SIEVE THEORY 2015
ASSIGNMENT 1
DUE DATE: MONDAY, MARCH 30, 2015

1. The von Mangoldt function is defined as $\Lambda(n) = \log p$, if $n = p^k$ is a power of a prime p ($k \geq 1$), and $\Lambda(n) = 0$ otherwise. Show that

a) The Dirichlet convolution of Λ with the constant function $\mathbf{1}$ is $\Lambda * \mathbf{1} = \log$.

b) The Dirichlet series associated to $-\Lambda$ is the logarithmic derivative of the zeta function: For $\mathrm{Re}(s) > 1$,

$$-\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \frac{\zeta'(s)}{\zeta(s)} \ .$$

2. The Euler totient function $\varphi(n)$ is the number of integers $0 < a < n$ which are coprime to n. It is a consequence of the Chinese Remainder Theorem that it is a multiplicative function. Show that

$$\varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d} \ .$$

3. The indicator function of the squarefree integers is $\mathbf{1}_{\mathrm{SF}}(n) = |\mu(n)|$. Show that the associated Dirichlet series is

$$\sum_{n=1}^{\infty} \frac{\mathbf{1}_{\mathrm{SF}}(n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)}, \quad \mathrm{Re}(s) > 1 \ .$$

4. The Möbius function is defined for square-free integers $n = p_1 \cdots\cdots p_k$ as $\mu(n) = (-1)^k$ (p_i are distinct primes) and $\mu(n) = 0$ otherwise. The summatory function of $\mu(n)$ is $M(x) := \sum_{n \leq x} \mu(n)$. Show that <u>if</u> we are given $0 < \delta < 1$ so that $M(x) = O(x^\delta)$ for all $x \gg 1$, then $\zeta(s) \neq 0$ for all s in the half-plane $\mathrm{Re}(s) > \delta$.

5. a) Use partial summation to show that if $\mathrm{Re}(s) > 1$ and $x \geq 1$, then

$$\sum_{n>x} \frac{1}{n^s} = \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s \int_x^{\infty} \frac{\{u\}}{u^{s+1}} \, du,$$

where $\{x\} = x - \lfloor x \rfloor$ and the floor function $\lfloor x \rfloor$ equals the largest integer $\leq x$. Explain why this implies that for $\mathrm{Re}(s) > 0$ and $x \geq 1$

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s \int_x^\infty \frac{\{u\}}{u^{s+1}}\, du.$$

b) Use the equation above to show that for $t \geq 2$

$$|\zeta(\sigma + it)| \ll \begin{cases} t^{1-\sigma} \text{ if } 0 < \sigma < 1, \\ \log t \text{ if } \sigma = 1, \end{cases}$$

where the implied constant depends on σ.