COURSE NOTES, 2015: MEAN VALUES OF MULTIPLICATIVE FUNCTIONS

1. Mean values of multiplicative functions

1.1. Motivation: the Selberg sieve. Recall the sieving problem from the last lecture: given the following

- $\mathcal{A} \subset \mathbb{Z}$ with $\#\mathcal{A} = X$;
- $\mathcal{P}$ a set of primes and
$$P(z) = \prod_{\substack{p \le z \\ p \in \mathcal{P}}} p;$$
- for each square-free d such that $p|d \Rightarrow p \in \mathcal{P}$ define
$$\mathcal{A}_d = \{n \in \mathcal{A} : d|n\}.$$
and <u>assume</u>
$$\#\mathcal{A}_d = \frac{X}{f(d)} + R_d \tag{1}$$
where f is a multiplicative function;
- $$\mathcal{S}(\mathcal{A}, \mathcal{P}, z) = \#\{n \in \mathcal{A} : \gcd(n, P(z)) = 1\}$$

The Selberg Sieve gave the bound
$$\mathcal{S}(\mathcal{A}, \mathcal{P}, z) \le \frac{X}{S(z)} + R(z)$$
where
$$S(z) = \sum_{\substack{d \le z \\ d|P(z)}} \frac{\mu^2(d)}{(\mu * f)(d)}$$
and
$$R(z) = \sum_{\substack{d_1, d_2 \le z \\ d_1, d_2 | P(z)}} |R_{[d_1, d_2]}|.$$
It will be convenient for us to write $f(d) = d/\omega(d)$.

We saw that when applying the Selberg sieve one requires a lower bound for
$$S(z) = \sum_{\substack{n \le z \\ n|P(z)}} \frac{\mu^2(n)}{(\mu * f)(n)}.$$

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In applications this can be difficult. In this lecture we will discuss the problem of evaluating mean values of multiplicative functions, and in particular estimating sums of the form above.

First let us rewrite $S(z)$ in a more convenient way. Define the multiplicative function $\omega(n)$ such that $\omega(p^k) = 0$ for $k \geq 2$ and

$$\omega(p) = \begin{cases} \frac{p}{f(p)} \text{ if } p \in \mathcal{P} \\ 0 \text{ if } p \notin \mathcal{P}. \end{cases}$$

It follows that for $n|P(z)$ and $d|n$ we have

$$\omega\left(\frac{n}{d}\right) = \frac{\omega(n)}{\omega(d)}$$

so for $n|P(z)$ we have

$$(\mu * f)(n) = \sum_{d|n} f(n/d)\mu(d) = n \sum_{d|n} \frac{\mu(d)}{d\omega(d)} = \frac{n}{\omega(n)} \prod_{p|n} \left(1 - \frac{\omega(p)}{p}\right)$$

Thus,

$$S(z) = \sum_{\substack{d \leq z \\ d|P(z)}} \frac{\mu^2(d)}{(\mu * f)(d)} = \sum_{d \leq z} \frac{\omega(d)}{d \prod_{p|d} \left(1 - \frac{\omega(p)}{p}\right)}.$$

Our main result is

Theorem 1.1. *Let $\omega(p)$ be as above. Suppose in addition that*

$$\sum_{p \leq z} \frac{\omega(p) \log p}{p} = \kappa \log z + O(1),$$

for some $\kappa \geq 0$. Then

$$S(z) \asymp (\log z)^\kappa.$$

In fact under these assumptions it is possible to give an asymptotic formula for $S(z)$. We will delay discussion of this for later. An immediate corollary is

Corollary 1.2. *In the notation as above, we have for $\omega(p)$ satisfying the hypotheses of the previous theorem that*

$$\mathcal{S}(\mathcal{A}, \mathcal{P}, z) \ll \frac{X}{(\log z)^\kappa} + \sum_{\substack{d_1, d_2 \leq z \\ d_1, d_2 | P(z)}} |R_{[d_1, d_2]}|,$$

