SIEVE THEORY 2015
ASSIGNMENT 2
DUE DATE: WEDNESDAY, APRIL 15, 2015

Exercise 1. Let $\omega(n)$ the number of distinct prime divisors of an integer n, so for instance $\omega(12) = 2$. Show that $\omega(n) \ll \log n/\log\log n$.

Exercise 2. Let $\Omega(n)$ be the number of prime divisors of n, counted with multiplicity, i.e. the number of prime powers dividing n, so for instance $\Omega(12) = 3$. Show that the mean value of $\Omega(n)$ is $\log\log n$:

$$\frac{1}{N}\sum_{n\le N}\Omega(n) = \log\log N + O(1)\ .$$

Exercise 3. Let $\tau(n) = \sum_{d|n} 1$ be the divisor function. Using $\sum_{n\le x}\tau(n) = x(\log x + C) + O(x^{1/2})$, show that

$$\sum_{n>Y}\frac{\tau(n)}{n^2} = \frac{\log Y + C + 2}{Y} + O(\frac{1}{Y^{3/2}})$$

Exercise 4. Let $a > 0$, b be coprime integers. Find the density of integers n for which $an + b$ is squarefree.

Exercise 5. Let $f(x) \in \mathbb{Z}[x]$ be a separable polynomial (i.e. with no repeated roots) of positive degree. Set $B := \gcd\{f(n) : n \in \mathbb{Z}\}$ and let B' be the smallest divisor of B so that B/B' is square-free. For each prime p, we denote by p^{q_p} the largest power of p dividing B', and by $r_f(p)$ the number of $a \bmod p^{2+q_p}$ for which $f(a)/B' = 0 \bmod p^2$. We set

$$c_f = \prod_p \left(1 - \frac{r_f(p)}{p^{2+q_p}}\right)$$

which is the conjectural density of integers n for which $f(n)/B'$ is squarefree.

For $f(x) = x(x+1)(x+2)(x+3)$, find B_f and B'_f, and show that $r_f(p) = 4$ for $p \ne 2, 3$.

Hence $c_f = R\prod_{p\ne 2,3}(1 - \frac{4}{p^2})$ for some rational number. Find R.

Exercise 6. Assume that $B' = 1$. Show that $c_f > 0$, i.e. that $r_f(p) < p^2$ for all primes p.