Chapter 7

Lecture 7

Advanced Topics in Graph Algorithms
©Tel-Aviv University
Spring Semester, 1994

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Lecture 7 : June 2

7.1 Uniquely Partially Orderable Graphs

Definition Given a graph H_0 with n vertices labeled $\{v_1, \ldots, v_n\}$ and n disjoint graphs $H_1, \ldots, H_n$, we denote by $H = H_0\Big[H_1, H_2, \ldots, H_n\Big]$ the *composition* graph, resulting from replacing each vertex in H_0, $v_i, 1 \leq i \leq n$ by the graph H_i, and replacing all edges $ij \in E(H_0)$ by the edges connecting all the vertices of H_i with the vertices of H_j. That is $V(H) = \cup_{i=1}^{n} V_i$ and $E(H) = \cup_{i=1}^{n} E_i \cup \{(v,u) | v \in V_i, u \in V_j \ \ and \ \ (i,j) \in E_0\}$.

The graph H_0 is called the *outer factor* of H, and the graphs $H_1, \ldots, H_n$ are called the *inner factors* of H.

Example 7.1 *See figure 7.1 for an undirected example of a composition graph. See figure 7.2 for a directed example of a composition graph.*

Theorem 7.2 *Given $n+1$ undirected graphs $G_0, \ldots, G_n$, let $G = G_0\Big[G_1, \ldots, G_n\Big]$ denote their composition graph. Then, G is a comparability graph if and only if, all the graphs $G_i, 0 \leq i \leq n$ are comparability graphs.*

Proof: ($\Longleftarrow$) Clearly, if one of the $G_i, i = 0, \ldots, n$ is not a comparability graph, then G can not be a comparability graph.

