Chapter 2

Lecture 2

Advanced Topics in Graph Algorithms

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2.1 Perfect Graphs

In this lecture the graphs we consider are undirected. When we refer to a graph G we mean the Graph $G(V, E)$, with vertex set V and edge set E.

2.1.1 Definition of perfect graph

The following three conditions define perfection properties of a graph $G(V, E)$.

- (P1) $\omega(G_A) = \chi(G_A), \ \forall A \subseteq V$
- (P2) $\alpha(G_A) = k(G_A), \ \forall A \subseteq V$
- (P3) $\omega(G_A)\alpha(G_A) \geq |A|, \ \forall A \subseteq V$

By duality, a graph G satisfies (P1) iff its complement $\overline{G}$ satisfies (P2).

In this lecture we'll prove the *perfect graph theorem* which says that (P1),(P2) and (P3) are equivalent. A graph which satisfies one of the perfection properties satisfies all (by the theorem) and is called *perfect graph.*

Remark: The equivalence holds for finite graphs only.

Lemma 2.1 *Let the graph $G(V, E)$ satisfy: $\alpha(G) = k(G)$. If $\overline{K}$ is a minimum clique cover, and S is a maximum independent set, then each clique in $\overline{K}$ has exactly one vertex in common with S.*

Proof: By definition, no clique can intersect an independent set in more than one element. Therefore the intersection of S and each clique in $\overline{K}$ may be empty, or one element. Remember that each vertex is contained in some clique. If there is a clique whose intersection with S is empty then, $|\overline{K}| > |S|$, *contradiction* to the assumption of equality. ■

Lemma 2.2 *Let G be an induced subgraph of H then:*

1. $\omega(H) \geq \omega(G)$
2. $\chi(H) \geq \chi(G)$
3. $\alpha(H) \geq \alpha(G)$
4. $k(H) \geq k(G)$

Lemma 2.3 *If $D \subseteq V$ intersects every maximum independent set of G in exactly one vertex then: $\alpha(G_{V-D}) = \alpha(G) - 1$*

Proof: Denote $\alpha = \alpha(G)$. Let U be a maximum independent set of G. By the assumption $U \cap D = \{i\}$. Therefore $U - \{i\}$ is an independent set of G_{V-D}. Therefore $\alpha(G_{V-D}) \geq \alpha - 1$. If $\alpha(G_{V-D}) > \alpha - 1$ then $\alpha(G_{V-D}) = \alpha$ (because of the monotonicity shown in Lemma 2.2). It follows that G_{V-D} contains a maximum independent set of size α, therefore G contains this maximum independent set of size α, which does not intersect D, *contradiction.* ■

2.1.2 Some Definitions and Properties

Given two graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$, we define their *union* and *join.*

Definition *Union*: $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$

Definition *Join*: $G_1 + G_2 = (V_1 \cup V_2, E = E_1 \cup E_2 \cup \{uv | u \in V_1, v \in V_2\})$

Lemma 2.4 *if G_1, G_2 are perfect, then their union and join are perfect.*

Proof: Denote the parameters of each graph G_i as: $\chi_i,\ \omega_i,\ \alpha_i,\ k_i \quad i = 1, 2$
The parameters value for $G_1 \cup G_2$ are:

$$\omega = \max(\omega_1, \omega_2) \qquad \chi = \max(\chi_1, \chi_2)$$

$$k = k_1 + k_2 \qquad \alpha = \alpha_1 + \alpha_2$$

The parameters value for $G_1 + G_2$ are:

$$\omega = \omega_1 + \omega_2 \qquad \chi = \chi_1 + \chi_2$$

$$k = \max(k_1, k_2) \qquad \alpha = \max(\alpha_1, \alpha_2)$$

These relations together with the perfection of $G1$, $G2$ imply the claim. ∎

Definition $G \circ x$: Let $G(V, E)$ be an undirected graph with vertex $x \in V$. The graph $G \circ x$ is obtained from G by adding a new vertex x' which is connected to all the neighbors of x: $G \circ x = (V \cup \{x'\}, E' = E \cup \{x'u | u \in Adj(x)\})$.

