THE ANALYTIC LARGE SIEVE

1. The analytic large sieve

In the last lecture we saw how to apply the analytic large sieve to derive an arithmetic formulation of the large sieve, which we applied to the problem of bounding the least quadratic non-residue. In this lecture we will derive the analytic large sieve. For more discussion of the large sieve I recommend the excellent survey article written by Montgomery [2]. Before stating the main result let us introduce the following notation. Given a real number α define the nearest distance from α to an integer as

$$\|\alpha\| := \min_{n \in \mathbb{Z}} |n - \alpha|.$$

Definition. *A subset of real numbers $\{\alpha_r\}$ is called δ-**spaced** if for $r \neq s$*

$$\|\alpha_r - \alpha_s\| \geq \delta.$$

Also, for complex numbers a_n write

$$\mathcal{L}(\alpha) = \sum_{M < n \leq M+X} a_n e(\alpha n).$$

Theorem 1.1 (The analytic large sieve 1)**.** *Let $\{\alpha_r\} \subset \mathbb{R}$ be δ-spaced then*

$$\sum_r |\mathcal{L}(\alpha_r)|^2 \leq (X + 1/\delta - 1) \sum_{M < n \leq M+X} |a_n|^2.$$

Specializing the points α_r gives the following

Corollary 1.2 (The analytic large sieve 2)**.** *We have*

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{a \,(\mathrm{mod}\ q)} \left| \mathcal{L}\left(\frac{a}{q}\right) \right|^2 \leq \left(X + Q^2 - 1\right) \sum_{M < n \leq M+X} |a_n|^2.$$

Proof. Consider the set of points

$$\bigcup_{1 \leq q \leq Q} \bigcup_{\substack{1 \leq a < q \\ \gcd(a,q)=1}} \{a/q\}.$$

It suffices to show this set of points is $1/Q^2$-space. To see this observe that

$$\left\| \frac{a_1}{q_1} - \frac{a_2}{q_2} \right\| = \left\| \frac{a_1 q_2 - a_2 q_1}{q_1 q_2} \right\| \geq \frac{1}{q_1 q_2} \geq \frac{1}{Q^2}.$$

□

Date: June 11, 2015.

2. Duality

As a first attempt, let us try to directly estimate the second moment of $\mathcal{L}(\alpha_r)$. We get that

$$\sum_r |\mathcal{L}(\alpha_r)|^2 = \sum_{M<m,n\leq M+X} a_m\overline{a_n}\sum_r e(\alpha_r(m-n))$$

However, at this stage we do not have enough information about the points $\{\alpha_r\}$ to proceed further.

We now proceed in a different direction. Consider a linear map

$$\Theta : \mathbb{C}^X \to \mathbb{C}^R,$$

which can be expressed as $\Theta = \{\theta(n,r)\}_{n=M+1,r=1}^{M+X,R}$. Observe that for $\theta(n,r) = e(\alpha_r n)$ and $\alpha = (a_{M+1},\dots,a_{M+X})$

$$\|\Theta\alpha\|_2^2 = \sum_r \left|\sum_n a_n e(\alpha_r n)\right|^2.$$

Our goal is to show that

$$\|\Theta\alpha\|_2^2 \leq (X + 1/\delta - 1)\|\alpha\|_2^2.$$

Let us now recall that the norm of Θ is given by

$$\|\Theta\| = \inf\{c : \|\Theta\alpha\|_2 \leq c\|\alpha\|_2 \text{ for all } \alpha\}.$$

The adjoint of Θ which is denoted by Θ^* is defined via the formula $\langle\Theta\alpha,\beta\rangle = \langle\alpha,\Theta^*\beta\rangle$ and such an operator exists by the Riesz representation theorem. In our setting Θ^* is just the conjugate transpose, which we will denote by $\Theta^\dagger$. Observe that for $\beta = (b_1,\dots,b_R)$

$$\|\Theta^\dagger\beta\|_2^2 = \sum_n \left|\sum_r b_r e(-\alpha_r n)\right|^2.$$

Recall the following result

$$\|\Theta\| = \|\Theta^\dagger\|,$$

which we state as

Proposition 2.1 (Duality). *Suppose that there exists Δ such that for $b_n \in \mathbb{C}$*

$$\sum_{M<n\leq M+X} \left|\sum_r b_r e(\alpha_r n)\right|^2 \leq \Delta \sum_r |b_r|^2. \tag{1}$$

Then for $a_n \in \mathbb{C}$

$$\sum_r \left|\sum_{M<n\leq M+X} a_n e(\alpha_r n)\right|^2 \leq \Delta \sum_{M<n\leq M+X} |a_n|^2. \tag{2}$$

The converse holds as well.