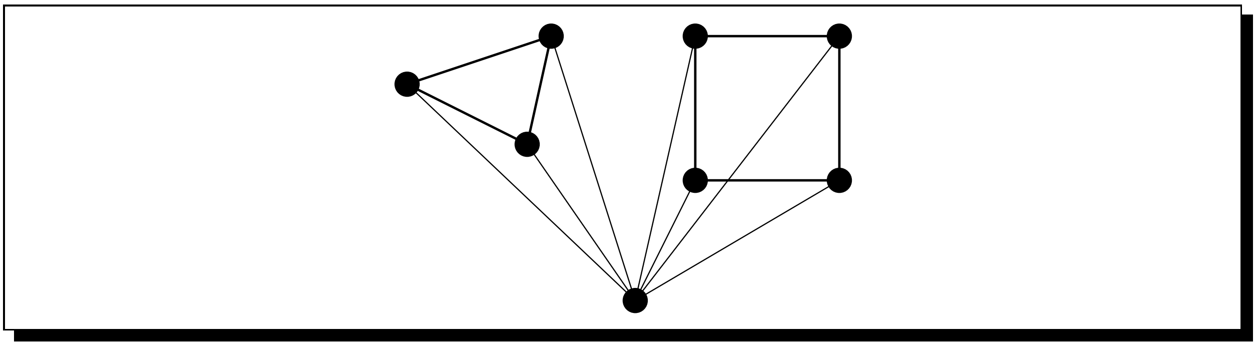

Figure 7.1: The composition graph of $P_3[K_3, K_1, C_4]$

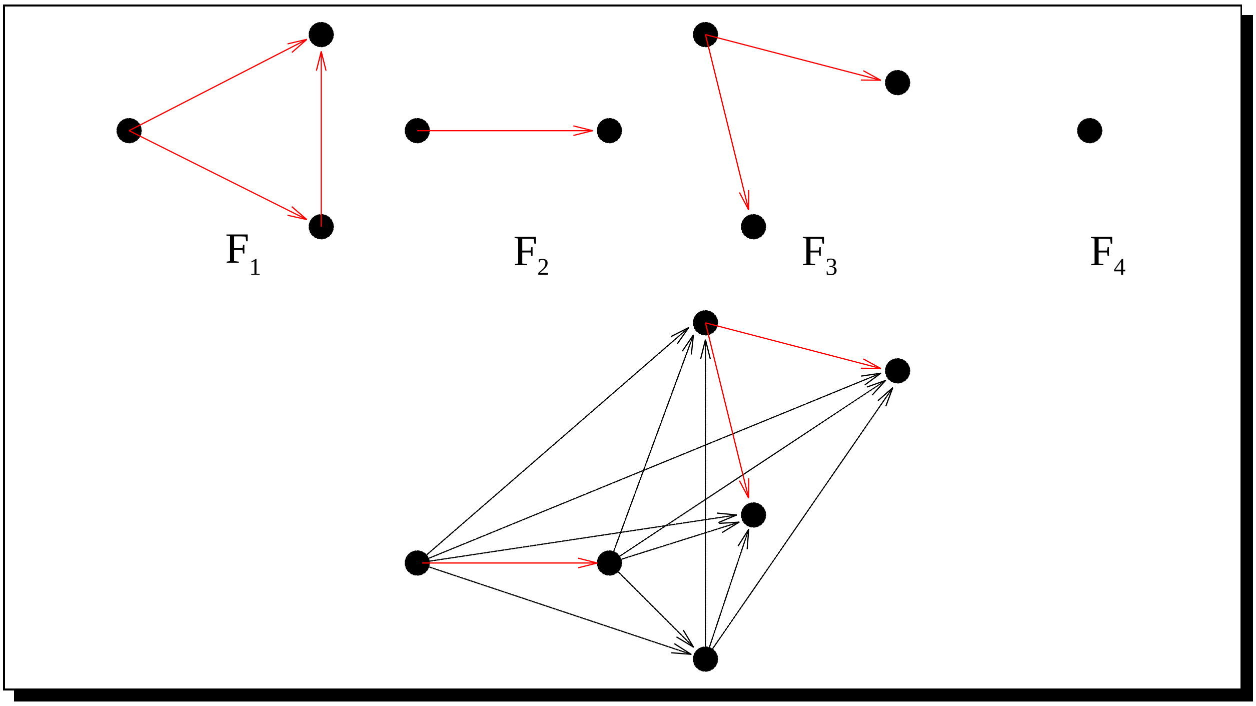

Figure 7.2: The composition graph of $F_1[F_2, F_3, F_4]$

($\Longrightarrow$) Let $F_0, \ldots, F_n$ be transitive orientations of the graphs $G_0, \ldots, G_n$, let $F = F_0\big[F_1, \ldots, F_n\big]$ be an orientation of G. Given two edges $a \to b, b \to c \in F$. We have to show that $a \to c \in F$. One of the following hold:

- The vertices $a, b, c \in V_i, 1 \leq i \leq n$. Since F_i is a transitive orientation it follows that $a \to c \in F$.
- The vertices $a, b \in V_i, c \in V_j, i \neq j$. Then, it follows that $i \to j \in F_0$, but then $a \to c \in F$.
- The vertices $a \in V_i$ $b, c \in V_j, i \neq j$, then $i \to j \in F_0$, thus $a \to c \in F$.
- The vertices $a \in V_i, b \in V_j, c \in V_k$, then $i \to j, j \to k \in F_0$, since F_0 is transitive it follows $i \to k \in F_0$, thus $a \to c \in F$.

Thus F is a transitive orientation of G. ∎

Definition A graph $G = (V, E), V = \{v_1, \ldots, v_n\}$ is *decomposable* if it can be presented as a non-trivial composition of induced subgraphs of G. The trivial decompositions of G are $G = K_1\big[G\big]$, $G = G\big[\{v_1\}, \ldots, \{v_n\}\big]$. A graph which has only trivial decompositions is called *prime*.

If there exists a partition of the vertex set of G: $V = V_1 + V_2 + \cdots + V_r$, $1 < r < |V|$, such that $G = G_R\big[G_{V_1}, \ldots, G_{V_r}\big]$, where R is a set of r representatives from $V_1, \ldots, V_r$, then we say the the partition *induces a proper decomposition* of G.

Example 7.3 *See figure 7.3 for an example of various decompositions of a graph.*

Theorem 7.4 *Let $G_0, \ldots, G_n$ be n disjoint, undirected graphs, and let $G = G_0[G_1, \ldots, G_n]$ be their composition. Let $\widehat{A}$ be a color class of G, then exactly one of the following holds:*

- *$\widehat{A} \subseteq E_j$ for a single value of $1 \leq j \leq n$.*
- *$\widehat{A} \cap E_j = \emptyset$ for all $1 \leq j \leq n$.*

Proof: Any color-class in G is, by definition, a connected subgraph of G. Assume that $\widehat{A} \cap E_j \neq \emptyset$ for a vertex $1 \leq j \leq n$. Look at an edge $ab \in \widehat{A} \cap E_j$, and let $cd \in E(G)$ be an edge, such that $cd \Gamma ab$. There are three possibilities:

1. The edge $cd \in E_k, k \neq j$. This is not possible, because internal edges of distinct inner factors do not share a vertex.

