THE LIST-RAMSEY THRESHOLD FOR FAMILIES OF GRAPHS

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Abstract. Given a family of graphs $\mathcal{F}$ and an integer r, we say that a graph is r-Ramsey for $\mathcal{F}$ if any r-colouring of its edges admits a monochromatic copy of a graph from $\mathcal{F}$. The threshold for the classic Ramsey property, where $\mathcal{F}$ consists of one graph, was located in the celebrated work of Rödl and Ruciński.

In this paper, we offer a twofold generalisation to the Rödl–Ruciński theorem. First, we show that the list-colouring version of the property has the same threshold. Second, we extend this result to finite families $\mathcal{F}$, where the threshold statements might also diverge. This also confirms further special cases of the Kohayakawa–Kreuter conjecture. Along the way, we supply a short(-ish), self-contained proof of the 0-statement of the Rödl–Ruciński theorem.

Contents

1. Introduction

Given an integer $r \geqslant 2$ and graphs F and G, we say that G is r-Ramsey for F if any r-colouring of the edges of G admits a monochromatic copy of F. We denote this property by $G \in \mathrm{Ram}(F, r)$. Ramsey's theorem [12] states that, for all r and F, the clique K_n is r-Ramsey for F whenever n is sufficiently large.

One of the efforts of probabilistic combinatorics in the last few decades has been to understand the behaviour of such Ramsey properties in random structures. More concretely, one can ask how the probability that the binomial random graph $G_{n,p}$ is r-Ramsey for some graph F changes as p traverses the interval $[0, 1]$. In a major breakthrough, Rödl and Ruciński [13] answered this question by finding thresholds for the Ramsey property for all graphs F and every number of colours r. To state the theorem, first define the maximum 2-density of a graph F to be

$$m_2(F) \coloneqq \max \left\{ \frac{e_{F'} - 1}{v_{F'} - 2} \colon \emptyset \neq F' \subseteq F,\ v_{F'} \geqslant 3 \right\} \cup \left\{ \frac{1}{2} \right\}.$$

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Theorem 1.1 (Rödl–Ruciński [13])**.** *Given $r \geqslant 2$ and a graph F, there are positive constants c_0, c_1 such that*

$$\lim_{n\to\infty} \mathbb{P}\left(G_{n,p} \in \mathrm{Ram}(F,r)\right) = \begin{cases} 1, & p \geqslant c_1 \cdot n^{-1/m_2(F)}, \\ 0, & p \leqslant c_0 \cdot n^{-1/m_2(F)}, \end{cases}$$

unless F is a star forest or $r = 2$ and F is the path on four vertices[1]*.*

In the two exceptional cases not covered by Theorem 1.1, the corresponding statement requires some amendment. This is because, in both cases, the threshold is determined by the appearance of a fixed graph G that is r-Ramsey for our forbidden graph. More precisely, the Ramsey threshold coincides with the threshold $\hat{p}_G = \hat{p}_G(n)$ for the property that $G \subseteq G_{n,p}$.

In some cases, Theorem 1.1 can be strengthened still to yield a sharp threshold. The following theorem is a combination of the works of Friedgut and Krivelevich [2] and Friedgut, Kuperwasser, Samotij, and Schacht [3].

Theorem 1.2 ([2, 3])**.** *Let $r \geqslant 2$ and let F be a graph that is either a clique, a cycle, or a tree*[2] *(excluding the exceptional cases from before). Then there are positive constants c_0, c_1 and a function $c_0 \leqslant c(n) \leqslant c_1$ such that, for every $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(G_{n,p} \in \mathrm{Ram}(F,r)\right) = \begin{cases} 1, & p \geqslant (1+\varepsilon)\cdot c(n) \cdot n^{-1/m_2(F)}, \\ 0, & p \leqslant (1-\varepsilon)\cdot c(n) \cdot n^{-1/m_2(F)}. \end{cases}$$

We note here that, in the exceptional cases, the Ramsey property does not have a sharp threshold. Further, even though $G_{n,p}$ is a.a.s. not Ramsey when $p \ll \hat{p}_G$, the probability that $G_{n,p}$ contains the fixed Ramsey graph G is still bounded from below by some positive constant for any $p = \Theta(\hat{p}_G)$.

