# Chapter 10

# Lecture 10


Advanced Topics in Graph Algorithms
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Spring Semester, 1994

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Lecture 10 : June 16


## 10.1 Comparability invariants

In lecture # 9 we quoted the following theorem:

**Theorem 10.1 ( Trotter, Moore, Sumner [1976], Gysin [1977] )** *If two partial orders have the same comparability graph then their dimensions are same.*

Hence, the dimensions of all posets which have the same comparability graph is the same. This leads to the following definition:

**Definition** Let $\psi$ be a property of partially ordered sets (posets). Then for poset $P$ $\psi$ (P) := 1 if $\psi$ holds on $P$ and $\psi(P) := 0$ otherwise. Property $\psi$ is called *comparability invariant* (CI) if for every two posets $P, P'$ $G(P) = G(P')$ implies $\psi(P) = \psi(P')$.

Some of the known comparability invariants are: the dimension, length of maximum chain (because it is $\omega(G)$), minimum number of chains ( or anti-chains ) needed for covering of $P$ (because they are $k(G)$, $\alpha(G)$ in correspondence), and the number of linear extensions of $P$.

In 1991 *Brightwell* and *Winkler* showed that the problem of counting the number of linear extensions of a poset is $\#P$ - complete.

**Definition** Let $P = (V, E)$ be poset, $L = (V, E')$ is linear extension. Without loss of generality order for $E'$ is $v_1 < v_2 < \ldots < v_n$. The *jump number* (JP) of $L$ is the number of pairs $(v_i, v_{i+1})$ such that $\{v_i, v_{i+1}\} \notin E$ .

More generally:

$$J(P,L) = \mid \{i \mid (L^{-1}(i), L^{-1}(i+1)) \notin E\} \mid$$

The jump number of $P$ is the least jump number over all linear extensions of $P$ i.e.

$$J(P) = min\{J(P,L) \mid L \text{ is a linear extension of } P\}.$$

The jump number is known to be a comparability invariant. In 1991 *Syslo* proved that calculating $J(P)$ is NP - complete.

**Theorem 10.2 ( Habib, [1982] )** *Property $\psi$ is comparability invariant iff the following is true:*

*(1) $\psi(P) = \psi(P^{-1})$ $\forall$ poset $P$.*

*(2) Let $P = P_0[P_1, \dots, P_m]$, $Q = Q_0[Q_1, \dots, Q_m]$ be proper decompositions and $G(P_i) = G(Q_i)$, $0 \le i \le m$. If $\psi(P_i) = \psi(Q_i)$ $\forall i$, then $\psi(P) = \psi(Q)$.*

**Definition** Let $P_1 = (\ V_1,\ <_1\ )$, $P_2 = (\ V_2, <_2\ )$ be posets where $V_1, V_2$ are disjoint. Then we'll define a *parallel composition of* $P_1, P_2$ by $P_1 + P_2 = (\ V_1 + V_2,\ <_+\ )$ where

$$x <_+ y \Leftrightarrow ((x, y \in V_1,\ x <_1 y) \vee (x, y \in V_2,\ x <_2 y)).$$

The *serial composition of* $P_1, P_2$ is $P_1 * P_2 = (V_1 + V_2, <_*)$ where

$$x <_* y \Leftrightarrow ((x, y \in V_1,\ x <_1 y) \vee (x, y \in V_2,\ x <_2 y) \vee (x \in v_1,\ y \in v_2)).$$

Thus, $P_1 + P_2 \equiv 2K_1[P_1, P_2]$, $P_1 * P_2 \equiv P_1[P_1, P_2]$.

**Definition** A poset $P = (V, <)$ is called *series parallel* (SP) if either $\mid V \mid= 1$ or one can derive it from any series parallel posets $P_1, P_2$ by parallel or series compositions.

For examples see Figure 10.1.

