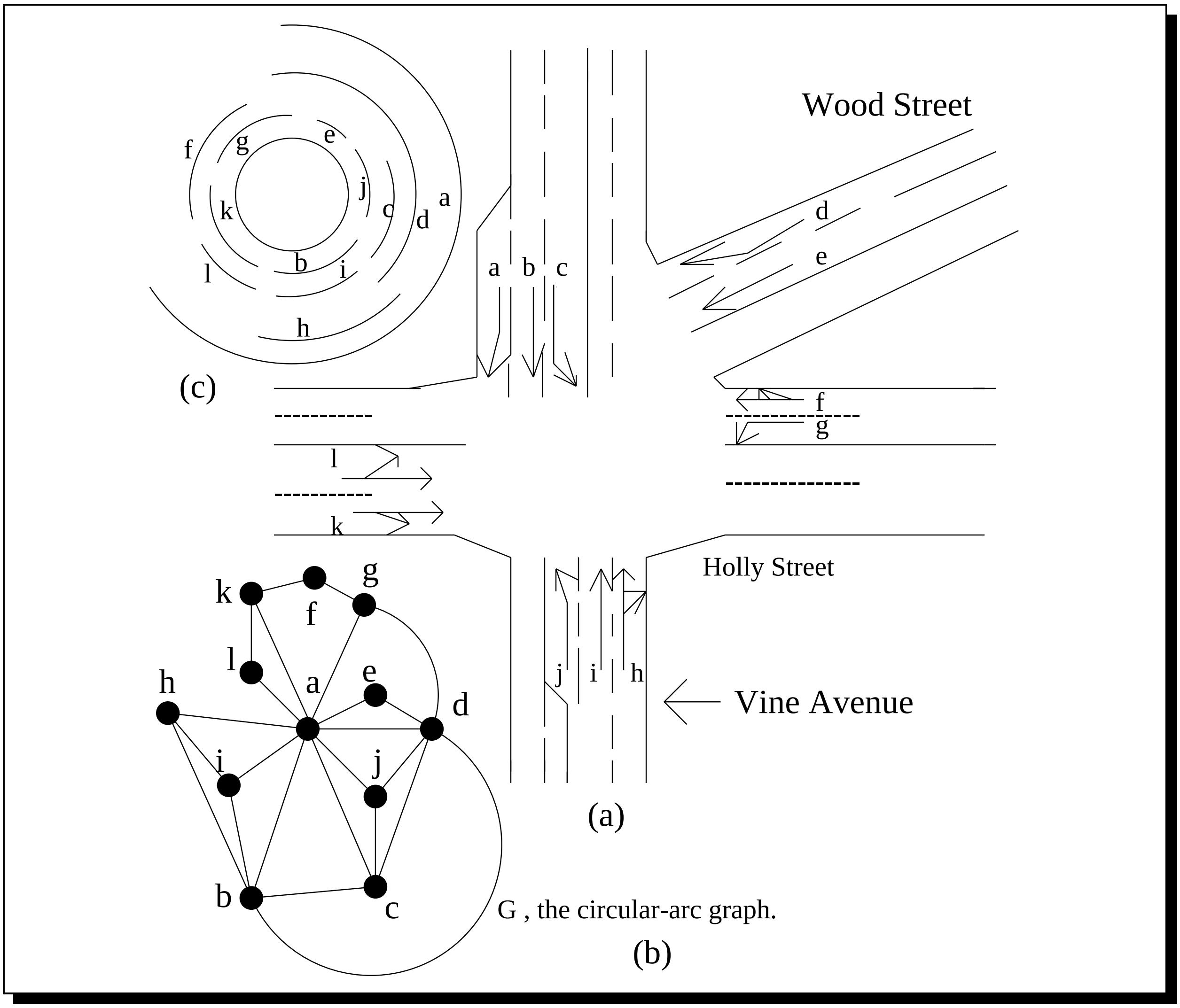

Figure 1.9: (a) The traffic flow. (b) The intersection graph of the arcs. (c) The clock cycle.

draw the chord connecting its two endpoints. Clearly two arcs overlap iff their corresponding chords intersect.