In this paper, we will be interested in two extensions to the graph Ramsey property. The first is a generalisation of the property to families of graphs. The second is a list-colouring variant that was recently introduced in [1].

Definition 1.3 (Family-Ramsey)**.** Given an integer $r \geqslant 2$ and a family $\mathcal{F}$ of graphs, we say that a graph G is r-Ramsey with respect to $\mathcal{F}$ if any r-colouring of the edges of G admits a monochromatic copy of one of the members of $\mathcal{F}$. We denote this property as $G \in \mathrm{Ram}(\mathcal{F}, r)$.

Definition 1.4 (List-Ramsey)**.** Given an integer $r \geqslant 2$ and a family $\mathcal{F}$ of graphs, we say that a graph G is r-list-Ramsey with respect to $\mathcal{F}$ if there is an assignment of lists of r colours to the edges of G, not necessarily identical, such that any colouring from these lists admits a monochromatic copy of some $F \in \mathcal{F}$. We denote this property by $G \in \mathrm{Ram}_\ell(\mathcal{F}, r)$.

In [3, Section 7], it was shown that, for all graphs F bar the exceptional cases, if $p = \Theta(n^{-1/m_2(F)})$, then any constant-sized graph that appears in $G_{n,p}$ with some positive probability, as n tends to infinity, cannot be 2-list-Ramsey for F. This sparks some hope that we can extend Theorem 1.1 to the list-Ramsey threshold.

[1]This last case which was originally missed, was later noticed in [2].

[2]The work in [2] establishes sharp thresholds for trees, and [3] proves the result to the family of *collapsible* graphs, which includes cliques and cycles; see [3] for the exact definition of collapsibility.

Such hope turned out to be true for other Ramsey-type properties, such as those corresponding to Schur's theorem [15] on sums and to van der Waerden's theorem [16] on arithmetic progressions. The usual thresholds for these properties had been established in the works of Graham, Rödl, and Ruciński [4] and Rödl and Ruciński [13, 14]. The list-colouring variants for these properties were considered in [3], in the context of establishing sharp thresholds, where it was shown that the list-colouring threshold coincides with the usual threshold.

Remark. In fact, both of these results are immediate corollaries of [3, Theorem 1.7], which gives sufficient conditions for locating the threshold for list-colourability of random induced subgraphs of 'almost-linear' hypergraphs.

Returning to graphs and to the results of this paper, we divide our statement for graph families into two cases. First, we treat the 'generic' case, where we look at finite families of graphs that do not include forests. Second, we allow forests, which invites some of the exceptional behaviour that we encountered in the statement of the original random Ramsey theorem. Given a family of graphs $\mathcal{F}$ define

$$m_2(\mathcal{F}) \coloneqq \min \{m_2(F) \colon F \in \mathcal{F}\}.$$

Note that forbidding forests is the same as asserting that $m_2(\mathcal{F}) > 1$.

Theorem 1.5 (Generic case)**.** *Given an integer $r \geqslant 2$ and a finite family of graphs $\mathcal{F}$ with $m_2(\mathcal{F}) > 1$, there are positive constants c_0, c_1 such that*

$$\lim_{n\to\infty} \mathbb{P}\left(G_{n,p} \in \mathrm{Ram}_\ell(\mathcal{F}, r)\right) = \begin{cases} 1, & p \geqslant c_1 \cdot n^{-1/m_2(\mathcal{F})}, \\ 0, & p \leqslant c_0 \cdot n^{-1/m_2(\mathcal{F})}. \end{cases}$$

In other words, as long as we exclude forests, we have a semi-sharp threshold corresponding to the minimum 2-density amongst the graphs in $\mathcal{F}$.