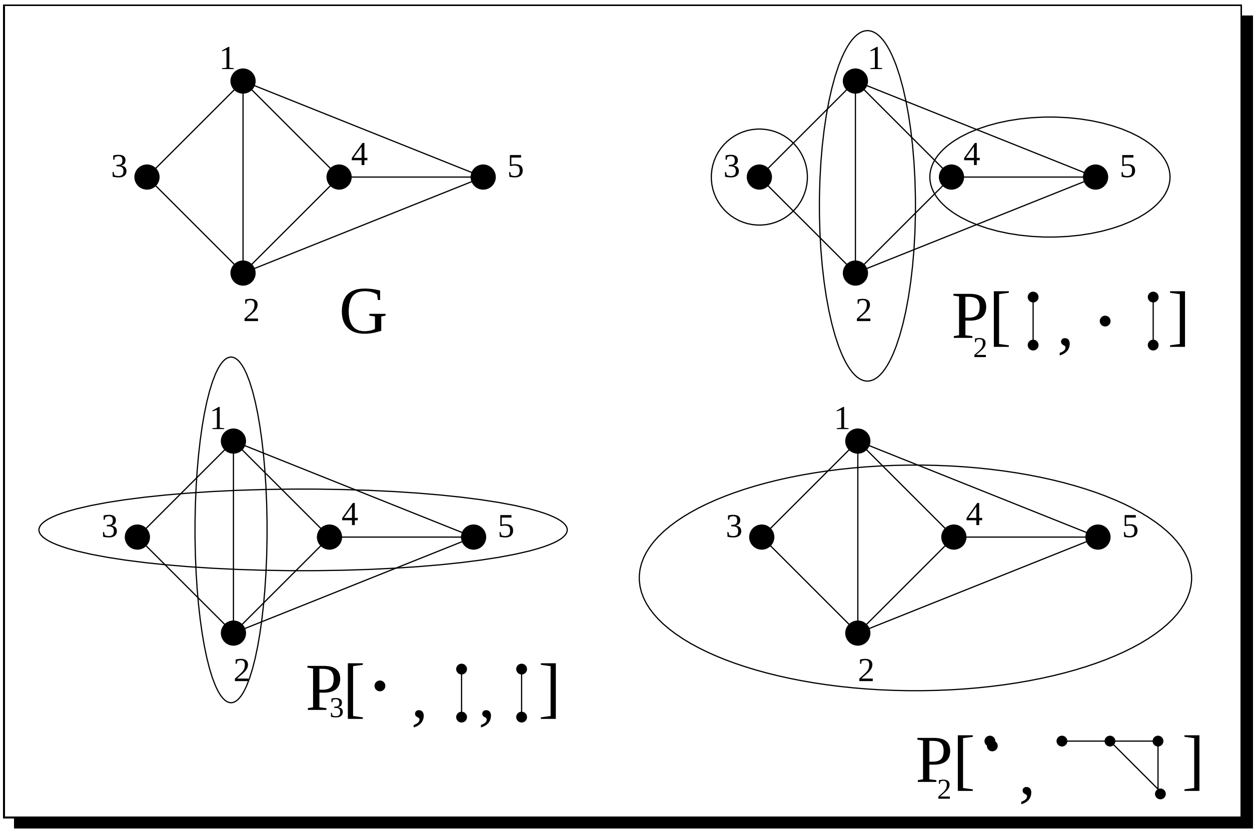

Figure 7.3: The decomposition of a graph

2. The edge cd is an external edge. This is not possible, because then the set $C = \{a, b\} \cup \{c, d\}$ having *three* vertices, induce a triangle in G, contradicting the fact that $cd\Gamma ab$.

Thus the only remaining possibility is that $cd \in E_j$, and by the connectivity of $\widehat{A}$, it follows that $\widehat{A} \subseteq E_j$. ∎

Definition Let $G = (V, E)$ be an undirected graph. A set $Y \subseteq V$ is called a *module* or a *partitive set* if for all $x \in V \setminus Y$ either $Y \cap Adj(x) = \emptyset$ or $Y \subseteq Adj(x)$. The set Y is a *non-trivial* module if $1 < |Y| < |V|$.

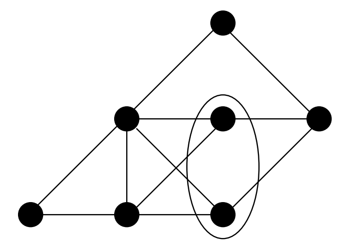

Figure 7.4: Module in a graph

Example 7.5 *See figure 7.4 for an example of a module in a graph.*

Lemma 7.6 *The graph G possess a non-trivial module $Y \subset V$, if and only if G is decomposable.*

Proof: ($\Longleftarrow$) Let $G = G_0[G_1, \ldots, G_n]$ is a decomposition of G, with $G_i = (V_i, E_i)$ a non-trivial inner factor. If $x \in G_j, j \neq i$ then either $V_i \subseteq adj(x)$ (if $ij \in E_0$) or $v_i \cap Adj(x) = \emptyset$ (if $ij \notin E_0$). Hence G_i is a module.

($\Longrightarrow$) If Y is a non-trivial module of G, then the division -

$$V = \{v_1\} + \{v_2\} + \cdots + \{v_k\} + Y$$

induce a proper decomposition of G. ∎

Corollary 7.7 *A set of vertices $Y \subset V$ in the graph $G = (V, E)$ with $1 < |Y| < |V|$ is a module, if and only if there exists a proper decomposition of $G = G_0[G_1, \ldots, G_n]$, such that, there is an inner factor of G, $G_i, 1 \leq i \leq n$, such that $G_i = G_Y$.*

Claim 7.8 *If Y is a set of vertices spanned by a color-class $\widehat{A}$ in the undirected graph $G = (V, E)$, then Y is a module.*

Proof: If $Y = V$ or $|Y| = 1$, then the set Y is a trivial module of G.

Else, let a be any vertex in $V \setminus Y$, and assume that $b \in Y \cap Adj(a)$, then $ab \in E \setminus \hat{A}$. Since Y is defined by edges there exists a vertex $c \in Y$ such that $bc \in \hat{A}$. The edge $ac \in E$ exists (else, $ba \in \hat{A}$). The edge $ac \notin \hat{A}$ (else, $a \in Y$).

Given any vertex $d \in Y$, let $de \in \hat{A}$ be an edge connected to d. Now, using the triangle lemma, it follows that $ad \in E$, thus $Y \subseteq Adj(a)$. ■

Claim 7.9 *An undirected graph $G = (V, E)$ has at most one color-class which spans all of V.*

Proof: Assume that $\hat{A}, \hat{B}$ are two distinct color-classes that both span V. Given any vertex $b \in V$, there are edges $ab \in \hat{B}, bc \in \hat{A}$. Since $\hat{A} \neq \hat{B}$ the edge $ac \in E$.

Let $\hat{C}$ be the color-class that contain the edge ac. If $\hat{C} \neq \hat{A}, \hat{C} \neq \hat{B}$, from the triangle lemma (iii), it follows that no edge in $\hat{A}$ is incident to the vertex a. A contradiction.