Remark. *Duality is a key component of the proof that we will give. It is natural to want to give an asymptotic for*

$$\sum_r |L(\alpha_r)|^2$$

and one might hope that if a_n has extra structure say for example $a_n = \tau(n)$ this might be possible. However, this cannot be done if we use duality since in order to establish (1) *we need to know that* (2) *holds for more general b_n and we lose the extra structure of the a_n.*

By other methods, recent work of Conrey, Iwaniec, and Soundararajan [1] *gives an* ***asymptotic*** *large sieve in a special, and important, setting.*

Proof. This is a special instance of the fact that the norm of a linear operator is equal to the norm of its adjoint. For clarity we will give a direct proof.

We will show (1) implies (2) the converse follows from a similar argument, which we will omit. Define

$$\Psi = \sum_n \sum_r a_n b_r e(n\alpha_r)$$

and take

$$b_r = \overline{\sum_m a_m e(m\alpha_r)}.$$

So that

$$\Psi = \sum_{n,m} a_n \overline{a_m} \sum_r e((n-m)\alpha_r) = \sum_r \left| \sum_n a_n e(\alpha_r n) \right|^2.$$

Applying Cauchy-Schwarz we also have

$$\begin{aligned} |\Psi|^2 =& \sum_n \left| \sum_r a_n b_r e(n\alpha_r) \right|^2 \\ \leq& \sum_n |a_n|^2 \sum \left| \sum_r b_r e(n\alpha_r) \right|^2 \\ \leq& \Delta \sum_n |a_n|^2 \sum_r |b_r|^2. \end{aligned}$$

Now observe that

$$\sum_r |b_r|^2 = \sum_r \left| \sum_n a_n e(m\alpha_r) \right|^2 = \Psi.$$

Combining estimates gives

$$\sum_r \left| \sum_n a_n e(m\alpha_r) \right|^2 \leq \Delta \sum_n |a_n|^2$$

as claimed.

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3. A problem in Fourier analysis

3.1. **Background on Fourier analysis.** For $f \in L^1(\mathbb{R})$ define the Fourier transform of f by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e(-x\xi)\,dx.$$

Recall the space of Schwarz functions, $S(\mathbb{R})$ is the set of smooth functions such that for any $i, j \geq 0$

$$x^i f^{(j)}(x) \ll_{i,j} 1.$$

We now recall some basic facts

- For $f \in S(\mathbb{R})$ the mapping $f \to \widehat{f}$ is a continuous injective map of S onto itself.
- The Fourier transform is an isometry on $S(\mathbb{R})$, i.e. $\|f\|_2 = \|\widehat{f}\|_2$. and Plancherel's theorem extends this isometry to $f \in L^1 \cap L^2$.
- A special case of the uncertainty principle is that both $f, \widehat{f}$ cannot both have extremely rapid decay. In particular, both cannot be compactly supported.
- For $f \in S(\mathbb{R})$ the Poisson summation fomula states

  $$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m).$$

  In fact the above formula is valid for f such that $f(x), \widehat{f}(x) \ll (1 + |x|)^{-1-\varepsilon}$.

3.2. **An extremal problem arising from the large sieve.** From the previous section we know that by duality it suffices to prove that for complex numbers b_n

$$\sum_{M<n\leq M+X} \left|\sum_r b_r e(n\alpha_r)\right|^2 \leq (X + 1/\delta - 1)\sum_r |b_r|^2.$$

We now proceed directly to see that

$$\sum_{M<n\leq M+X} \left|\sum_r b_r e(n\alpha_r)\right|^2 = \sum_{r,s} b_r\overline{b_s} \sum_{n\in\mathbb{Z}} \mathbf{1}_{[1,X]}(n+M)e(n(\alpha_r - \alpha_s)).$$

The sum on the right-hand side has been written in a suggestive way. We would like to apply the Poisson summation formula

$$\sum_{n\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m).$$

to the function

$$f(x) = \mathbf{1}_{[1,X]}(n+M)e(n(\alpha_r - \alpha_s)).$$

Unfortunately for us

$$\widehat{1}_{[-1,1]}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi},$$

is not in $L^1(\mathbb{R})$ so applying Poisson summation is not valid.