Remark. The finiteness assumption is necessary. For example, if we just allow $\mathcal{F}$ to contain all graphs with 2-density at least t, then $G_{n,p}$ will become r-Ramsey for $\mathcal{F}$ as soon as $p \gg n^{-1}$. Indeed, if $p \gg n^{-1}$, then a.a.s. $e(G_{n,p}) \gg n$. Further, since any r-colouring of $G_{n,p}$ admits a colour class C taking up at least a $(1/r)$-fraction of the edges, the ratio $\frac{e_C-1}{n-2}$ tends to infinity with n, which implies that $C \in \mathcal{F}$.

The 1-statement follows from the usual 1-statement for the Ramsey property. Indeed, it just means that $G_{n,p}$ cannot be properly coloured from a specific choice of lists, namely, where all the lists are identical. Following Nenadov and Steger [11], we can prove this very quickly using the hypergraph container lemma and a 'supersaturated' version of Ramsey's theorem. Therefore, we only need to prove the 0-statement. Moreover, as $\mathrm{Ram}_\ell(\mathcal{F}, r) \subseteq \mathrm{Ram}_\ell(\mathcal{F}, 2)$ for every $r \geqslant 2$, we may also assume that $r = 2$.

Theorem 1.6 (0-statement)**.** *For every finite family $\mathcal{F}$ of graphs with $m_2(\mathcal{F}) > 1$, there exists a positive constant c_0 such that, for all $p \leqslant c_0 \cdot n^{-1/m_2(\mathcal{F})}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(G_{n,p} \in \mathrm{Ram}_\ell(\mathcal{F}, 2)\right) = 0.$$

In contrast, allowing forests into your life only invites chaos and strife. While we are able to handle infinite families of graphs in this case, we face a vast array of new exceptions. These stem from interactions between two types of graphs that we define below.

Definition 1.7. We say that a graph F is a *$\mathcal{B}$-graph* if every component of F is a subgraph of a broom. (A *broom* is a tree made from a path of length two by connecting to one of its endpoints an arbitrary number of leaves, called *hairs* of the broom.) We say that F is a *$\mathcal{C}^*$-graph* if every component of F is either a subgraph of an odd cycle or a star.

Theorem 1.8 (Allowing forests: list-Ramsey)**.** *Let $r \geqslant 2$ be an integer and let $\mathcal{F}$ be an arbitrary family of graphs. There exists a positive constant c_0 such that, for all $p \leqslant c_0 \cdot n^{-1}$,*

$$\lim_{n\to\infty} \mathbb{P}(G_{n,p} \in \mathrm{Ram}_\ell(\mathcal{F}, r)) = 0,$$

unless $r = 2$ and $\mathcal{F}$ contains both a $\mathcal{B}$-graph and a $\mathcal{C}^*$-graph, in which case $\mathbb{P}(G_{n,p} \in \mathrm{Ram}_\ell(\mathcal{F}, r))$ is bounded away from zero for every $p = \Omega(n^{-1})$.

Interestingly, this is a case where the statements for the list-Ramsey threshold and the usual Ramsey threshold diverge. There are families containing $\mathcal{B}$-graphs and $\mathcal{C}^*$-graphs for which the usual Ramsey threshold is still semi-sharp. This behaviour is determined by the 2-colourability of an auxiliary hypergraph $\mathrm{Aux}(\mathcal{F})$, defined in Section 5.

Theorem 1.9 (Allowing forests: usual Ramsey)**.** *Let $r \geqslant 2$ be an integer and let $\mathcal{F}$ be an arbitrary family of graphs. There exists a positive constant c_0 such that, for all $p \leqslant c_0 \cdot n^{-1}$,*

$$\lim_{n\to\infty} \mathbb{P}(G_{n,p} \in \mathrm{Ram}(\mathcal{F}, r)) = 0,$$

unless $r = 2$ and either $\mathcal{F}$ contains a star forest or $\mathrm{Aux}(\mathcal{F})$ is not 2-colourable.