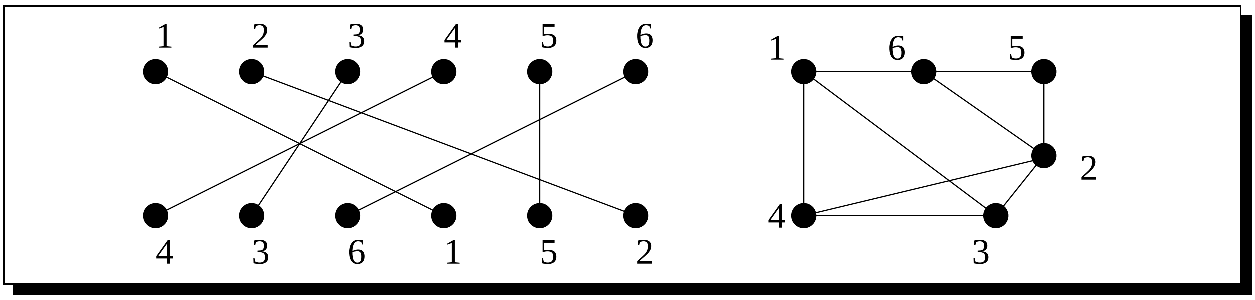

Figure 1.10: The matching diagram of [4,3,6,1,5,2], and the corresponding permutation graph.

1.1.4 Interval Graphs

An undirected graph G is called an interval graph if its vertices can be put into one-to-one correspondence with a set of intervals f of a linearly ordered set (like the real line) such that two vertices are connected by an edge of G iff their corresponding intervals have nonempty intersection. We call f an *interval representation* for G.

Application to scheduling :

Consider a collection $c = c_i$ of courses being offered by a major university. Let T_i be the time interval during which course c_i is to take place. We would like to assign courses to classrooms so that no two courses meet in the same room at the same time. This problem can be solved by properly coloring the vertices of the graph $G = (C, E)$ where :

$$c_i c_j \in E \leftrightarrow T_i \cap T_j \neq \emptyset.$$

Each color corresponds to a different classroom. The graph G is obviously an interval graph, since it is represented by time intervals. This example is especially interesting because efficient linear-time algorithms are known for coloring interval graphs with a minimum number of colors. (The minimum coloring problem is NP-complete for general graphs).

An application of interval Graph

Suppose $c_1, c_2, ..., c_n$ are chemical compounds which must be refrigerated under closely monitored conditions. If compound c_i must be kept at a constant temperature between t_i and t'_i degrees, how many refrigerators will be needed to store all the compounds ?

Let G be the interval graph with vertices $c_1, c_2, ..., c_n$ and connect two vertices whenever the temperature intervals of their corresponding compounds intersect. Intervals on a line satisfy the *Helly property* : if a set $S = \{T_i ... T_k\}$ of intervals such that, $T_i \cap T_j \neq \emptyset$ for every $i \neq j$ then $\bigcap_{t_i \in S} T_i \neq \emptyset$. Hence, if $\{c_{i1}, c_{i2}, ..., c_{ik}\}$ is a clique of G, then the intervals

$\{[t_{ij}, t'_{ij}] \mid j = 1, 2, ..., k\}$ will have a common point of intersection, say t. A refrigerator set at a temperature of t will be suitable for storing all of them. Thus, a solution to the problem will be obtained by finding a minimum clique cover of G. This is a polynomial problem on interval graph, although it is NP-hard for arbitrary graphs.

Interval graphs satisfy many interesting properties, one of them is that interval graph is an *hereditary property*: An induced subgraph of an interval graph is an interval graph.