Lemma 2.5 $\forall x \neq y \quad (G \circ x) - y = (G - y) \circ x$

Proof: Let x' denote the new vertex (the copy of x). Both graphs has the same vertex set: $V + x' - y$. In both graphs, uv is an edge iff one of the following exist:

1. $uv \in E, u \neq y, v \neq y$
2. $v = x', ux \in E, u \neq y$

∎

Definition More generally, if $x_1, \ldots, x_n$ are the vertices of G and $h = (h_1, \ldots, h_n)$ is a vector of non-negative integers, then $H = G \circ h$ is constructed by substituting for each x_i a stable set of h_i vertices $x_i^1, \ldots, x_i^{h_i}$ and joining x_i^s with x_j^t iff x_i and x_j are adjacent in G. We say that H is obtained from G by *multiplication of vertices*.

Remark. The definition allows $h_i = 0$, in which case H includes no copy of x_i. Thus, every induced subgraph of G can be obtained by multiplication of G by appropriate (0,1)-valued vector. For example $G = G \circ (1, \ldots, 1)$.

2.1.3 Perfect Graph Theorem

We now reach the main part of the lecture, in which we prove two lemmas that will help to prove the perfect graph theorem.

Lemma 2.6 *(Berge[1961]). Let H be obtained from G by multiplication of vertices.*

1. *If G satisfies (P1), then H satisfies (P1).*
2. *If G satisfies (P2), then H satisfies (P2).*

Proof: The proof is by induction on the number of vertices in G. The lemma is true if G has only one vertex. We shall assume that (1) and (2) are true for all graphs with fewer vertices than G. Let $H = G \circ h$. If one of the coordinates of h equals zero, say $h_i = 0$, then H can be obtained from $G - x_i$ by multiplication of vertices.

But, if G satisfies (P1) [respectively (P2)], then $G - x_i$ also satisfies (P1) [respectively (P2)] ($G - x_i$ is an induced subgraph of G, therefore (P1) on G implies (P1) on all induced subgraphs of G). Then by the induction hypothesis H satisfies (P1) [respectively (P2)].

Thus, we may assume that each coordinate $h_i \geq 1$, and since H can be built up from a sequence of single-copy multiplications, it is sufficient to prove the result for $H = G \circ x$. Let x' denote the added "copy" of x.

Assume that G satisfies (P1). Since x and x' are nonadjacent, $\omega(G \circ x) = \omega(G)$. Let G be colored using $\omega(G)$ colors. Color x' the same color as x. This will be a coloring of $G \circ x$ in $\omega(G \circ x)$ colors. Hence, $G \circ x$ satisfies (1).

Next assume that G satisfies (P2). We must show that $\alpha(G \circ x) = k(G \circ x)$. Let $\overline{K}$ be a (minimum) clique cover of G with $|\overline{K}| = k(G) = \alpha(G)$, and let K_x be the clique in $\overline{K}$ containing x. There are two cases.

Case 1: x is contained in a maximum stable set S of G, i.e., $|S| = \alpha(G)$. In this case $S \cup \{x'\}$ is a stable set of $G \circ x$, so

$$\alpha(G \circ x) = \alpha(G) + 1. \tag{2.1}$$

Since $\overline{K} \cup \{\{x'\}\}$ covers $G \circ x$, we have that

$$k(G \circ x) \overset{\overline{K}\cup\{\{x'\}\}\ covers\ G\circ x}{\leq} k(G) + 1 \overset{(P2)}{=} \alpha(G) + 1 \overset{equation\ \ 2.1}{=} \alpha(G \circ x) \overset{always}{\leq} k(G \circ x).$$

Thus, $\alpha(G \circ x) = k(G \circ x)$.

Cases 2: No maximum stable set of G contains x. In this case,

$$\alpha(G \circ x) = \alpha(G). \tag{2.2}$$

(G is an induced subgraph of $G \circ x$, therefore $\alpha(G \circ x) \geq \alpha(G)$. If $\alpha(G \circ x) > \alpha(G)$ it means that we switched one vertex from a maximum stable set in G with x, x', to obtain a maximum stable set in $G \circ x$ with cardinality greater by one. But that means x must be an element in a maximum stable set in G, contradicting case 2 assumption).