**Definition** A graph is called *cograph* if it is $P_4$ - free.

**Theorem 10.3** *$G$ is a cograph iff for every induced subgraph $H$ of $G$ either $H$ or $\bar{H}$ is disconnected.*

*Proof:* Exercise. ∎

**Theorem 10.4** *Every cograph is a permutation graph.*

**Theorem 10.5** *$A$ is a cograph iff $A$ is the comparability graph of a series-parallel poset.*

**Corollary 10.6** *The property "being a series-parallel poset" is a comparability invariant.*

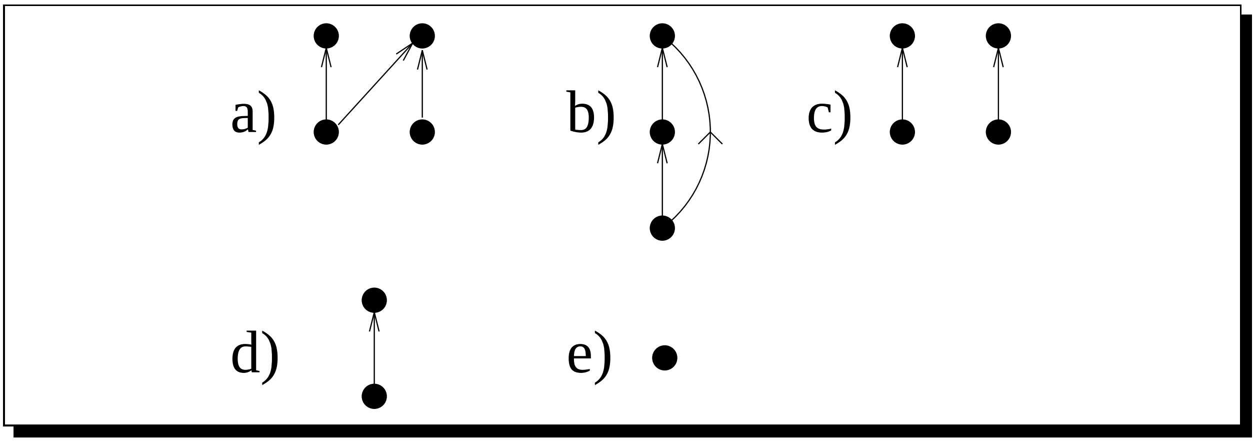



Figure 10.1: a) not SP poset; b) - e) SP posets

## 10.2 Interval graphs

**Definition** graph $G = (V, E)$ is called an *interval graph* if one can assign to each vertex $v \in V$ an interval on the real line $I_v$ such that $(v_1, v_2) \in E \Leftrightarrow I_{v_1} \bigcap I_{v_2} \neq \emptyset$.

Interval graph were introduced by *Hajös(1957)* and*Benzer(1958)*.

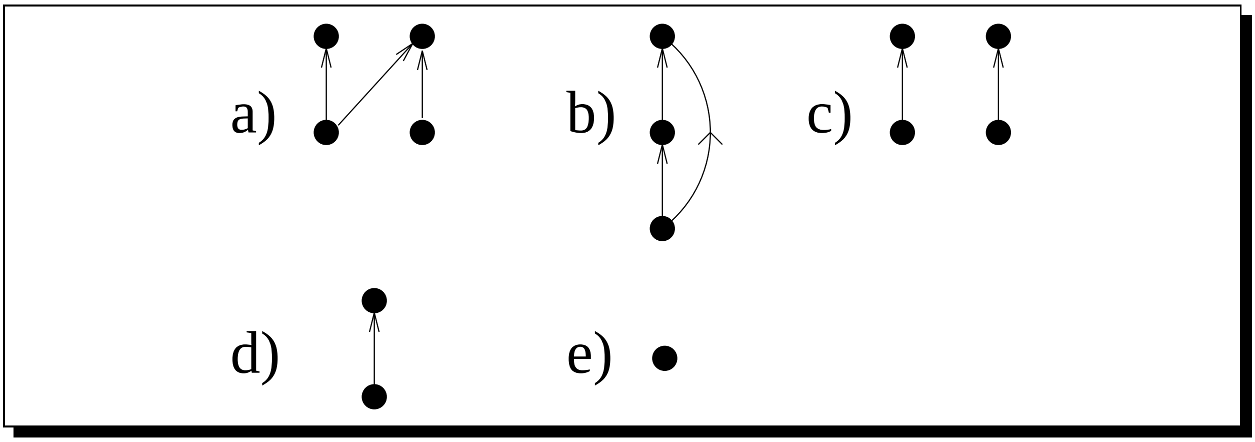



Figure 10.2: a) interval graph; b) it's interval representation; c) chordal, but not interval graph

For examples see Figure 10.2.

