Chapter 5

Lecture 5

Advanced Topics in Graph Algorithms

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5.1 Triangulated Graphs Are Perfect

5.1.1 Proving This Property

In some graph families, the minimum graph coloring and the maximum clique problem, can be simplified using the principle of *separation into pieces*, as follows:

Theorem 5.1 (Berge, 1973) *Let G be a connected, undirected graph, let S be a vertex separator in G, and let $G_{A_1}, G_{A_2}, ..., G_{A_t}$ be the connected components of G_{V-S}. If S is a clique (not necessarily maximal), then*

$$\chi(G) = max_i \chi(G_{S+A_i})$$

$$w(G) = max_i \; w(G_{S+A_i})$$

Proof: Clearly $\chi(G) \geq \chi(G_{S+A_i})$ for each i, (because G_{S+A_i} is an induced subgraph of G) , so
$\chi(G) \geq k = max_i \; \chi(G_{S+A_i})$ On the other hand, G can be colored using exactly k colors, as follows:

- First, color G_S, (S is a clique, so there is only one way to do it).

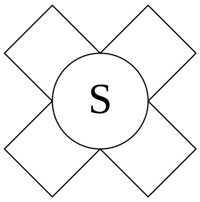

Figure 5.1: The Graph

- Then, independently extend the coloring to each piece G_{S+A_i}. Each piece can be colored with at most k colors, and S is always colored using $|S|$ colors , so it is possible to extend the coloring on S to G_{S+A_i} (renaming the colors if necessary).
- There will be no contradictions because if $v \in A_i$, $u \in A_j$, $i \neq j$ have the same color they are disconnected (because S separates them).

Next, consider the case of clique. Again $w(G) \geq w(G_{S+A_i})$ for each i, so
$w(G) \geq max_i \; w(G_{S+A_i}) = m$.
Let X be a maximum clique of G, i.e. $|X| = w(G)$. It is impossible that two vertices of X lie in G_{A_i} and G_{A_j}, $i \neq j$, since they are not connected . Thus, X is wholly contained in one of the pieces, say G_{S+A_r} . Hence $m \geq w(G_{S+A_r}) \geq |X| = w(G)$.
Therefore, $w(G) = m$. ∎

Corollary 5.2 *Let S be a separating set in a connected graph $G = (V, E)$, and let $G_{A_1}, \ldots, G_{A_t}$ be the connected components of G_{V-S}. If S is a clique, and if each subgraph G_{S+A_i} is perfect, then G is perfect.*

Proof: The proof is by induction on n, the number of vertices in the graph.

Claim 5.3 *It is sufficient to prove $\chi(G) = w(G)$*

Proof: It is trivial if $n = 1$. Suppose $|V(G)| = n$ and the claim is true for all graphs with less then n vertices. Suppose S is a separating set, and a clique in G, and each G_{S+A_i} is perfect. Pick $v \in V$, and let $G' = G - \{v\}$. If $v \in S$, define $S' = S - \{v\}$. In G' S' is also a clique , and also each $G_{S'+A_i}$ is perfect (because this is an induced subgraph of a perfect graph). If $v \in A_i$, S is still a clique , and $G_{S+A_i-\{v\}}$ is perfect. In both cases, from the induction hypothesis , G' is perfect . This implies that for every proper induced subgraph H , $\chi(H) = w(H)$. ∎

Now prove that $\chi(G) = w(G)$.

$\chi(G) = max_i \; \chi(G_{S+A_i})$ (using theorem 5.1).
$max_i \; \chi(G_{S+A_i}) = max_i \; w(G_{S+A_i})$ (because every G_{S+A_i} is perfect),
$w(G) = max_i \; w(G_{S+A_i})$ (using theorem 5.1 again),

Thus, $\chi(G) = w(G)$. ∎

Theorem 5.4 *(Berge 1960, Hajnal-Suranyi 1958)*
Every triangulated graph is perfect .

Proof: By induction on the number of vertices in the graph. Let G be a triangulated graph. We shall assume it is connected, otherwise we consider each component individually.