Since each clique of $\overline{K}$ intersects a maximum stable set exactly once (Lemma 2.1), this is true in particular for K_x. But x is not a member of any maximum stable set. Therefore, $D = K_x - \{x\}$ intersects each maximum stable set of G exactly once, so by Lemma 2.3

$$\alpha(G_{V-D}) = \alpha(G) - 1. \tag{2.3}$$

This implies that

$$k(G_{V-D}) \overset{induction}{=} \alpha(G_{V-D}) \overset{equation\ 2.3}{=} \alpha(G) - 1 \overset{equation\ 2.2}{=} \alpha(G \circ x) - 1.$$

Taking a clique cover of G_{V-D} of cardinality $\alpha(G \circ x) - 1$ along with the extra clique $D \cup \{x'\}$, we obtain a cover of $G \circ x$ of cardinality $\alpha(G \circ x)$. Therefore we have a clique cover with the lowest possible number of cliques, because always $k \geq \alpha$, so

$$k(G \circ x) = \alpha(G \circ x).$$

■

Lemma 2.7 *(Fulkerson[1971], Lovász[1972]). Let G be an undirected graph each of whose proper induced subgraphs satisfies (P2), and let H be obtained from G by multiplication of vertices. If G satisfies (P3), then H satisfies (P3).*

Proof: Let G satisfy (P3) and choose H to be a graph having the smallest possible number of vertices which can be obtained from G by multiplication of vertices but which fails to satisfy (P3) itself. Thus,

$$\omega(H)\alpha(H) < |X|, \tag{2.4}$$

where X denotes the vertex set of H, yet (P3) does hold for each proper induced subgraph of H.

As in the proof of the preceding lemma, we may assume that each vertex of G was multiplied by at least 1 and that some vertex u was multiplied by $h \geq 2$. Let $U = \{u^1, \dots u^h\}$ be the vertices of H corresponding to u. The vertex u^1 plays a distinguished role in the proof. By the minimality of H, (P3) is satisfied by H_{X-u^1}, which gives

$$\begin{aligned} |X| - 1 = |X - u^1| &\overset{(P3)}{\leq} \omega(H_{X-u^1})\alpha(H_{X-u^1}) \\ &\leq \omega(H)\alpha(H) \\ &\overset{equation\ 2.4}{\leq} |X| - 1 \end{aligned}$$

Thus, equality holds throughout, and we can define

$$p = \omega(H_{X-u^1}) = \omega(H),$$

$$q = \alpha(H_{X-u^1}) = \alpha(H),$$

and

$$pq = |X| - 1. \tag{2.5}$$

Since H_{X-U} is obtained from $G-u$ by multiplication of vertices, and using the assumption that every proper induced subgraph of G satisfies (P2), Lemma 2.6 (Berge) implies that H_{X-U} satisfies (P2). Thus, $k(H_{X-U}) = \alpha(H_{X-U}) \leq \alpha(H_{X-u^1}) = q$. Therefore, H_{X-U} can be covered by a set of q complete subgraphs of H, say $K_1, \ldots, K_q$ (this is not necessarily a minimum cover). Without lost of generality we may assume that these q cliques are pairwise disjoint and that $|K_1| \geq |K_2| \geq \ldots \geq |K_q|$. Obviously,

$$\sum_{i=1}^{q} |K_i| = |X - U| = |X| - h \overset{equation\ \ 2.5}{=} pq - (h-1)$$

Since $|K_i| \leq p$, at most $h-1$ of the cliques fail to contribute p to the sum. Hence,

$$|K_1| = |K_2| = \ldots = |K_{q-h+1}| = p.$$

Let H' be the subgraph of H induced by $X' = K_1 \cup \ldots \cup K_{q-h+1} \cup \{u^1\}$. Thus

$$|X'| = p(q-h+1) + 1 \overset{h \geq 2}{<} pq + 1 \overset{equation\ \ 2.5}{=} |X|. \tag{2.6}$$

Therefore H' is a proper induced subgraph of H. So by the minimality of H, H' satisfies (P3):

$$\omega(H')\alpha(H') \geq |X'|. \tag{2.7}$$

But $p = \omega(H) \geq \omega(H')$, so

$$\alpha(H') \overset{equation\ \ 2.7}{\geq} |X'|/p \overset{equation\ \ 2.6}{>} q - h + 1$$

Let S' be a stable set of H' of cardinality $q - h + 2$. Certainly $u^1 \in S'$, for otherwise S' would contain two vertices of a clique (by the definition of H'). Therefore, $S = S' \cup U = S' + (U - \{u'\})$ is a stable set of H with $(q-h+2) + (h-1) = q+1$ vertices, contradicting the definition of q. ∎

Theorem 2.8 *The Perfect Graph Theorem (Lovász [1972]). For an undirected graph $G = (V, E)$, the following statements are equivalent:*

- *(P1)* $\omega(G_A) = \chi(G_A), \ \forall A \subseteq V$
- *(P2)* $\alpha(G_A) = k(G_A), \ \forall A \subseteq V$
- *(P3)* $\omega(G_A)\alpha(G_A) \geq |A|, \ \forall A \subseteq V$

Proof: The three statements hold for a one vertex graph. We may assume that the theorem is true for all graphs with fewer vertices than G.