Thus $\hat{C} = \hat{A} \neq \hat{B}$, and by the same argument no edge from $\hat{B}$ is incident to the vertex c. A contradiction. ■

Definition A comparability graph is *uniquely partially orderable* graph (abbreviated **UPO** graph), if it has only two transitive orientations, where each orientation is the reversal of the other.

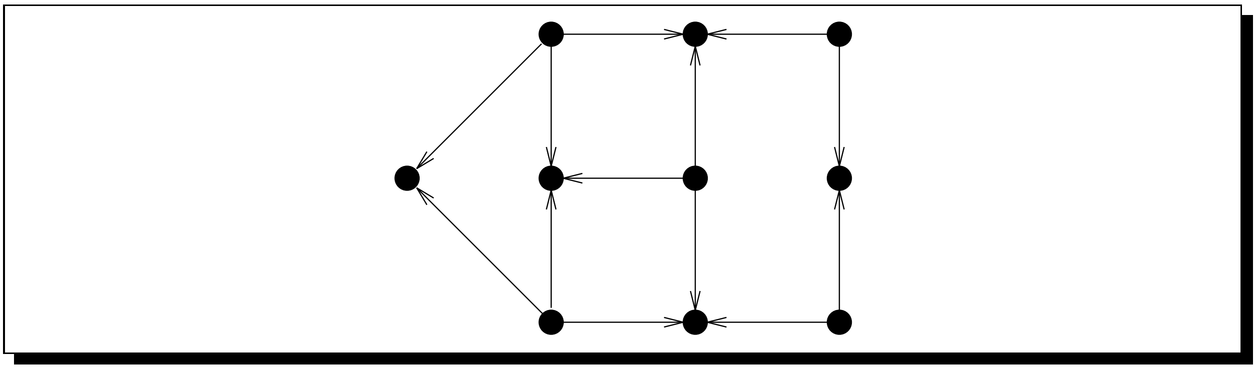

Figure 7.5: Uniquely partially orderable graph

Example 7.10 *See figure 7.5 for an example of a uniquely orderable graph*

Corollary 7.11 *A graph G is* **UPO iff** *G have a single color-class.*

Theorem 7.12 (Shevrin-Filippov '70) *Let G be a connected comparability graph. The following conditions are equivalent:*

1. *The graph G is* **UPO**.

2. *Every non-trivial module in G is an independent set.*

3. *In any proper decomposition of G, every inner-factor is an independent set (Explicitly, all the edges in G are external).*

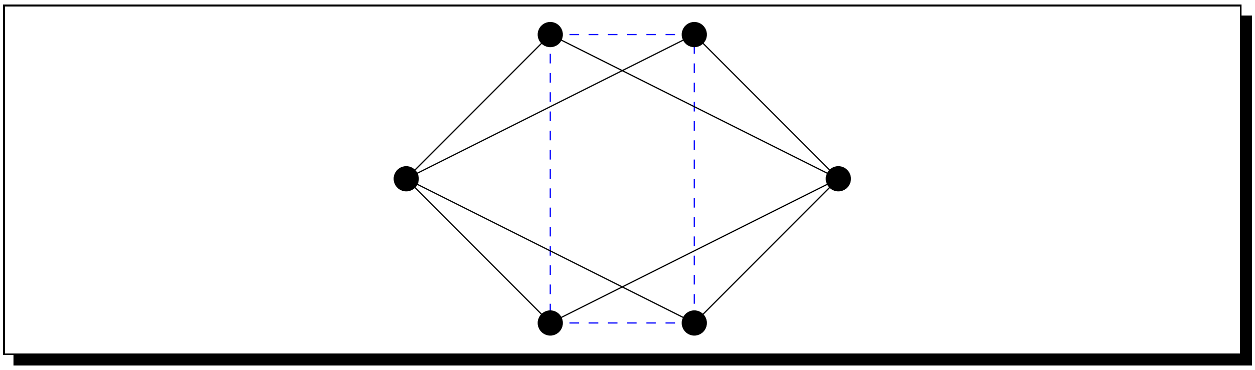

Figure 7.6: A graph with a color class that induce all vertices

Proof: (2) $\iff$ (3) Follow immediately from the natural equivalence between a module and a inner factor in a decomposition, as stated in corollary 7.7.

If there is a non-trivial module Y, which is not stable, there is a decomposition of G, with inner factor G_Y, which is not stable.

If G_Y is an inner factor in a given proper decomposition, then Y is a module, thus if G_Y is not stable, then the set Y is a non-stable module.

(1) $\Longrightarrow$ (3) If G is **UPO**, then in G there is a color-class, $\widehat{A}$, which contains all the edges of G. Thus by theorem 7.4, it follows that all the edges are external in any non-trivial decomposition of G (because no non-trivial inner factor can contain all the edges of $\widehat{A}$, thus such an inner factor, has a n empty intersection with $\widehat{A} = E(G)$).

(2) $\Longrightarrow$ (1) Assume, for the sake of contradiction, that G is not **UPO**. Then G has more than one color-classes (Since G is a comparability graph). From claim 7.9 it follows that there is a color-class which spans a proper subset Y of V. By claim 7.8 it follows that Y is a module of G. But Y is not a stable set (Y is defined as these vertices that are spanned by a color-class, which is connected). A contradiction. ■

Corollary 7.13 *A prime comparability graph G is* **UPO**.