- If G is complete, it is perfect.
- Otherwise let S be a minimal vertex separator for a pair a, b of nonadjacent vertices. By theorem 3.3, S is a clique. Each of the subgraphs G_{S+A_i} (defined in theorem 5.1) is triangulated (a hereditary property) and thus by induction hypothesis using corollary 5.2, G is perfect.

■

5.1.2 Other Results

Definition A *Meyniel graph* is a graph in which there are at least two chords in every odd cycle of length ≥ 5.

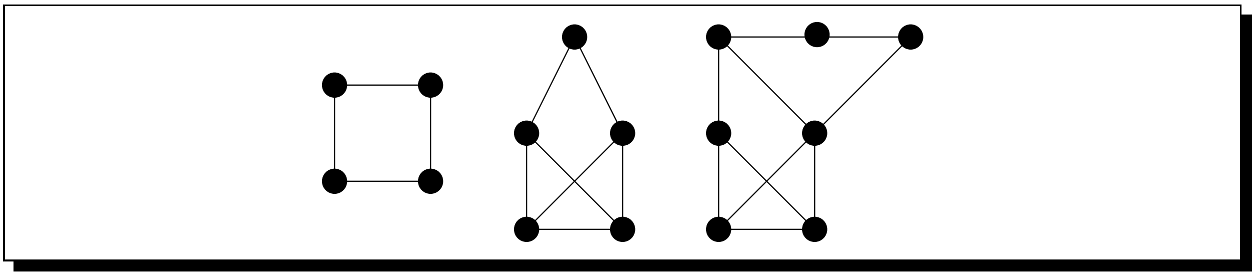

Figure 5.2: Some examples of Meyniel graphs (which are not chordal!)

Theorem 5.5 *(Meyniel 1976) Every Meyniel graph is perfect.*
(The proof can be found in Golumbic's book section 6.4).

Definition G is *weakly chordal graph* if G and $\bar{G}$ do not contain an induced C_k , $k \geq 5$.

Theorem 5.6 *(Hayward 1984) Weakly Chordal graphs are perfect.*

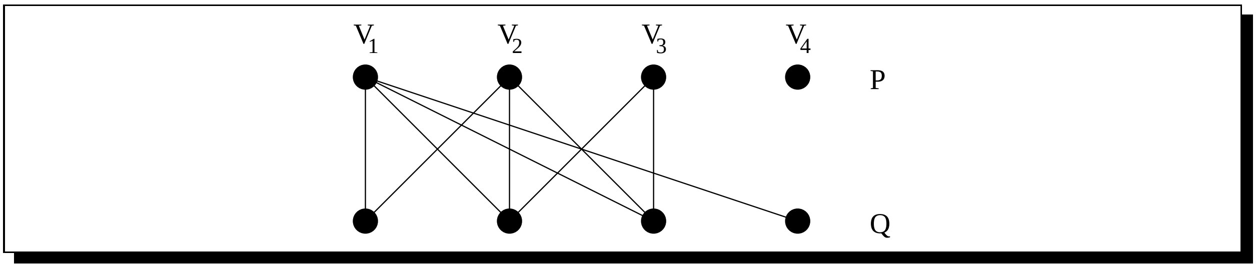

Figure 5.3: Example-A Chain Graph

5.2 Computing the Minimum Fill In

5.2.1 The Problem

The Minimum fill-in problem is defined as follows: Given a graph, find the minimum number of edges (fill in) whose addition makes the graph triangulated. i.e. for $G = (V, E)$ find F, such that $|F|$ is minimal and $G' = (V, E + F)$ is triangulated. This problem has to do with the Gaussian elimination of symmetric matrices. (When looking for an elimination with a maximal number of zeros in the matrix in each step - the problem arises in numerical algebra). In the book of Garey and Johnson, the minimum fill in problem (called there Chordal Graph Completion) is mentioned as a major open problem, shortly after the book was published, Yannakakis showed that the problem is NP complete.

5.2.2 Fill In is NP-Hard

Chain Graphs

Definition A bipartite graph $G = (P, Q, E)$ is a *chain graph* if one of the parts, say P, can be ordered v_1,v_2,..,v_k so that $Adj(v_1) \supseteq Adj(v_2) \supseteq ... \supseteq Adj(v_k)$ (the chain property).