1.1. **Corollaries and applications.**

Applying these theorems to the case where $\mathcal{F}$ is a singleton gives us the following strengthening of Theorem 1.1.

Corollary 1.10 (The list-Ramsey threshold)**.** *Given $r \geqslant 2$ and a graph F, there are positive constants c_0, c_1 such that*

$$\lim_{n\to\infty} \mathbb{P}\left(G_{n,p} \in \mathrm{Ram}_\ell(F, r)\right) = \begin{cases} 1, & p \geqslant c_1 \cdot n^{-1/m_2(F)}, \\ 0, & p \leqslant c_0 \cdot n^{-1/m_2(F)}, \end{cases}$$

unless F is a star forest, or $r = 2$ and F is the path on four vertices.

This follows from the fact that the only exceptions can come from graphs F that are simultaneously $\mathcal{B}$-graphs and $\mathcal{C}^*$-graphs. The only such graphs are star forests and the path on four vertices.

Another corollary of this work concerns asymmetric Ramsey thresholds. For graphs H, L with $m_2(H) \geqslant m_2(L)$, define their *mixed* 2*-density* as

$$m_2(H, L) \coloneqq \max\left\{\frac{e_{H'}}{v_{H'} - 2 + 1/m_2(L)} : \emptyset \neq H' \subseteq H\right\}.$$

Given graphs $F_1, \ldots, F_k$, we say that a graph G is Ramsey for $(F_1, \ldots, F_k)$ if any r-colouring of the edges of G admits, for some $i \in [\![r]\!]$, a copy of the graph F_i in the colour i. We denote this by $G \in \mathrm{Ram}(F_1, \ldots, F_k)$. Kohayakawa and Kreuter [7] conjectured that the threshold for this property is governed by the mixed 2-density of the two densest graphs.

Conjecture 1.11 (Kohayakawa–Kreuter [7])**.** *Let $F_1, \ldots, F_k$ be graphs with $m_2(F_1) \geqslant \cdots \geqslant m_2(F_k) > 1$. Then there are positive constants c_0 and c_1 such that*

$$\lim_{n\to\infty} \mathbb{P}\left(G_{n,p} \in \mathrm{Ram}(F_1, \ldots, F_k)\right) = \begin{cases} 1, & p \geqslant c_1 \cdot n^{-1/m_2(F_1,F_2)}, \\ 0, & p \leqslant c_0 \cdot n^{-1/m_2(F_1,F_2)}. \end{cases}$$

The 1-statement of this conjecture was proved by Mousset, Nenadov, and Samotij [10], but the 0-statement remains a formidable foe. The original work of Kohayakawa and Kreuter [7] gives a proof in the case where the two densest graphs are cycles. Since then, the 0-statement in Conjecture 1.11 has been proved in the case where both F_1 and F_2 are cliques [9] and where F_1 is a clique and F_2 is a cycle [8]. More recently, Hyde [5] provided a proof for most pairs of regular graphs.

Note that $m_2(F_2) \leqslant m_2(F_1, F_2) \leqslant m_2(F_1)$ so when F_1 and F_2 have the same 2-density, it also coincides with their mixed 2-density. As a result, we can prove the following additional special case of Conjecture 1.11.

Corollary 1.12. *The Kohayakawa–Kreuter conjecture holds whenever* $m_2(F_1) = m_2(F_2)$*.*

Indeed, if we can colour $G_{n,p}$ only using the first two colours without creating a copy of F_1 or F_2 in either colour, then $G_{n,p}$ is not Ramsey for $(F_1, \dots, F_k)$. Therefore, the 0-statement for this case follows from the 0-statement for the property $\mathrm{Ram}(\{F_1, F_2\}, 2)$.