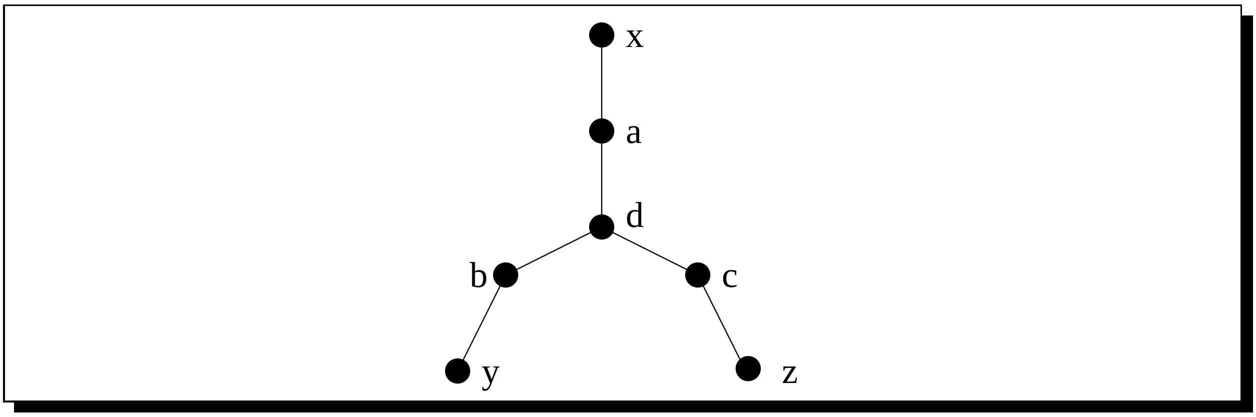

Figure 1.11: A triangulated graph which is not an interval graph.

Hereditary properties abound in graph theory, such as planarity and bipartiteness. The next property of interval graphs is also an hereditary property.

Triangulated graph property :

Every simple cycle of length strictly greater than 3 possesses a chord. Graphs which satisfy this property are called *triangulated graphs.*

Theorem 1.2 *(Hajös[1958]) An interval graph satisfies the triangulated graph property.*

Proof: Suppose the interval graph G contains a chordless cycle $[v_0, v_1, v_2, ...v_{l-1}, v_0]$ with $l > 3$. Let I_k denote the interval corresponding to V_k. For $i = 1, 2, ..., l-1$ choose a point $p_i \in (I_{i-1} \cap I_i)$. Since I_{i-1} and I_{i+1} do not overlap, the P_i constitute a strictly increasing or strictly decreasing sequence. Therefore, it is impossible for I_0 and I_{l-1} to intersect, contradicting the criterion that $V_0 V_{l-1}$ is an edge of G. ∎

Not every triangulated graph is an interval graph. Consider the tree T given in figure 1.11, which certainly has no chordless cycles. The intervals $I_a, I_b,$ *and* I_c of a representation for T would have to be disjoint, and I_d would properly include the middle interval, say I_b. Where, then could we put I_y so that it intersects I_b but not I_d ? Clearly we would be stuck.

Transitive orientation property.

Each edge can be assigned a one-way direction, in such a way that the resulting oriented graph $G = (V, E)$ satisfies the following condition :

$$(\ ab \in E \ and\ bc \in E\ ) imply\ ac \in E\ (\forall a, b, c \in V) \quad \star$$