To see a simple application of the result let us recall that when bounding the number of twin primes we saw that

$$\pi_2(x) \ll \frac{x}{S(z)} + (z \log z)^2$$

where

$$S(z) = \sum_{d \le z} \frac{\rho(d)}{d \prod_{p|d}(1 - \rho(p)/p)}$$

and $\rho(d)$ is supported on square frees and is given by

$$\rho(p) = \begin{cases} 1, & \text{if } p = 2 \\ 2, & \text{otherwise.} \end{cases}$$

From the above definition one gets by Mertens' theorem

$$\sum_{p \le z} \frac{\rho(p) \log p}{p} = 2 \log z + O(1).$$

Theorem 1.1 implies $S(z) \asymp (\log z)^2$. Taking $z = x^{1/2 - o(1)}$ we get

$$\pi_2(x) \ll \frac{x}{(\log x)^2}.$$

1.2. **Mean values of multiplicative functions.** For a multiplicative function f define

$$M_f(x) = \sum_{n \le x} f(n)$$

to be its mean value. Let's first start with a simple example. Recall $\tau(n) = (1 * 1)(n) = \sum_{d|n} 1$, so that

$$\begin{aligned} M_\tau(x) =& \sum_{n \le x} \sum_{d|n} 1 \\ =& \sum_{d \le x} \sum_{\substack{n \le x \\ d|n}} 1 \\ =& \sum_{d \le x} \left(\frac{x}{d} + O(1) \right) \\ =& x \log x + O(x). \end{aligned}$$

This example suggests the following strategy for estimating $M_f(x)$. Write $f = 1 * g$, where $g = \mu * f$ so that h is multiplicative following the same argument gives

$$\begin{aligned}
M_f(x) =& \sum_{n\le x}\sum_{d|n} g(d) \\
=& \sum_{d\le x} g(d) \sum_{\substack{n\le x \\ d|n}} 1 \\
=& \sum_{d\le x} g(d)\left(\frac{x}{d}+O(1)\right) \\
=& x\sum_{d\le x}\frac{g(d)}{d} + O\left(\sum_{d\le x}|g(d)|\right).
\end{aligned}$$

At this point we now argue heuristically. Since h is multiplicative one may expect that

$$\begin{aligned}
\sum_{d\le x}\frac{g(d)}{d} \sim& \prod_{p\le x}\left(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\cdots\right) \\
=& \prod_{p\le x}\left(1+\frac{f(p)-1}{p}+\frac{f(p^2)-f(p)}{p^2}+\cdots\right) \\
=& \prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\right)\left(1-\frac{1}{p}\right)
\end{aligned}$$

Remark. *In the case of the divisor function $\tau(n)$ the heuristic accurately predicts $M_\tau(x) \asymp x\log x$, however for the Möbius function the heuristic gives $M_\mu(x) \asymp x/(\log x)^2$, which is known to be false. So for general multiplicative functions the heuristic will not be able to be made rigorous. However, if $g = \mu * f$ is small then the heuristic can be made rigorous.*

1.3. **Upper bounds.** Even though the heuristic does not hold in general, it is possible to prove upper bounds of this form for certain multiplicative functions.

Proposition 1.3. *Let f be a multiplicative function. Then if $f(p^k) - f(p^{k-1}) \ge 0$ for all $k \ge 1$*

$$M_f(x) \le x\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\right)\left(1-\frac{1}{p}\right).$$

Proof. Write $f = 1 * g$ so that $g = \mu * f$. This implies that g is multiplicative and

$$g(p^k) = \sum_{d|p^k}\mu\left(\frac{p^k}{d}\right)f(d) = f(p^k) - f(p^{k-1}) \ge 0.$$

Thus, $h(n) \geq 0$ so that

$$\begin{aligned} M_f(x) =& \sum_{n\leq x} \sum_{d|n} g(d) \\ \leq& x \sum_{n \leq x} \frac{g(d)}{d} \\ \leq& x \prod_{p\leq x} \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots \right). \end{aligned}$$