Proof: If G is not decomposable, then it must be connected. Condition (3) in theorem 7.12 holds trivially thus G is **UPO**. ■

7.2 Characterizations and Recognition Algorithms

Definition For the undirected graph $G = (V, E)$, a partition of E into k sets: $E = \widehat{B_1} + \widehat{B_2} + \cdots + \widehat{B_k}$, is called a *G-decomposition* if B_i is an implication class of the graph $G = (V, \widehat{B_i} + \widehat{B_{i+1}} + \cdots + \widehat{B_k})$. See figure 7.7 for an example.

A sequence of edges $[x_1y_1, \ldots, x_ky_k]$ is a *decomposition scheme* for G, if there exists a G-decomposition $E = \widehat{B_1} + \widehat{B_2} + \cdots + \widehat{B_k}$, such that $x_iy_i \in \widehat{B_i}$, $i = 1, \ldots, k$.

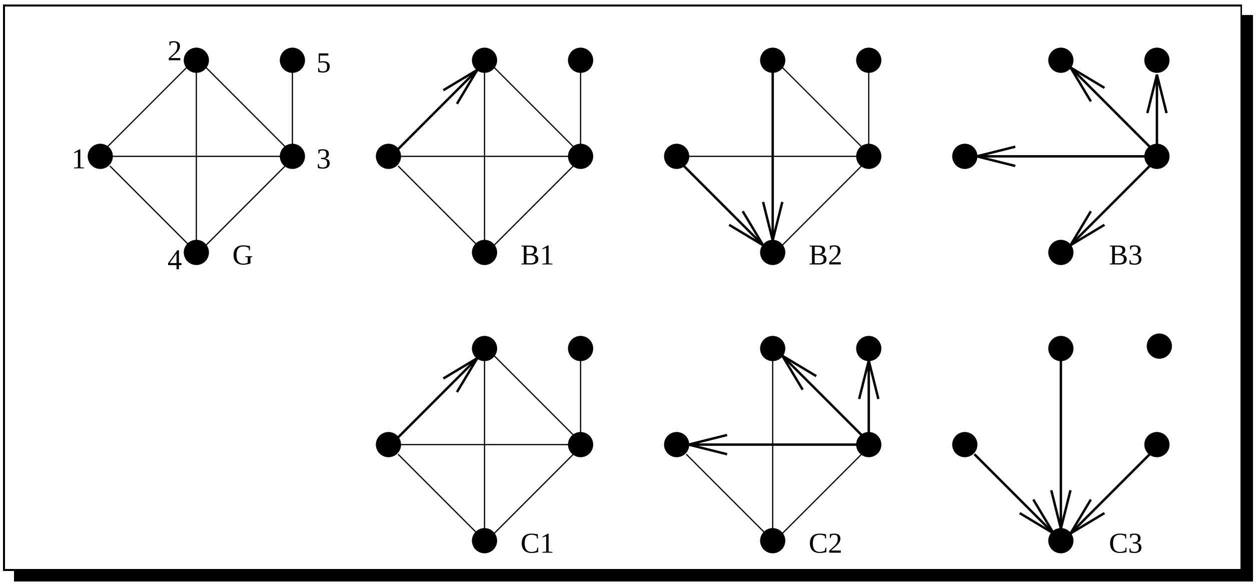

Figure 7.7: A graph G-decomposition

Example 7.14 *The following are examples for the decomposition schemes for figure 7.7:*

- *For the top G-decomposition:* $\left[\, 12, \begin{array}{c} 24 \\ 14 \end{array}, \begin{array}{c} 35 \\ 13 \\ 23 \\ 43 \end{array} \right].$
- *For the bottom G-decomposition:* $\left[\, 12, 13, 43 \,\right]$

Remark 7.15 *A G-decomposition of a graph can have several decomposition-schemes, but the decomposition scheme uniquely defines the G-decomposition. We will present algorithms to calculate both.*

See figure 7.8 for a description of the G-decomposition algorithm.

Algorithm: *construct G-decomposition*

```
    Input: A graph G = (V, E).
    Output: A G-DECOMPOSITION: A NUMBER k
            k SETS OF EDGES OF G: B̂_1, ..., B̂_k.
begin
    Let E_1 ← E, i ← 1.

Loop:
    Arbitrarily pick an edge e_i = x_i y_i ∈ E_i.

    Enumerate the implication class B_i of E_i containing x_i y_i.

    Let E_{i+1} ← E_i \ B̂_i.

    if E_{i+1} = ∅ then
        Let k ← i;
        stop.
    else
        Increase i by 1;
        go to Loop
    end if
end    construct G-decomposition
```

Figure 7.8: Algorithm to construct G-decomposition

Theorem 7.16 (Golumbic 77) *Let A be an implication class of an directed graph $G = (V, E)$. Let D be an implication class in $E \setminus \widehat{A}$. One of the following holds:*

1. *D is an implication class in E, and A is an implication class in $E \setminus \widehat{D}$.*
2. *$D = B + C$, where C, B are implication classes in E, and G contain a triangle with edge taken from each implication class: A, B, C.*

Proof: In the graph remaining after the removal of $\widehat{A}$ several implication classes of G may had been united. Let D be an implication class in $E \setminus \widehat{A}$. Assume that D is a union of several implication classes of E, including B and C.