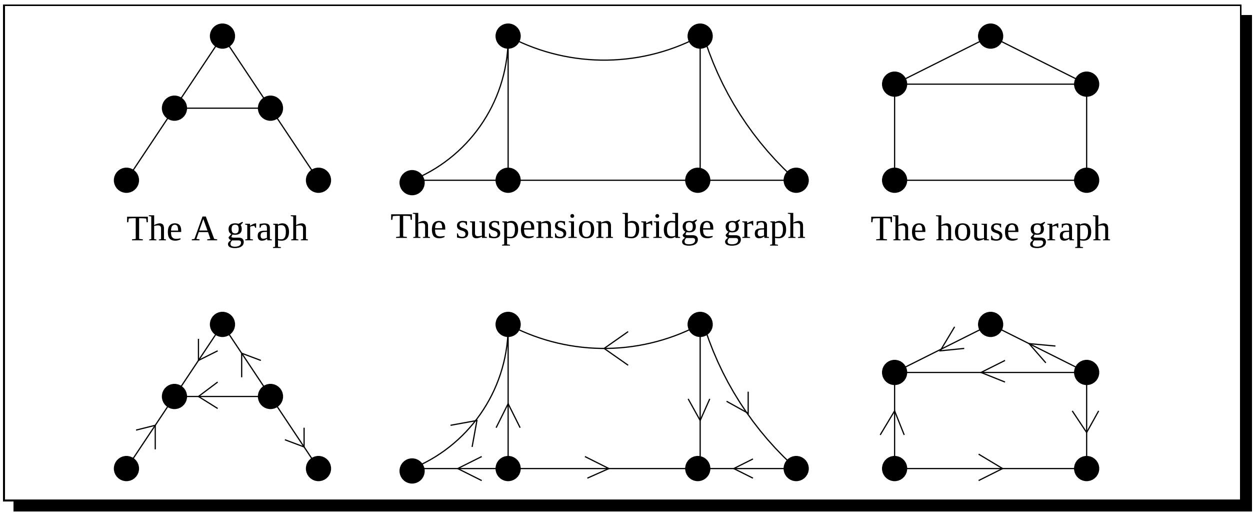

Figure 1.12: Transitive orientations of Three comparability graphs.

An undirected graph which is transitively orientable is sometimes called a *comparability graph.* Figure 1.12 shows a transitive orientation of the A graph and of the suspension bridge graph. The odd length chordless cycles $C_5, C_7, C_9...$ and the bull's head graph (see Figure 1.13) cannot be transitive oriented.

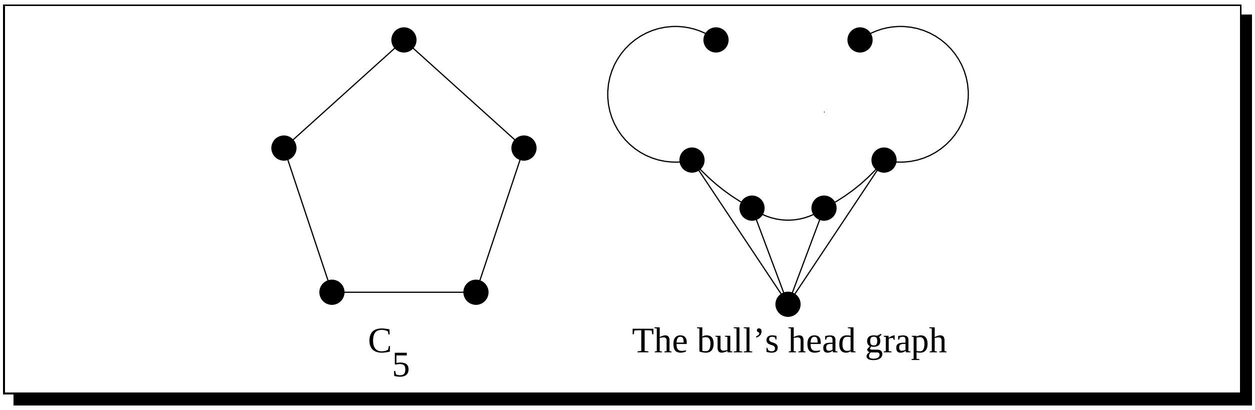

Figure 1.13: Two graphs which are not transitively orientable.

Theorem 1.3 *(Ghouila - Houri [1962]). The complement of an interval graph satisfies the transitive orientation property.*

Proof: Let $\{I_v \mid v \in V\}$ be an interval representation for $G = (V, E)$. Define an orientation F of the complement $\overline{G} = (V, \overline{E})$ as follows :

$$xy \in F \leftrightarrow I_x < I_y \quad (\forall xy \in \overline{E}).$$

Here $I_x < I_y$ means that the interval I_x lies entirely to the left of the interval I_y. (Remember, they are disjoint). Clearly $(\star)$ is satisfied, since $I_x < I_y < I_z$ implies $I_x < I_z$. Thus, F is a transitive orientation of $\overline{G}$. ■

Theorem 1.4 (Gilmore and Hoffman 1964) *An undirected graph G is an interval graph iff G is a triangulated graph and its complement $\overline{G}$ is a comparability graph.*

We will see the proof in a later lecture.