(P1) $\Rightarrow$ (P3). Suppose we can color G_A in $\omega(G_A)$ colors. Every optimal coloring (a partition to stable sets) $C_1, \ldots, C_m$ satisfies:

$$|A| = \sum_{i=1}^{m} |C_i| \leq m \max_i |C_i| \overset{at\ most\ \alpha(G_A)\ vertices\ of\ a\ given\ color}{\leq} \chi(G_A)\alpha(G_A) \overset{(P1)}{=} \omega(G_A)\alpha(G_A)$$

Therefore,

$$|A| \leq \omega(G_A)\alpha(G_A)$$

(P3) $\Rightarrow$ (P1). Let $G = (V, E)$ satisfy (P3). Then each proper induced subgraph of G satisfies (P3), and by induction satisfies (P1) (and also (P2)). It is sufficient to show that $\omega(G) = \chi(G)$.

If we had a stable set S of G such that $\omega(G_{V-S}) < \omega(G)$, we could then paint S in one color and paint G_{V-S} in $\omega(G) - 1$ other colors, and we would have $\omega(G) = \chi(G)$.

Now let's assume that such S does not exist. It means that for every stable set S of G G_{V-S} has a clique $K(S)$ of size $\omega(G)$. Let $\mathcal{I}$ be the collection of all stable sets of G. Keep in mind that $\forall S \in \mathcal{I} \;\; \exists\, \omega-clique\ K(S)$ in G_{V-S} and in particular $S \cap K(S) = \emptyset$.

Construct a (0,1)-matrix M with rows corresponding to vertices, and a column i corresponding to each independent set $S_1, \ldots, S_{|\mathcal{I}|}$. $M_{ij} = 1$ iff vertex i belongs to $K(S_j)$. The matrix is illustrated in figure 2.1.

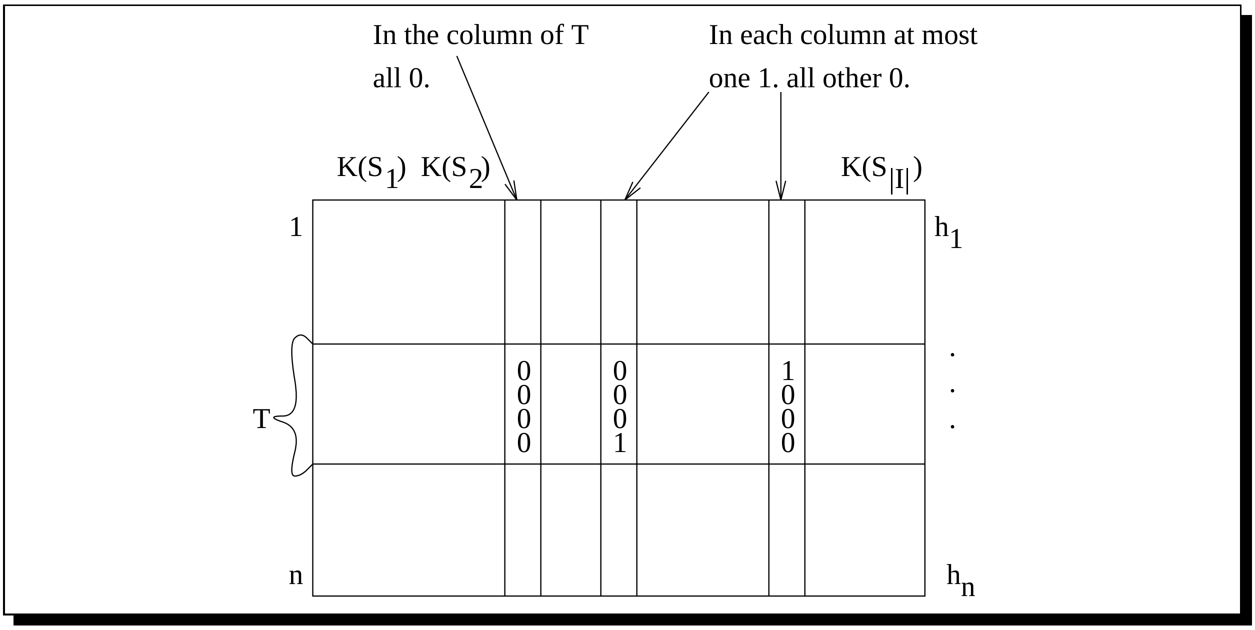

Figure 2.1: The matrix M: each row correspond to a vertex in V, and each column to an $\omega-clique\ K(S)$