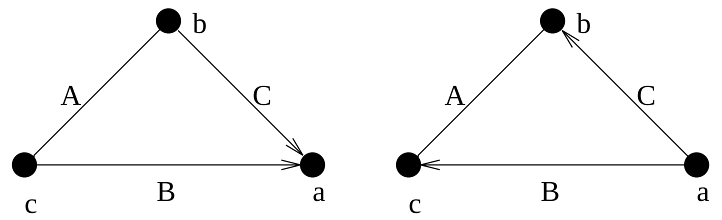

Figure 7.9: A triangle forcing the merge of two implication classes

Since B, C are two distinct implication classes in E, that were united in $E \setminus \widehat{A}$, there exist edges $ab, ac \in E \setminus \widehat{A}$ such that $ab \in C, ac \in B, bc \in \widehat{A}$ or $ba \in C, ca \in B, bc \in \widehat{A}$ (see figure 7.9). Assume without limiting generality that the former holds, since the second case is the inverse of the first case.

If $B = C^{-1}$ then $ba \in B$. It is known that $ac \in B$ but $bc \notin B$. Thus B is not transitive, and using theorem 6. it follows that $C^{-1} = \widehat{B}$, explicitly $B = C$. A contradiction.

Thus $\widehat{B} \cap \widehat{C} = \emptyset$ and thus the triangle $\triangle abc$ have three-colors.

Is it possible that D is a union of more than 2 implication classes? Assume that there is another implication class F, distinct from B, C, in $E \setminus \widehat{A}$ that is contained in D. From the above discussion, it follows that there is a three color triangle with $\widehat{A}, F, B$ as its colors (See figure 7.10). Now, using the triangle lemma it follows that triangles containing edges from both $\widehat{A}, \widehat{B}$, have their third edge from the same color-class. Thus $D = B + C$.

For the other case, suppose D is a single implication class in E. If A is not an implication class in $E \setminus \widehat{D}$, then there is a triangle with A and D as two of its edges. This triangle exist because where removing D from E, the implication class A was merged with another implication class. Let A_1 denote the implication class in $E - \widehat{D}$ that contains the third edge of the triangle (see figure 7.11).

But when removing A from E, this triangle implies the merge of the implication classes A_1, D into a single implication class. This contradicts the fact that D is an implication class in $E \setminus \widehat{A}$. Thus A is an implication class in $E \setminus \widehat{D}$. ∎

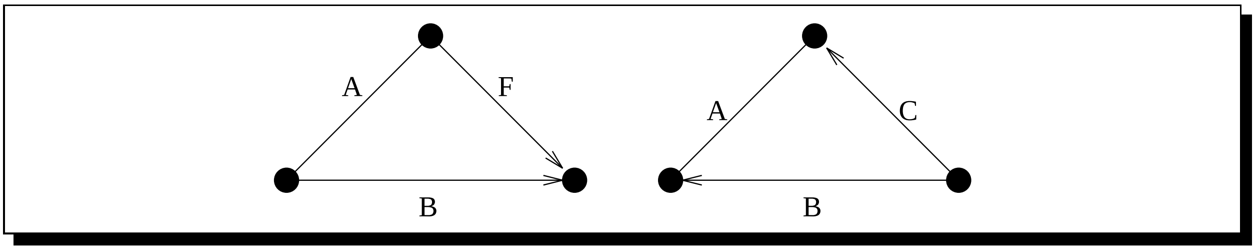

Figure 7.10: The classes $\widehat{F}, \widehat{C}$ must be equal.

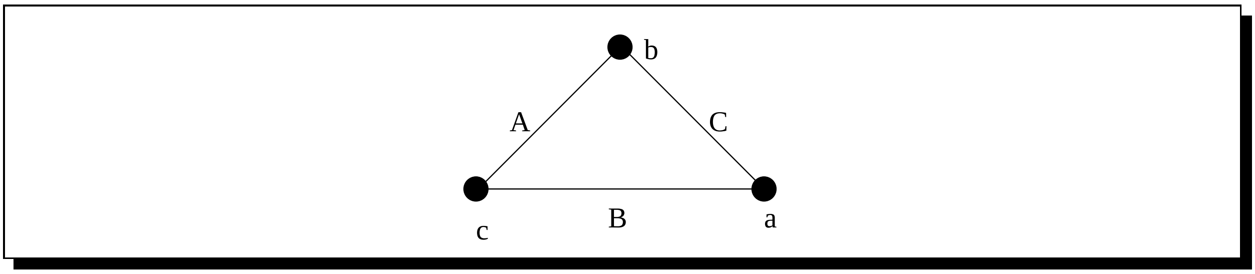

Figure 7.11: $\widehat{A} + \widehat{A_1}$ is an implication class in $E \setminus \widehat{D}$.

Corollary 7.17 *If A is an implication class, such that $A = \widehat{A}$, then there is no triangle with three colors such that $\widehat{A}$ is one of its colors.*

We leave this proof as an exercise. This claim implies, that for any triangle containing $\widehat{A}$, the two other edges in the triangle are colored by the same color.