**Theorem 10.7 (Gilmore - Hoffman [1964])** *The following are equivalent:*

*(1) G is an interval graph.*

*(2) G contains no induced $C_4$ and $\bar{G}$ is a comparability graph.*

*(3) It is possible to order the maximal cliques in G so that for every vertex $v \in V$ the cliques that contain v appear consecutively in this order.*

*Proof:*

(1) ⇒(2) Immediate, because it is easy to see that an interval graph $G$ contains no induced $C_4$ and a transitive orientation for $\bar{G}$ is obtained by taking an interval realization of $G$, $\{I_v\}_{v \in V}$ and setting $(u, v) \in E(\bar{G}) \Leftrightarrow I_u$ is completely to the left of $I_v$.

(2) ⇒(3) Let $F$ be a transitive orientation for $\bar{G}$.

**Lemma:** Let $A_1, A_2$ be distinct non-empty maximal cliques of $G$, and let $(V, F)$ be a transitive orientation of $\bar{G}$. Then 1) there exists an arc from $F$ between $A_1$ and $A_2$;

2) all such arcs between $A_1$ and $A_2$ have the same direction.

*Proof:*

(1) It is trivial because otherwise $B := A_1 \bigcup A_2$ is a clique in $G$ and $\mid B \mid > \mid A_1 \mid, \mid A_2 \mid$, in contradiction to the definition of $A_1, A_2$.

(2) Assume that $a, c \in A_1, b, d \in A_2, (a, b), (d, c) \in F$. For example see Figure 10.3. If $a \equiv c$ (or $b \equiv d$) then we get (d,b) $\in F$, a contradiction to $A_2$ being a clique in $G$. Therefore the vertices are different. The situation $\{a, d\} \in E$ and $\{b, c\} \in E$ is impossible, because otherwise $G$ contains an induced $C_4$. Without loss of generality assume that $\{ad\} \notin E$. Then we have two possible cases now:

(*i*) $(a, d) \in F$.Then the transitivity of $F$ implies $(a, c) \in F$.

(*ii*) $(d, a) \in F$. Transitivity of $F$ implies $(d, b) \in F$.

In both cases we get a contradiction.

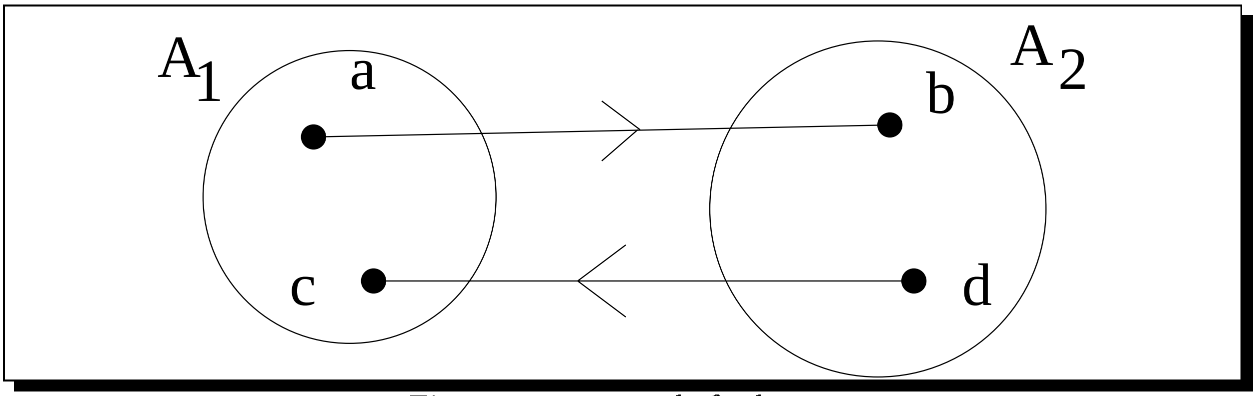



Figure 10.3: example for lemma

■

We now return to the theorem. Define a relation among maximal cliques in $G$ as follows: for max.cliques $A_1$ and $A_2$, $A_1 < A_2 \Leftrightarrow$ there is exists arc from $F$ whose tail is in $A_1$ and it's head is in $A_2$. According to lemma, this relation defines a tournament on the set of maximal cliques of $G$. We'll prove that it is transitive and therefore defines a complete order on G. Assume $A_1 < A_2 < A_3$. Then $\exists\ (w, x) \in F : w \in A_1, x \in A_2$ and $\exists\ (y, z) \in F : y \in A_2, z \in A_3$. For example see Figure 10.4.