Since we are only concerned with upper bounds we will now modify the problem by taking a smooth majorant of $\mathbf{1}$, which has a Fourier transform that is compactly supported (hence the majorant itself *will not* be compactly supported). Notice that if $F(x) \geq \mathbf{1}_{[1,X]}(x)$ is a Schwarz function (or is such that both $F(x), \widehat{F}(x) \ll (1+|x|)^{-1-\varepsilon}$) then we get by applying Poisson summation

$$\begin{aligned}\sum_{M<n\leq M+X}\left|\sum_r b_r e(n\alpha_r)\right|^2 &\leq \sum_n F(n+M)\left|\sum_r b_r e(n\alpha_r)\right|^2\\ &= \sum_{r,s} b_r\overline{b_s}\sum_m e(-mM)\widehat{F}(m-(\alpha_r-\alpha_s)).\end{aligned}$$

If in addition we have that $\widehat{F}(\xi) = 0$ for $\xi \geq \delta$ then since $\{\alpha_r\}$ is δ-spaced

$$\begin{aligned}\sum_{M<n\leq M+X}\left|\sum_r b_r e(n\alpha_r)\right|^2 &\leq \sum_{r,s} b_r\overline{b_s}\sum_m e(-mM)\widehat{F}(m-(\alpha_r-\alpha_s))\\ &= \widehat{F}(0)\sum_r |b_r|^2.\end{aligned} \tag{3}$$

3.3. **The extremal problem.** Note that

$$\widehat{F}(0) = \int_R F(x)\,dx.$$

and $F(x) \geq 1_{[1,X]}(x)$. So we have the following extremal problem in Fourier analysis. Given a function whose Fourier transform that is compactly support on $|\xi| \leq \delta$ how small can

$$\int_{\mathbb{R}} \left(F(x) - 1_{[1,X]}(x)\right)\,dx$$

be?

Independently, Beurling and Selberg solved this extremal problem. Moreover, they gave explicit constructions of the minimizing function $F(x)$ which solves this problem.

Theorem 3.1. *Let $\delta > 0$. There exists a function $F_\delta(x)$ such that*

i) $F_\delta(x) \geq \mathbf{1}_{[1,X]}(x)$,
ii) $\widehat{F_\delta}(\xi) = 0$ *for* $|\xi| \geq \delta$,
iii) $\displaystyle\int_{\mathbb{R}} \left(F_\delta(x) - \mathbf{1}_{[1,X]}(x)\right)\,dx = 1/\delta$.
iv) $F_\delta(x) \ll_{\delta,X} (1+|x|)^{-2}$.

Remark. *For further discussion of these functions see section 20 of Selberg's lectures* [3] *on sieves . Additionally, Montgomery's survey article* [2] *on the large sieve gives a quick overview of these functions and Vaaler* [4] *gives a more extensive study.*

Using the Beurling-Selberg function in (3) gives

$$\sum_{M<n\leq M+X}\left|\sum_r b_r e(n\alpha_r)\right|^2 \leq \widehat{F}_\delta(0)\sum_r |b_r|^2 = (X-1+1/\delta)\sum_r |b_r|^2.$$

Therefore, applying Proposition 2.1 we have proved

$$\sum_r \left|\sum_{M<n\leq M+X} a_n e(\alpha_r n)\right|^2 \leq (X+1/\delta-1)\sum_{M<n\leq M+X}|a_n|^2.$$

References

[1] Brian Conrey, Henryk Iwaniec, and Soundararajan. *Asymptotic large sieve.* Available on the arXiv at arXiv:1105.1176.

[2] Hugh Montgomery. *The analytic principle of the large sieve.* Bull. Amer. Math. Soc. **84** (1978), no. 4, 547-567.

[3] Atle Selberg. *Collected papers. II.* Reprint of the 1991 edition [MR1295844]. With a foreword by K. Chandrasekharan. Springer Collected Works in Mathematics. Springer, Heidelberg, 2014. viii+253 pp.

[4] Jeffrey Vaaler. *Some extremal functions in Fourier analysis.* Bull. Amer. Math. Soc. **12** (1985), no. 2, 183-216.