1.2. **Organization of the paper.** We continue with Section 2 in which we outline the steps for proving Theorem 1.6 and Theorem 1.8. The proof of Theorem 1.6, the generic case, is completed in two steps, corresponding to Section 3 and Section 4. Finally, Section 5 provides the proof for Theorem 1.8, the case where forests are allowed.

Another feature of our work is a rather speedy proof of the original random Ramsey theorem of Rödl and Ruciński (i.e., Theorem 1.1) that does not rely on the Nash-Williams theorem. Because dealing with families adds certain complications for one of the stages of the proof, carried out in Section 4, we begin that section by first considering the singleton case. Consequently, the (semi-)interested reader could stop their perusal at page 14.

2. Outline of the proof

We start with a general discussion of our proof strategy. Let $\mathcal{P}$ be a monotone property of graphs and suppose we wish to show that $G_{n,p}$ avoids $\mathcal{P}$ with high probability whenever $p \leqslant cn^{-1/\alpha}$, for some positive constants α, c. Suppose that G is a fixed graph that has the property $\mathcal{P}$. If $m(G) \leqslant \alpha$, then $G_{n,p} \supseteq G$, and thus $G_{n,p} \in \mathcal{P}$ by monotonicity, with probability $\Omega(1)$. As a result, to have any hope of proving our statement, we must be able to show the following:

(D) Any G with $m(G) \leqslant \alpha$ avoids $\mathcal{P}$.

This is a necessary condition and sometimes it is also sufficient. Indeed, if $p \leqslant cn^{-1}$ for c sufficiently small, then $m(G_{n,p}) \leqslant 1$ with high probability. Actually, this will be the case for our statement whenever $m_2(\mathcal{F}) = 1$, i.e., when we allow forests. However, if $p \gg n^{-1}$, then, with high probability, the density of $G_{n,p}$ tends to infinity.

In the latter case, we require another structural statement about $G_{n,p}$. One way to go about this is to find a family of graphs $\mathcal{Q}$ that contains all the minimal elements of $\mathcal{P}$ and such that the following holds with high probability:

(P) Every graph $G \in \mathcal{Q}$ with $G \subseteq G_{n,p}$ has $m(G) \leqslant \alpha$.

We refer to this type of statement as a *probabilistic lemma* and to the previous statement (D), which does not involve any randomness, as a *deterministic lemma*.

Given these statements the result follows readily. Indeed, if $G_{n,p} \in \mathcal{P}$, then it must also contain some minimal element $G \in \mathcal{P}$. However, this would mean that $G \in \mathcal{Q}$ and therefore, with high probability, that $m(G) \leqslant \alpha$. As a result, G avoids $\mathcal{P}$, a contradiction.

This is the usual approach for establishing 0-statements for Ramsey thresholds, which was taken, for example, in the short proof of the Rödl–Ruciński theorem [13] by Nenadov and Steger [11]. This is also how we prove Theorems 1.8 and 1.9. For the generic case, we will take a slightly different approach (similar to the approach of Liebenau–Mattos–Mendonça–Skokan [8]), which allows us to weaken the requirement of the probabilistic lemma at the cost of strengthening the deterministic lemma. Indeed, instead of arguing about the maximum density of members of $\mathcal{Q}$ we will only ask that, with high probability,

(P^-) Every graph $G \in \mathcal{Q}$ with $G \subseteq G_{n,p}$ has $e_G/v_G \leqslant \alpha$.

Subsequently, for the same proof to work, we need to show that the minimal elements of $\mathcal{P}$ are themselves dense (and not just that they contain a dense subgraph):

(D^+) Any minimal element $G \in \mathcal{P}$ has $e_G/v_G > \alpha$.