This property is hereditary, (if we take out a vertex from P, the chain of containment doesn't change, if we take out a vertex from Q, the containment order does not change).

Claim 5.7 *If a bipartite graph is a chain graph, both parts of the graph have the chain property.*

Proof: We have to find an order for Q. Let $Q_i = \cap_{j \le i} Adj(v_j)$ In this way,
$Q_1 \supseteq Q_2 \supseteq ... \supseteq Q_k$. Let $R_i = Q_i - Q_{i+1}$, $i = 1, ..., k-1$, $R_k = Q_k$. Then $R_1 + R_2 + ... + R_k$ is a partition of Q. Impose a linear order $\prec$ on the vertices in Q such that if $u \in R_i$,$v \in R_j$ and $i < j$ then $v \prec u$. Thus if $v \prec u$, $u, v \in Q$ then either $u, v \in R_i$ and $Adj(u) = Adj(v) = \{v_1, ..., v_i\}$, or $u \in R_i$,$v \in R_j$ and $i < j$, in which case $Adj(v) = \{v_1, ..., v_j\} \supseteq \{v_1, ..., v_i\} = Adj(u)$. ∎

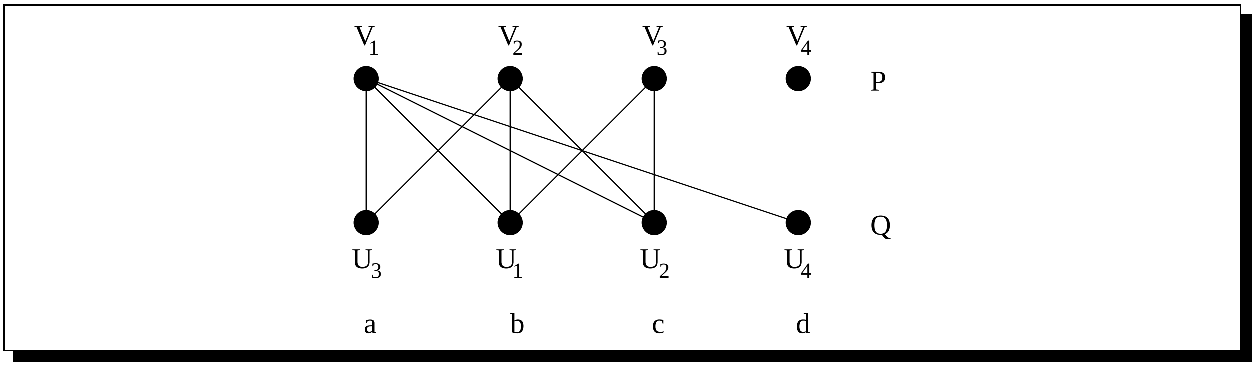

Figure 5.4: Ordering Q, in the same chain graph

For the example in Figure 5.4, we have:

$$Q_1 = \{a, b, c, d\};\, R_1 = \{d\}$$

$$Q_2 = \{a, b, c\};\, R_2 = \{a\}$$

$$Q_3 = \{b, c\};\, R_3 = \{b, c\}$$

$$Q_4 = \phi;\, R_4 = \phi$$

Definition Two edges ij,kl are independent if the graph induction on $\{i, j, k, l\}$ induces $2K_2$. (see figure 5.5)

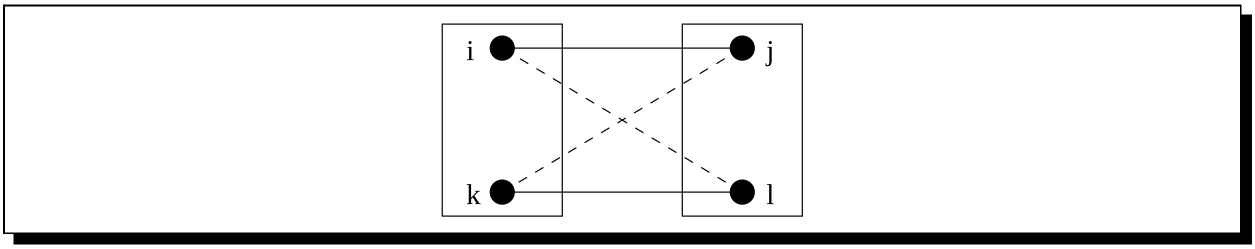

Figure 5.5: Independent edges (broken lines denote non-edges)