Corollary 7.18 *If A is an implication of $G = (V, E)$ such that $A = \widehat{A}$ then all the other implication classes of G are implication classes in $E \setminus \widehat{A}$.*

Proof: By theorem 7.16 if $E \setminus \widehat{A}$ contains two implication classes from E that where merged, then these two class with $\widehat{A}$ induce a three colored triangle in G, which is impossible by Corollary 7.17. ∎

Theorem 7.19 (characterizations of comparability graphs) (Gilmore and Hoffman 1964, Ghouilla-Houri 1962)

Let $G = (V, E)$ be an undirected graph with G-decomposition, $E = \widehat{B_1} + \cdots + \widehat{B_k}$. The following conditions are equivalent:

1. *G is a comparability graph.*
2. *For all implication classes A of E, $A \cap A^{-1} = \emptyset$.*
3. *$B_i \cap B_i^{-1} = \emptyset$ for $i = 1, \ldots, k$.*

4. *Any circuit $v_1v_2, v_2v_3, \dots, v_qv_1$ in E (not necessarily simple), such that $v_{q-1}v_1 \notin E$, $v_qv_2 \notin E$ and $v_{i-1}v_{i+1} \notin E, i = 2, \dots, q-1$ has* even *length,*

   Define a short chord *or a* triangular chord *in a cycle to be a chord connecting two vertices which are distance 2 along the cycle. Condition* 4 *is equivalent to requiring that every odd circuit in G contains a short chord.*

Proof: (1) $\Longrightarrow$ (2) Proved in theorem 6.6.

(2) $\Longrightarrow$ (3) By induction of k. For $k = 1$, B_1 is an implication class in G and it is given that $B_1 \cap B_1^{-1} = \emptyset$.

We assume the claim holds for all G-decomposition of graphs that include less than k sets. Specifically, the claim holds for $E \setminus \widehat{B_i}$. Let D by an implication class of $E \setminus \widehat{B_i}$. By theorem 7.16 two cases are possible:

- The class D is an implication class of E, then by the induction hypothesis it follows $D \cap D^{-1} = \emptyset$.
- $D = B + C$ where B, C are implication classes in E. Then, $D \cap D^{-1} = (B \cup C) \bigcap (B^{-1} \cup C^{-1}) = (B \cap B^{-1}) \cup (C \cap C^{-1}) = \emptyset$, because $B \cap C = \emptyset$ and $B \cap B^{-1} = \emptyset$ and $C \cap C^{-1} = \emptyset$ by the induction hypothesis.

(3) $\Longrightarrow$ (1) Let $E = \widehat{B_1} + \cdots + \widehat{B_k}$ a G-decomposition of E such that $B_i \cap B_{i-1} = \emptyset$. By theorem 6.9 it follows that B_1 is transitive. If $k = 1$ then we are done, else assume correctness to all G-decomposition of less than k sets. It follows that $F = B_2 + \cdots + B_k$ is a transitive orientation of $E \setminus \widehat{B_1}$. We have to prove that $B_1 + F$ is transitive orientation of G. Let $ab, bc \in B_1 + F$. If both edges belong to B_1 or to F, then the edge ac exists and must belong to the corresponding set by the fact that both B_1, F are transitive. Thus, assume that $ab \in B_1$ and $bc \in F$. Since $B_1 \cap B_1^{-1} = \emptyset$, if $ac \notin E$ then $ab\Gamma cb$ thus $cb \in B_1$ thus $bc \in \widehat{B_1}$ but $bc \in F$ by assumption, a contradiction. Thus $ac \in E$.

Assume that $ac \notin B_1 + F$, then $ca \in B_1 + F$, there are two possibilities :

- If $ca \in B_1$, since $ab \in B_1$ then $cb \in B_1$, a contradiction.
- If $ca \in F$, since $bc \in F$ then $ba \in F$, a contradiction.

Thus $ac \in B_1 + F$. In the same manner, if we assume that $ab \in F, bc \in B_1$ it follows that $ac \in B_1 + F$. Thus $B_1 + \cdots + B_k$ is a transitive orientation of E.

(1) $\Longleftrightarrow$ (4) Assume that G is not a comparability graph. By the fact that (1) $\Longleftrightarrow$ (2) it follows that there exists $v_1v_2 \in A \cap A^{-1} \neq \emptyset$. By the lemma ensuring the existence of a canonical implication chain, there is a canonical implication chain such that -

$$v_1v_2 \Gamma v_3v_2 \Gamma v_3v_4 \Gamma v_5v_4 \cdots \Gamma v_qv_{q-1} \Gamma v_qv_{q+1} = v_2v_1.$$

The length of the chain is odd (because of its special structure), and there is no short chord between two vertices in cycle, because then $v_i v_{i+1} \not\Gamma v_{i+2} v_{i+1}$.

For the other direction, assume that there is an odd cycle in the graph with no short chords, then it holds -

$$v_1 v_2 \Gamma v_2 v_3 \Gamma v_3 v_4 \cdots \Gamma v_q v_{q-1} \Gamma v_q v_1 \Gamma v_2 v_1$$

is a canonical implication chain. Thus $v_1 v_2 \in A \cap A^{-1} \neq \emptyset$ and G is not a comparability graph. ∎

Definition Given an undirected graph $G = (V, E)$ we define a graph $G' = (V', E')$ in the following manner -

$$V' = \{(i,j), (j,i) | ij \in E\}$$

$$E' = \{(x,y) \leftrightarrow (y,z) | xz \notin E\}$$

(Note that by our convention $xx \notin E$ so $(x,y) \leftrightarrow (y,x)$ for every $xy \in E$.) An orientation of G' such that $(x \to y, y \to z)$ imply that $xz \in E$ is called *quasi-transitive* orientation of G.

Example 7.20 *See figure 7.12 for an example of a graph and the resulting quasi-transitive graph. The figure omits some the edges of the form (ij, ji).*

Theorem 7.21 (Ghouilla-Houri, 1962) *The following claims are equivalent:*

1. *The graph G is a comparability graph*
2. *The graph G have a quasi-transitive orientation.*
3. *The graph G' is bipartite.*

Proof: (1) $\Longrightarrow$ (2) Trivial, since G has a transitive orientation and this orientation is also quasi-transitive.