□

Consider the function $\tau_k(n) = (\tau_{k-1} * 1)(n)$ where $\tau_2(n) = \tau(n)$ is the divisor function. Then

$$\tau_k(p^j) = \frac{(k+j-1)!}{j!(k-1)!}.$$

Corollary 1.4. *For each $k \geq 2$ and $j \geq 1$*

$$\sum_{n\leq x} \tau_k(n)^\ell \ll x(\log x)^{k^\ell - 1}.$$

Consequently,

$$\tau_k(n) \ll n^\varepsilon.$$

Proof. Clearly $\tau_k(p^j) - \tau_k(p^{j-1}) \geq 0$ so the previous lemma implies that

$$\begin{aligned} M_{\tau_k^j}(x) \leq& x \prod_{p\leq x} \left(1 + \frac{k^\ell}{p} + O\left(\frac{1}{p^2}\right)\right)\left(1 - \frac{1}{p}\right) \\ =& x \exp\left((k^\ell - 1) \sum_{p\leq x} \left(\frac{1}{p} + O\left(\frac{1}{p^2}\right)\right)\right) \\ \ll& x(\log x)^{k^\ell - 1}. \end{aligned}$$

Using this we have that

$$\tau_k(n)^\ell \leq \sum_{1\leq m\leq 2n} \tau_k(m)^\ell \ll n(\log n)^{k^\ell - 1} \ll n^2$$

so that for ℓ sufficiently large

$$\tau_k(n) \ll n^{2/\ell} < n^\varepsilon.$$

□

For our application to sieves we needed to estimate sums of the form

$$M_h(x,z) = \sum_{\substack{n\leq x \\ n|P(z)}} h(n)$$

where h is a non-negative multiplicative function and is supported on square free integers.

Lemma 1.5. *For h as above*

$$M_h(x,z) \le \prod_{p\le z}(1+h(p))$$

Proof. We have

$$M_h(x,z) = \sum_{\substack{n\le x\\ n|P(z)}} h(n) \le \sum_{n|P(z)} h(n) = \prod_{p|P(z)} (1+h(p)).$$

□

Lower bounds are much more difficult. In the next section we will make an extra assumption on h and establish a lower bound

$$M_h(x,z) \ge \frac{1}{2}\prod_{p\le z}(1+h(p))$$

which holds for z sufficiently small relative to x.

2. Lower bounds

To obtain lower bounds we will need more information about our multiplicative function in particular we will need some information about its behavior on the prime numbers. Let ω be a non-negative multiplicative function supported on square-frees. In this section we will make the following hypothesis

$$\prod_{w<p\le z}\left(1-\frac{\omega(p)}{p}\right)^{-1} \le C\left(\frac{\log z}{\log w}\right)^{\kappa} \tag{2}$$

where $\kappa \ge 0$. In applications we will often know that

$$\sum_{p\le z}\frac{\omega(p)\log p}{p} = \kappa\log z + O(1). \tag{3}$$

It is possible to show that this condition implies (2) with $C = 1+o(1)$.

Lemma 2.1. *Let $\delta > 0$. Then for f satisfying* (2)

$$\sum_{p\le z}\frac{\omega(p)}{p}(p^{\delta}-1) \le (\kappa+\log C)(z^{\delta}-1).$$

Proof. Using the inequality $\log(1-x)^{-1} \ge x$ we get that

$$\sum_{y<p\le z}\frac{\omega(p)}{p} \le \kappa\log\frac{\log z}{\log y} + \log C.$$

Write

$$A(x) = \sum_{p\le x}\frac{\omega(p)}{p}.$$

and $F(t) = t^\delta - 1$. Therefore, by partial summation

$$\begin{aligned}
\sum_{p\le z} \frac{f(p)}{p} F(p) =& F(z)A(z) - \int_2^z F'(t)A(t)\,dt \\
=& \int_2^z F'(t)(A(z) - A(t))\,dt \\
\le& \int_2^z \left(\kappa \log \frac{\log z}{\log t} + \log C\right) F'(t)\,dt \\
\le& F(z)\log C + \kappa \int_2^z F'(t) \log \frac{\log z}{\log t}\,dt
\end{aligned}$$

Integrating by parts and making the change of variables $v = \delta \log t$

$$\begin{aligned}
\int_1^z F'(t) \log \frac{\log z}{\log t}\,dt =& \int_1^z \frac{F(t)}{t \log t}\,dt \\
=& \int_0^{\delta \log z} \frac{e^v - 1}{v}\,dv \le z^\delta - 1.
\end{aligned}$$