Theorem 5.8 *A bipartite graph G is a chain graph iff G does not contain two independent edges.*

Proof: ($\Longrightarrow$) Let ij and kl be independent edges. Then $Adj(i) \not\subseteq Adj(k)$, $Adj(i) \not\supseteq Adj(k)$ and G is not a chain graph.
($\Longleftarrow$) If this isn't a chain graph, there must be two vertices: $i, k \in P$ so that

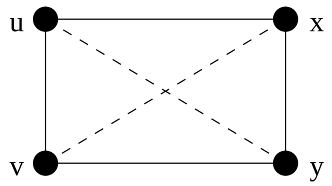

Figure 5.6: Independent edges that form an induced C_4

$Adj(i) \not\subseteq Adj(k)$ and $Adj(i) \not\supseteq Adj(k)$
there exist: $j \in Adj(i) - Adj(k)$ and $l \in Adj(k) - Adj(i)$
and the edges ij and kl are independent. ∎

Definition For a bipartite graph $G = (P, Q, E)$ we define a new graph $C(G) = (N, E')$, where $N = P \cup Q$, $E' = E + \{uv \mid u, v \in P\} + \{uv \mid u, v \in Q\}$. (i.e. we place the independent set on each side of the clique. See figure 5.7).

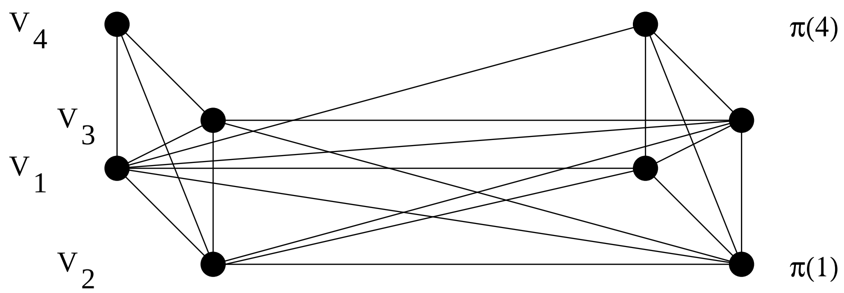

Figure 5.7: $C(G)$ for the graph G of Figure 5.3

Lemma 5.9 *G is a chain graph iff $C(G)$ is triangulated.*

Proof: ($\Longrightarrow$) If G is a chain graph, by lemma G contains two independent edges: ux , vy .

Thus, $\{u, v, y, x\}$ is an induced C_4 in $C(G)$.
($\Longleftarrow$) Let G be a chain graph, let π be an ordering of P , such that

$$Adj(\pi(1)) \supseteq Adj(\pi(2)) \supseteq ... \supseteq Adj(\pi(p))$$

where $p = |P|$. Since the property of being a chain graph is hereditary, it suffices to show that $C(G)$ has a simplicial vertex. The neighborhood $Adj'(\pi(p))$ of $\pi(p)$ in $C(G)$ is