(2) $\Longrightarrow$ (1) Let F be a quasi transitive of G, and let C be an odd cycle. The cycle C must contain two adjacent edges in F: $y \to z, x \to y$ and then because of the quasi-transitive property $xz \in E$. Thus (1) follows from theorem 7.19 since we found a short chord.

(1) $\Longleftrightarrow$ (3) We will use GH characterization (Any odd cycle contains a short-chord). The vertices of a cycle in G' correspond to adjacent edges in G (with common vertex), that induce a cycle in G with no short chords. Thus G' is bipartite $\Longleftrightarrow$ G is a comparability graph. ∎

By the above theorem it is possible to check if a graph is a comparability graph, as follows: Construct the graph G', and check if it is bipartite. The size of G' is : $|V'| = 2|E|$, and -

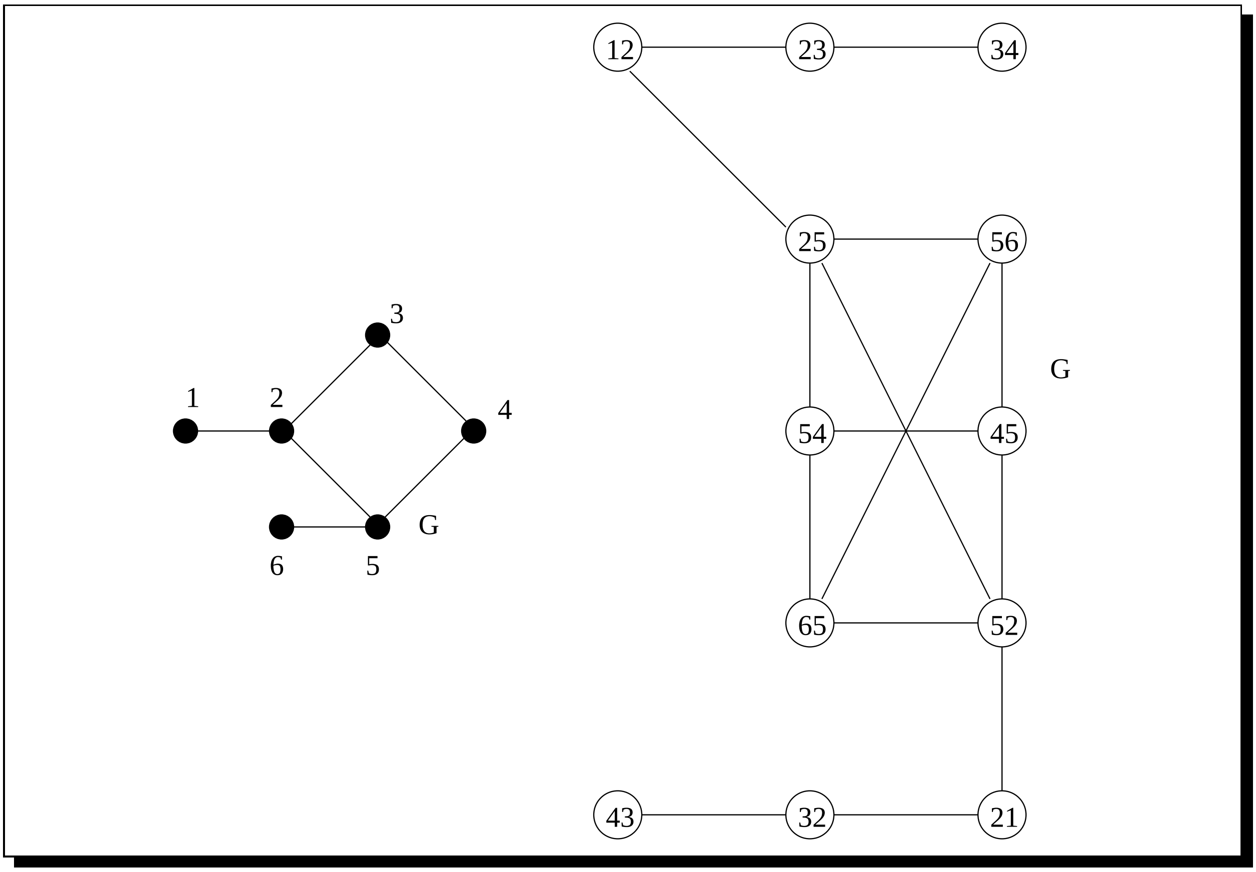

Figure 7.12: A graph G and its quasi-transitive graph G' (in G', edges $(x, y) \leftrightarrow (y, x)$ are not drawn)

$$|E'| = |E| + 2\sum_{v\in V}\left|\left\{w, x \;\middle|\; wv \in E, vx \in E, wx \notin E\right\}\right| = O(|V|^3)$$

Thus the complexity of constructing the graph is $O(|V|^3)$. Check if G' is bipartite can be performed in linear time in the size of G', i.e. in $O(|V|^3)$.

One can obtain a transitive orientation of G by taking the orientation of edges whose vertices belong to one part of the bipartite graph G'. In particular, G is **UPO** *iff* G' is connected and bipartite. We shell give more efficient algorithms for recognizing comparability graph later in the course.