□

Proposition 2.2. *Suppose $h(n)$ is supported on square frees and given by*

$$h(p) = \frac{\omega(p)}{p(1 - \frac{\omega(p)}{p})}$$

where $\omega(n)$ satisfies (2) and $\omega(p) = 0$ if $p \notin \mathcal{P}$. Then

$$M_h(x, z) \ge (1 - c(r)) \prod_{p\le z} (1 + h(p)).$$

where $z = x^{1/r}$ and $c(r) < 1$ whenever $r > 2(\kappa + \log C)$ and for $r > 4(\kappa + \log C + 1)$, $c(r) < 4/5$.

Proof. Let

$$E(x, z) = \prod_{p\le z} (1 + h(p)) - M_h(x, z)$$

It suffices to show that with $z = x^{1/r}$ we have for $r > 2(\kappa + \log C)$

$$E(x, z) \prod_{p|P(z)} (1 + h(p))^{-1} \le c(r) < 1.$$

We now apply "Rankin's trick" (i.e. the condition $n > x$ implies $(n/x)^\varepsilon > 1$ for $\varepsilon > 0$)

$$\begin{aligned} E(x,z) &= \sum_{\substack{n\ge x\\ n|P(z)}} h(n) \\ &\le \frac{1}{x^\varepsilon} \sum_{n|P(z)} h(n)n^\varepsilon \\ &= \frac{1}{x^\varepsilon} \prod_{p\le z} (1+h(p)p^\varepsilon)\,. \end{aligned}$$

Since

$$h(p) = \frac{\omega(p)}{p(1-\frac{\omega(p)}{p})}$$

we get

$$\prod_{p\le z} (1+h(p))^{-1} = \prod_{p|P(z)} \left(1-\frac{\omega(p)}{p}\right).$$

Thus,

$$\begin{aligned} E(z)\prod_{p\le z}(1+h(p))^{-1} &\le \frac{1}{x^\varepsilon}\prod_{p\le z}(1+h(p)p^\varepsilon)\left(1-\frac{\omega(p)}{p}\right) \\ &= \frac{1}{x^\varepsilon}\prod_{p\le z}\left(1-\frac{\omega(p)}{p}+\left(1-\frac{\omega(p)}{p}\right)h(p)p^\varepsilon\right) \\ &= \frac{1}{x^\varepsilon}\prod_{p\le z}\left(1-\frac{\omega(p)}{p}+\frac{\omega(p)}{p}p^\varepsilon\right) \end{aligned}$$

Now use the previous lemma to get that

$$\begin{aligned} E(x,z)\prod_{p\le z}(1+h(p))^{-1} &\le \frac{1}{x^\varepsilon}\prod_{p\le z}\left(1-\frac{\omega(p)}{p}(p^\varepsilon-1)\right) \\ &\le \frac{1}{x^\varepsilon}\exp\left(\sum_{p\le z}\frac{\omega(p)}{p}(p^\varepsilon-1)\right) \\ &\le \frac{1}{x^\varepsilon}\exp\Big((\kappa+\log C)(z^\varepsilon-1)\Big). \end{aligned}$$

For $0 \le t \le 1$ one has $e^t - 1 \le 2t$ so that for $\varepsilon \le 1/\log z$ one has $z^\varepsilon - 1 \le 2\varepsilon \log z$. We deduce that for $\varepsilon \le 1/\log z$

$$E(x,z)\prod_{p\le z}(1+h(p))^{-1} \le \exp((\kappa+\log C)2\varepsilon\log z - \varepsilon\log x) < 1$$

when $z = x^{1/r}$ and $r > 2(\kappa + \log C)$. If we choose $\varepsilon = 1/\log z$ and if $r > 4(\kappa + \log C + 1)$ then the above quantity is $< 4/5$. □

Before proving Theorem 1.1 we need one more technical lemma

Lemma 2.3. *Suppose that* $\omega(n)$ *satisfies* (3)*. Then*

$$\prod_{p\le z}\left(1-\frac{\omega(p)}{p}\right)\sim c_\omega\frac{e^{-\gamma\kappa}}{(\log z)^\kappa}.$$

where

$$c_\omega=\prod_p\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-\kappa}$$

is a convergent product.