$$Adj'(\pi(p)) = Adj_G(\pi(p)) \ + \ P \ - \ \pi(p)$$

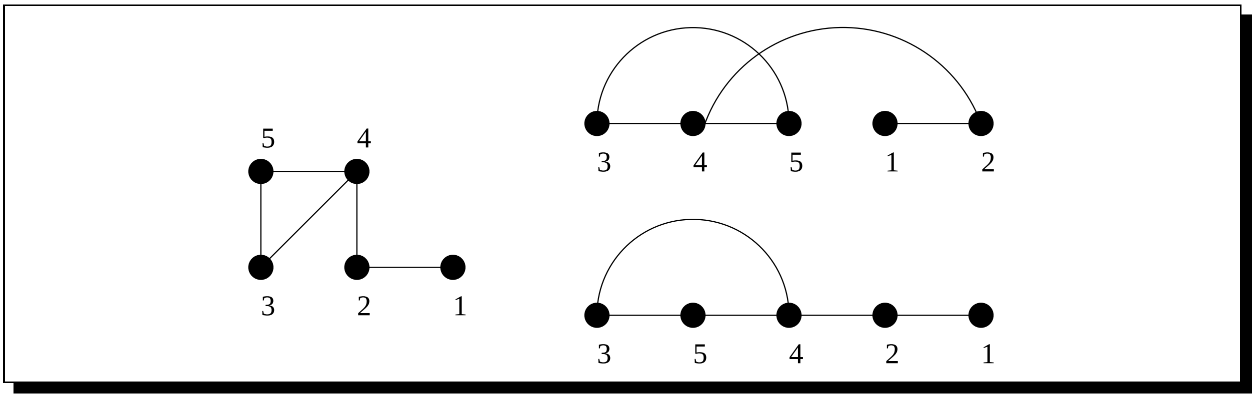

Figure 5.8: a graph (left) with two possible layouts of cost 8 (center) and 6 (right)

$Adj_G(\pi(p)) \subseteq Q$ and Q is a clique, so $Adj_G(\pi(p))$ is a clique in $C(G)$. $P - \pi(p) \subseteq P$ and P is a clique, so $P - \pi(p)$ is a clique in $C(G)$. Also, since $Adj(\pi(p)) \subseteq Adj(v)$ for every $v \in P$, all the nodes of P are adjacent to all nodes of $Adj(\pi(p))$. Therefore $Adj'(\pi(p))$ is a clique in $C(G)$, and $\pi(p)$ is a simplicial vertex of $C(G)$. ■

Optimal Linear Arrangement (OLA)

A linear arrangement (or a layout) of a graph $G = (N, E)$ is a $1 - 1$ mapping $\pi : N \to \{1, 2, .., |N|\}$. For an edge $e = uv$ of G , let $\delta(e, \pi) = |\pi^{-1}(u) - \pi^{-1}(v)|$. The cost $c(\pi)$ of the linear arrangement π is $c(\pi) = \sum_{e \in E} \delta(e, \pi)$. The *optimal linear arrangement problem* is to decide, given a graph G and an integer k, whether there exists a linear arrangement π of G with cost $c(\pi) \leq k$. This problem is NP-complete.

Chain Graph Completion (CGC)

Given a bipartite graph $G = (P, Q, E)$ and a constant k, is there a set of non-edges F such that $|F| \leq k$ and $(P, Q, E + F)$ is a chain graph?

Theorem 5.10 *(Yannakakis , 1981)*
Chain graph completion is NP-complete.

Proof: Reduction from OLA: Given an instance of OLA: a graph $G = (N, E)$ and an integer k , we build an instance of CGC: a bipartite graph $G' = (P, Q, E')$ as follows: $P = N$, and Q contains two vertices e_1 and e_2 for every edge e of G, and a set $R(v)$ of $n - d(v)$ vertices for every vertex v of N, where $n = |N|$ and $d(v)$ is the degree of v in G. If $e = (u, v)$ is an edge of G, then the nodes e_1, e_2 that correspond to e are adjacent to u and v. The nodes in $R(v)$ are adjacent to v. In Fig 5.9 we see an example of this construction.