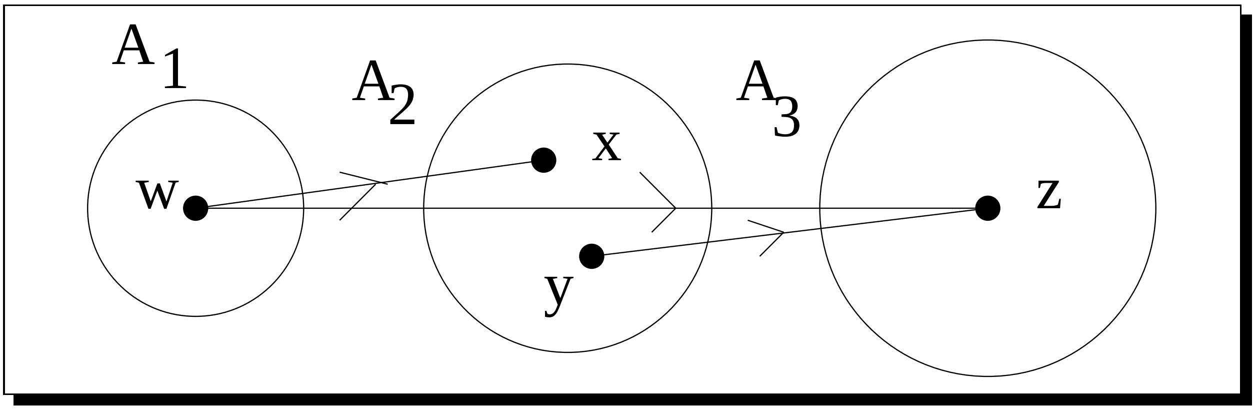



Figure 10.4: example for theorem of Gilmore-Hoffman

($\alpha$) if $\{x, z\} \notin E$ then the lemma implies that $(x, z) \in F$ and because $F$ is a transitive orientation for $G$ we get $(w, z) \in F$ and, hence, $A_1 < A_2$.

($\beta$) if $\{w, y\} \notin E$ then analogously we get $(w, y) \in F$ and $A_1 < A_2$.

($\gamma$) $\{x, z\}, \{w, y\} \in E$. $G$ contains no $C_4$ and therefore $\{w, z\} \notin E$. It follows that, since $F$ is transitive, $(w, z) \in F$. Hence, $A_1 < A_3$.

We have shown that $<$ is complete order (linear) on the maximal cliques of $G$. Let $x \in V$. Suppose that there exist $A_i < A_j < A_k$ such that $x \in A_i, A_k$ and $x \notin A_j$.

$$(x \notin A_j) \Rightarrow \exists y \in A_j : \{x, y\} \notin E$$

$$(A_i < A_j) \Rightarrow (x, y) \in F$$

$$(A_j < A_k) \Rightarrow (y, x) \in F$$

We have a contradiction. Therefore the claim is true.

(3) $\Rightarrow$(1) Let $I(x)$ be the set of maximal cliques of $G$ that contain $x$. By (3) each such set is an interval in complete order $(C, <)$ of maximal cliques. $\forall x, y \in V\ \{x, y\} \in E \Leftrightarrow I(x) \bigcap I(y) \neq \emptyset$, because two vertices are neighbors iff they are contained together in some maximal clique.