Proof (Sketch). Trivially one has

$$\prod_{p\le z}\left(1-\frac{\omega(p)}{p}\right)=\prod_{p\le z}\left(1-\frac{1}{p}\right)^\kappa\prod_{p\le z}\left(1-\frac{1}{p}\right)^{-\kappa}\left(1-\frac{\omega(p)}{p}\right)$$

By Mertens' theorem

$$\prod_{p\le z}\left(1-\frac{1}{p}\right)^\kappa\sim\frac{e^{-\gamma\kappa}}{(\log z)^\kappa}$$

Thus, it suffices to show that

$$\prod_{p>z}\left(1-\frac{1}{p}\right)^{-\kappa}\left(1-\frac{\omega(p)}{p}\right)\sim 1.$$

To see this we first make the following claims:

i)

$$\sum_{p\ge z}\frac{\omega(p)^2}{p^2}=O\left(\frac{1}{\log z}\right)$$

ii)

$$\sum_{w<p\le z}\frac{\omega(p)}{p}=\kappa\log\frac{\log z}{\log w}+O\left(\frac{1}{\log w}\right)$$

Both claims follow from partial summation. The first is less obvious so we will sketch the argument. Applying (3) twice at $z=p-1/2$ and $z=p+1/2$ gives

$$\frac{\omega(p)\log p}{p}\ll 1$$

Thus,

$$\sum_{p>z}\frac{\omega^2(p)}{p^2}\ll\sum_{p>z}\frac{\omega(p)}{p\log p}\ll\frac{1}{\log z}.$$

Now apply i), ii) and Mertens' theorem to see that

$$\prod_{z\leq p\leq N}\left(1-\frac{1}{p}\right)^{-\kappa}\left(1-\frac{\omega(p)}{p}\right)=\exp\left(\sum_{z<p\leq N}\frac{\omega(p)}{p}-\kappa\sum_{z<p\leq N}\frac{1}{p}+O\left(\frac{1}{\log z}\right)\right)$$
$$=1+O\left(\frac{1}{\log z}\right)$$

Taking $N\to\infty$ completes the proof.

□

Proof of Theorem 1.1. Recall that

$$S(z)=\sum_{\substack{n\leq z\\ n|P(z)}}h(n)=M_h(z,z)$$

where h is a multiplicative function supported on square frees and is given by

$$h(p)=\frac{\omega(p)}{p(1-\frac{\omega(p)}{p})},$$

where $\omega(p)=0$ for $p\notin\mathcal{P}$. The upper bound for $S(z)$ follows from Lemmas 2.3 and 1.5. Proposition 2.2 implies that for $r>3(\kappa+\log C)$

$$M_h(x,x^{1/r})\gg\prod_{p|P(z)}(1+h(p))=\prod_{p\leq x^{1/r}}(1+h(p))=\prod_{p\leq x^{1/r}}\left(1-\frac{\omega(p)}{p}\right)^{-1}.$$

So by Lemma 2.3

$$M_h(x,x^{1/r})\gg(\log x)^{\kappa}.$$

To complete the proof note that

$$M_h(z,z)\geq M_h(z,z^{1/r})\gg(\log z)^{\kappa}.$$

□

2.1. **More precise estimates.** By more careful arguments it is possible to give an asymptotic evaluation of $S(z)$. We will state this result without proof. The argument can be found in Halberstam and Richert's book (see Lemma 5.4 of [1])

Theorem 2.4. *Suppose that $\omega(p)$ is a non-negative multiplicative function supported on square frees and satisfies* (3). *Then*

$$\frac{1}{S(z)}=\frac{\Gamma(\kappa+1)}{(\log z)^{\kappa}}\prod_{p}\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-\kappa}\left(1+O\left(\frac{1}{\log z}\right)\right).$$

References

[1] H. Halberstam and H.E. Richert, *Sieve Methods.* Courier Dover Publications, 2013.