We have seen already that every interval graph satisfies the two conditions. Looking back each of the graph in Figure 1.12 can be properly colored using three colors and each contains a triangle. Therefore for these graphs, their chromatic number equals their clique number. We will see later in the course that any triangulated graph and any comparability graph also satisfies the following properties :

$\chi-$*Perfect property*

For each induced subgraph G_A of G, $\chi(G_A) = \omega(G_A)$.

The chordless cycles C_5, C_7, C_9 are not χ - perfect. A dual notion of χ - perfection is the following:

alphaperfect@$\alpha-$*Perfect property*

For each induced subgraph G_A of G, $\alpha(G_A) = k(G_A)$.

Theorem 1.5 *The perfect graph theorem (Lovász [1972]).*

A graph is $\chi-$perfect iff it is $\alpha-$perfect.

We will prove this theorem in lecture 2.

1.1.5 Line graphs of bipartite graphs

Let $G = (V, E)$. The *line graph* or the *edge graph* of G denoted $L(G)$ is the graph with $V(L(G)) = E(G)$ and

$$(e1, e2) \in E(L(G)) \leftrightarrow e1 \textit{ and } e2 \textit{ have a common endpoint in } G.$$

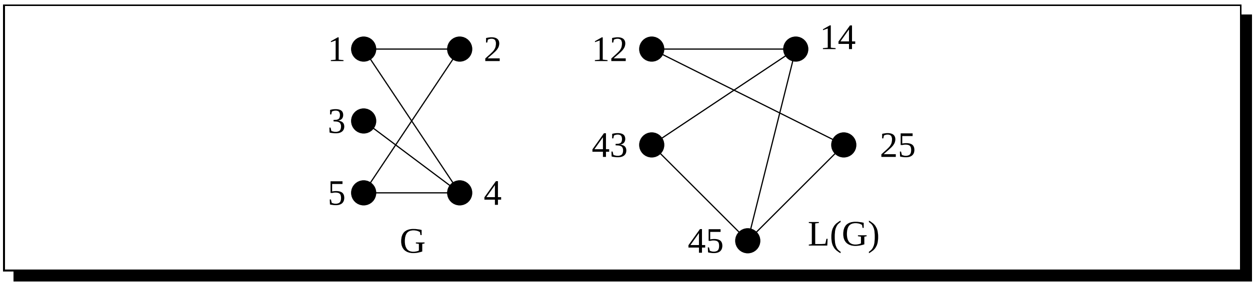

Figure 1.14: (a) A graph G.(b) The line graph of G.

Being a line graph of a bipartite graphs is an hereditary property. (Removing a vertex in $L(G)$ is the same as removing the corresponding edge in G, which remains bipartite).

An Independent set in $L(G)$ corresponds to a matching (a set of independent edges) in G. A Clique in $L(G)$ corresponds to a set of pairwise adjacent edges in G. If G is bipartite it contains no triangles, so every set of pairwise adjacent edges in G) is a star. Therefore, if G is bipartite, a clique cover of $L(G)$ corresponds to a vertex cover of G.

Let us call a graph $LGBG$ if it is the line graph of a bipartite graph. König (1931) proved that the edge chromatic number of a bipartite graph equals its maximum degree. By the above, this is equivalent to saying that if G is $LGBG$ then $\chi(G) = \omega(G)$. The König-Hall theorem (1931) states that the size of a maximal matching in a bipartite graph equals its minimum vertex cover. This is equivalent to saying that $\alpha(G) = K(G)$ if G is $LGBG$. Together with the fact that being $LGBG$ is hereditary, any of these two statements imply that $LGBG'S$ are perfect. From the perfect graph theorem the two statements are equivalent.