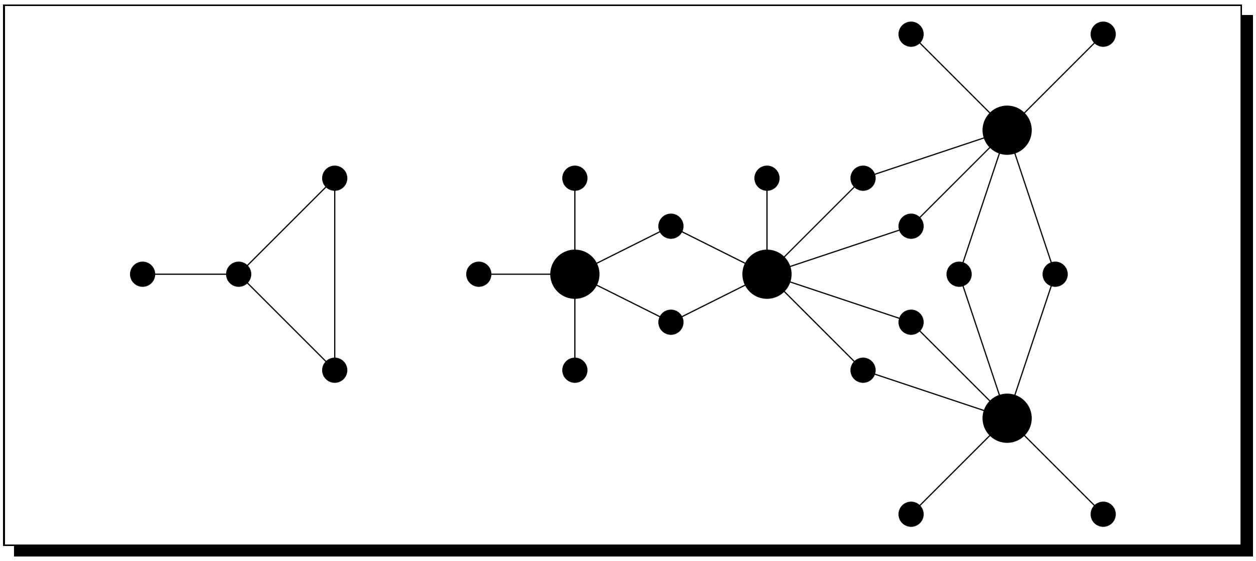

Figure 5.9: the graph G (left) and G' (right)

Let $A = [\frac{n^2(n-1)}{2}] - 2m$, where m is the number of edges in G. The constant for CGC will be $k' = k + A$. Let $l(G)$ be the minimum cost of a linear arrangement of G, and $h(G')$ the minimum number of edges whose addition to G' gives a chain graph. We shall prove that $h(G') = l(G) + A$ Thus $l(G) \leq k \iff h(G') \leq k + A$. which will prove the correctness of the reduction. First observe that an ordering π of N specifies uniquely a minimal set $H(\pi)$ of edges whose addition makes G' into a chain graph with the neighborhoods of the nodes in P ordered according to π. For every node $x \in Q$, let $\sigma(x) = max\{i \mid (x, \pi(i)) \in E'\}$. Then

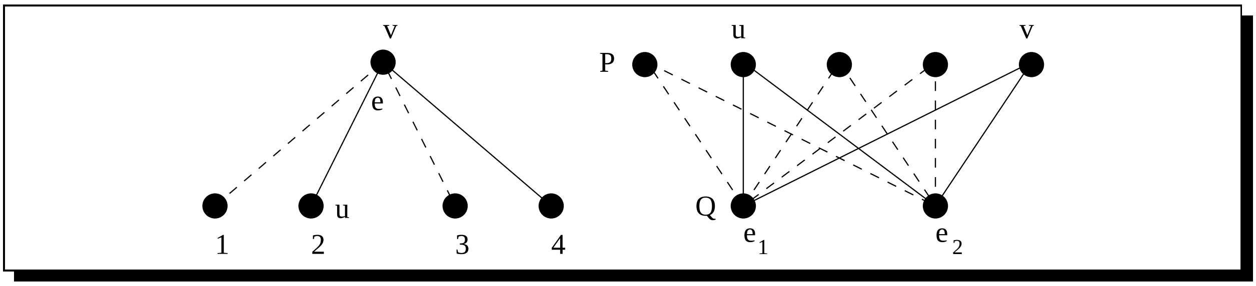

Figure 5.10: Example-Edges we have to add

$H(\pi) = \{(x, \pi(j)) \mid x \in Q, j < \pi^{-1}(x)\} - E'$.
Conversely, suppose that F is a set of edges such that $G'(F) = (P, Q, E' \cup F)$ is a chain graph and let π be an ordering of the nodes in P according to their neighborhoods in $G(F')$. It is easy to see that $F \supseteq H(\pi)$, and therefore if F is a minimal augmentation then $F = H(\pi)$. Let $h(\pi) = |H(\pi)|$. In order to complete the proof, it suffices to show that for every ordering π of N, $h(\pi) = c(\pi) + [\frac{n^2(n-1)}{2}] - 2m$,where $c(\pi)$ is the cost of the linear arrangement π of G.