For each $x_i \in V$, let h_i denote the number of cliques $K(S)$ which contain x_i(=the number of ones in the i'th row). Let $H = (X, F)$ be obtained from G by multiplying each x_i by h_i [if $h = (h_1, \ldots, h_n)$, then $H = G \circ h$]. On the one hand, by Lemma 2.7,

$$\omega(H)\alpha(H) \geq |X|. \tag{2.8}$$

On the other hand,

$$|X| = \sum_{x_i \in V} h_i \overset{\substack{\text{summing the matrix M} \\ \text{over rows and columns.}}}{=} \sum_{S \in \mathcal{I}} |K(S)| \overset{\substack{\text{every clique in } \mathcal{I} \\ \text{is of size } \omega.}}{=} \omega(G)|\mathcal{I}|. \tag{2.9}$$

Since every clique in $G \circ x$ contains at most one "copy" of vertex $x \in G$, we have $\omega(G \circ x) \leq \omega(G)$. Therefore,

$$\omega(H) \leq \omega(G). \tag{2.10}$$

If a maximum stable set contain some "copy" of x_i then it will contain *all* its "copies" (i.e. all h_i "copies"). To find $\alpha(H)$, we sum up the h_i's of the vertices which induce a stable set T (in G) , and do so to all the stable sets of G which are collected in $\mathcal{I}$, searching the one with a maximum sum (note that all maximum stable sets in H are obtained in this way). Hence,

$$\alpha(H) = \max_{T \in \mathcal{I}} \sum_{x_i \in T} h_i$$

(Note that the maximum stable set in H does not necessarily correspond to the maximum stable set in G. The multiplicities may play a role too).

For each T, the sum above can be computed in two ways (figure 2.1):

1. Move row by row, but only for rows of vertices in T, and sum up the ones in each row$(= h_i)$.
2. Move column by column, and sum up the the ones in each column which correspond to vertices in T.

Hence,

$$\max_{T \in \mathcal{I}} \sum_{x_i \in T} h_i = \max_{T \in \mathcal{I}} [\sum_{S \in \mathcal{I}} |T \cap K(S)|].$$

T is a stable set and $K(S)$ is a clique, so

$$T \cap K(S) \leq 1.$$

$K(T)$ is a clique in G_{V-T}, so

$$T \cap K(T) = 0.$$

In the second way of summing, at least one term will be zero, and all terms are at most one (compare figure 2.1). So we conclude:

$$\alpha(H) = \ldots = \max_{T \in \mathcal{I}} [\sum_{S \in \mathcal{I}} |T \cap K(S)|] \leq |\mathcal{I}| - 1. \tag{2.11}$$

Let us summarize what we have shown:

$$|X| = \omega(G)|\mathcal{I}| \qquad \textit{equation } 2.9$$

$$\alpha(H) \leq |\mathcal{I}| - 1. \qquad \textit{equation } 2.11$$

$$\omega(H) \leq \omega(G). \qquad \textit{equation } 2.10$$

Then,

$$\alpha(H)\omega(H) \leq \omega(G)[|\mathcal{I}| - 1] = |X|\frac{|\mathcal{I}| - 1}{|\mathcal{I}|} < |X|$$

contradicting equation 2.8.

(P2) $\Leftrightarrow$ (P3). G satisfies (P2) $\stackrel{duality}{\Leftrightarrow}$ $\overline{G}$ satisfies (P1) $\stackrel{proved\ above}{\Leftrightarrow}$ $\overline{G}$ satisfies (P3) $\stackrel{duality}{\Leftrightarrow}$ G satisfies (P3). ∎

Corollary 2.9 *A graph G is perfect iff its complement $\overline{G}$ is perfect.*

Proof: By duality, G satisfies (P2) $\Leftrightarrow$ $\overline{G}$ satisfies (P1) . By the Perfect Graph Theorem each of these conditions is sufficient and necessary for perfection. ∎

Corollary 2.10 *A graph G is perfect iff every graph H obtained from G by multiplication of vertices is perfect.*

Proof: One direction is trivial. The other direction follows immediately from Lemma 2.6 and the Perfect Graph Theorem. ∎

The last two corollaries are useful in checking perfection: we can choose to check the complement, and we can reduce non-adjacent vertices with identical neighborhoods and check on the resulting simpler graph.