■

**Corollary 10.8** *An undirected $G$ is an interval graph iff $G$ is chordal and $\bar{G}$ is comparability.*

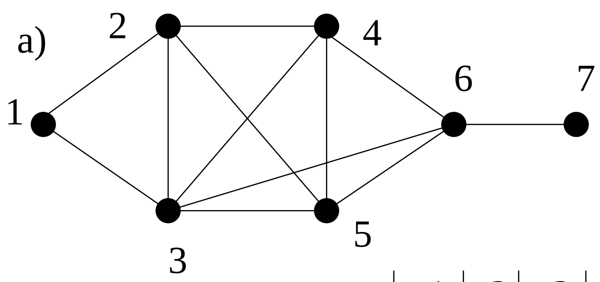
*Proof:* If $G$ is interval then it is chordal by the theorem of *Hajös* (1958) (lecture # 1).The rest holds according to the last theorem. Conversely, if $G$ is chordal then it contains no $C_4$ and therefore if $\bar{G}$ is a comparability graph then $G$ is interval. ■

**Reminder:** The *clique matrix of graph* is a (0,1) matrix $M$ such that its rows correspond to the maximal cliques of graph, its columns to the vertices and $M_{ij} = 1$ iff $v_j$ belongs to the $i$-th maximal clique. Theorem 10.4 implies that if rows are ordered in the order $<$ then the ones in every column will appear contiguously. This property is called the *consecutive ones property for columns.* For example see Figure 10.5.

**Theorem 10.9 (Fulkerson-Gross [1965])** *An undirected graph $G$ is interval iff its clique matrix has the consecutive ones property.*

*Proof:* Immediately follows from the third characterization of Theorem 10.4. ■

a)




b)

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| {1, 2, 3} | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| {2, 3, 4, 5} | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| {3, 4, 5, 6} | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| {6, 7} | 0 | 0 | 0 | 0 | 0 | 1 | 1 |

Figure 10.5: a) interval graph; b) it's clique matrix.

**Definition** In an undirected graph $G$ a triplet of vertices $< x, y, z >$ is called *an asteroidal triplet* if $G_{<x,y,z>}$ is stable and for each pair from $\{x, y, z\}$ there exists path between these two vertices which does not pass through neighbors of the third vertex.

**Theorem 10.10 (Lekkerkerker-Boland [1962])** *G is an interval graph iff it is chordal and asteroidal triplet-free.*

*Proof:* If $G$ interval then it is chordal. Consider an interval realization of $G$. Assume that $< z, y, x >$ is asteroidal triplet. Without loss of generalization suppose that the left endpoint of $I_z$ is smallest among left endpoints of the intervals of the triplet in this realization and right endpoint of $I_x$ is biggest among right endpoints of this intervals. Then there exists sequence of intervals $z = v_1, \ldots, v_k = x$ such that $\forall\ 1 \leq i \leq k-1\ I_{v_i} \bigcap I_{v_{i+1}} \neq \emptyset$. On other hand, $\forall\ 2 \leq j \leq k-1\ I_y \bigcap I_{v_i} = \emptyset$. It is contradiction. Therefore $G$ is asteroidal triplet-free. For example see Figure 10.6. Proof of other direction is omitted here. ∎

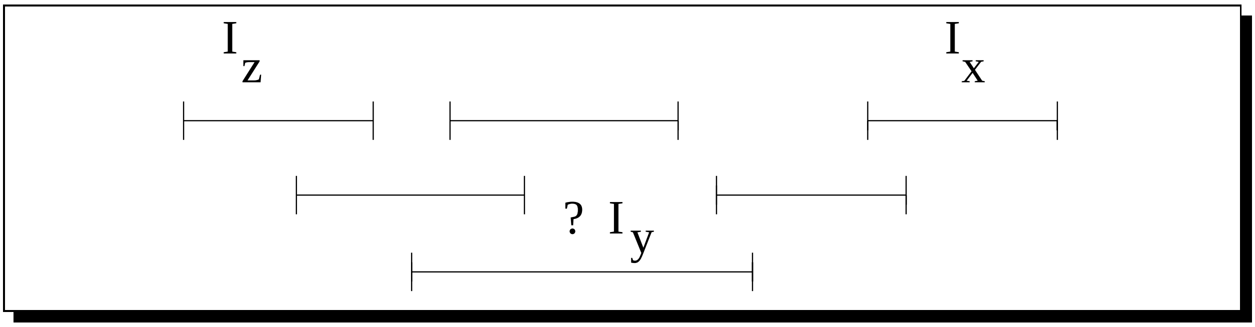



Figure 10.6: Example for Lekkerkerker - Boland theorem