Let π be an ordering of N. For every $v \in N$ and $x \in R(v)$, (see Figure 5.11) $H(\pi)$ contains $\pi^{-1}(v) - 1$ edges incident to x. Let $e = uv$ be an edge of G, and suppose without loss of generality that $\pi^{-1}(u) < \pi^{-1}(v)$. The number of edges of $H(\pi)$ incident to each of the two nodes e_1,e_2 that correspond to e is

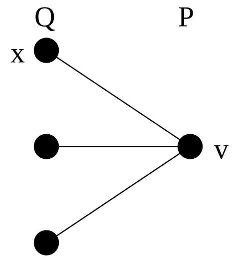

Figure 5.11: x and v

$$\pi^{-1}(v) - 2 = \pi^{-1}(u) + [\pi^{-1}(v) - \pi^{-1}(u)] - 2$$
$$= \pi^{-1}(u) + \delta(e,\pi) - 2;$$

thus the number of edges of $H(\pi)$ incident to e_1 and e_2 is
$2(\pi^{-1}(v) - 2) = \pi^{-1}(v) - 2 + \pi^{-1}(u) + \delta(e,\pi) - 2$
$= \pi^{-1}(u) + \pi^{-1}(v) + \delta(e,\pi) - 4$. Consequently:

$$h(\pi) = \sum_{v \in N} \sum_{x \in R(v)} [\pi^{-1}(v) - 1] \tag{5.1}$$

$$+ \sum_{e=uv \in E} [\pi^{-1}(v) + \pi^{-1}(u) + \delta(e,\pi) - 4] \tag{5.2}$$

$$= \sum_{v \in N} (n - d(v))(\pi^{-1}(v) - 1) \tag{5.3}$$

$$+ \sum_{v \in N} d(v)(\pi^{-1}(v)) + \sum_{e \in E} \delta(e,\pi) - 4m \tag{5.4}$$

$$= \sum_{v \in N} n[\pi^{-1}(v) - 1] + \sum_{v \in N} d(v) + c(\pi) - 4m \tag{5.5}$$

$$= c(\pi) + [\frac{n^2(n-1)}{2}] - 2m \tag{5.6}$$

the last equality follows

$$\sum_{v \in N} d(v) = 2m, \tag{5.7}$$

and

$$\sum_{v \in N} [\pi^{-1}(v) - 1] = 0 + 1 + 2 + ... + (n-1) = [\frac{n(n-1)}{2}] \tag{5.8}$$

■

The result for Fill in

Theorem 5.11 *Minimum Fill in is NP-hard*

Proof: Reduction from CGC: Given $G = (P, Q, E)$ and a constant k we build the graph $C(G)$, and give the same k (as an instance of the minimum fill in problem). By Lemma 5.8 Adding edges to $C(G)$ creates a triangulated graph iff adding the same edges to G creates a chain graph. ■

5.2.3 Problems

- **Maximum Chordal subgraph:** Remove minimum number of edges from a graph, until you get a triangulated graph. This problem is still open. There exists an algorithm for finding a maximal fill in (not maximum), it's time complexity is $O(mn)$ where $m = |E|$, $n = |V|$ (Ohtsuki, 1976).
- **Approximating the fill in**: Find in polynomial time a fill-in set (i.e. a set F' such that $(V, E+F')$ is chordal) such that if F is an optimal fill in, $[\frac{|E|+|F'|}{|E|+|F|}] \leq f(n)$. The best f(n) known today is $O(\sqrt{k} log^4 n)$ for graph with max degree k (Klein et al., 1990). Can one get better approximation, or prove lower bounds on the achievable approximation, assuming $P \neq NP$?
- **parametric fill-in problem:** How fast can we solve the fill in problem for fixed k? Simple enumeration of all possible sets of size $\leq$ takes $O(n^{2k}m)$. Two better algorithms are known (Kaplan,Shamir,Tarjan 94) of complexity $O(c_k m)$ and $O(k^5 mn + C_k)$. Both C_k and c_k are, of course, exponential in k.