While the difference between (D) and (D^+) might look superficial, not being able to control the maximum density disallows some of the tools that were essential in the argument of Rödl and Ruciński (as well as that of Nenadov and Steger), such as the Nash-Williams theorem. Instead, we build on the work from [3] and deploy a discharging argument that also allows us to deal with list-colouring. One feature of this approach is that we get a proof that is self-contained and—due to the weakened probabilistic lemma—relatively short. With more work, we could upgrade our proof of (P^-) to a proof of (P) by showing that, with high probability, every graphs in $\mathcal{Q}$ that appears in $G_{n,p}$ has a bounded number of vertices. However, we do not pursue this here, as the weaker assertion (P^-) is sufficient for our purposes.

We now turn to applying this scheme in our context. As we mentioned above, the case $m_2(\mathcal{F}) = 1$ only requires (D) and the 0-statement holds if and only if (D) holds. Since (D) fails for a larger class of families in the context of list-Ramsey property, as opposed to the classical Ramsey property, we obtain different threshold statements. This is discussed fully in Section 5.

Next, let $\mathcal{F}$ be a finite family with $m_2(\mathcal{F}) > 1$. We may assume that $\mathcal{F}$ consists of strictly 2-balanced graphs. (Recall that a graph F is *strictly 2-balanced* if $m_2(F) > m_2(F')$ for every nonempty subgraph $F' \subseteq F$.) Indeed, we may define $\mathcal{F}'$ by going over every $F \in \mathcal{F}$ and adding to $\mathcal{F}'$ a strictly 2-balanced subgraph $F' \subseteq F$ with the same 2-density as F; such F' exists since $m_2(F) > 1$. By construction, $m_2(\mathcal{F}) = m_2(\mathcal{F}')$. Moreover, any colouring of $G_{n,p}$ that avoids all graphs in $\mathcal{F}'$ also avoids all graphs in $\mathcal{F}$. We may therefore replace $\mathcal{F}$ and prove the result for $\mathcal{F}'$ in its stead.

Definition 2.1. Given a graph G and a family of graphs $\mathcal{F}$, define the *$\mathcal{F}$-hypergraph of G* to be the hypergraph on the vertex set $E(G)$ whose hyperedges correspond to copies of F in G, for all $F \in \mathcal{F}$.

Definition 2.2. A graph C is called an *$\mathcal{F}$-cluster* if the $\mathcal{F}$-hypergraph of C is connected. Let $\mathcal{C}_{\mathcal{F}}$ be the family of all $\mathcal{F}$-clusters.

The family $\mathcal{Q}$ that we are going to choose in (P^-) is the family of all $\mathcal{F}$-clusters; it is not hard to see that a minimal Ramsey graph for $\mathcal{F}$ must be an $\mathcal{F}$-cluster. This explicit description of $\mathcal{Q}$ will allow us to prove a probabilistic lemma without much effort.

To summarise, the following two lemmas give us Theorem 1.6.

Lemma 2.3 (The Probabilistic Lemma)**.** *Let $\mathcal{F}$ be a finite family of strictly 2-balanced graphs. There exists a positive constant c such that, if $p \leqslant cn^{-1/t}$, then a.a.s. every $\mathcal{F}$-cluster $C \subseteq G_{n,p}$ satisfies $e_C/v_C \leqslant t$.*

Lemma 2.4 (The Deterministic Lemma)**.** *Let $\mathcal{F}$ be a family of strictly 2-balanced graphs, each containing a cycle. Every graph G that is minimally list-Ramsey with respect to $\mathcal{F}$ satisfies $e_G/v_G > m_2(\mathcal{F})$.*

The proofs of these lemmas are presented in the next two sections. While we prove the Probabilistic Lemma straight away, we will take a more cautious approach with the Deterministic Lemma. Since our argument is somewhat technical, we first treat the case where $\mathcal{F}$ is a singleton[3]. The proof for single graphs already gives a good idea of how the general argument works and lays much of the groundwork towards it.

3. Proof of the Probabilistic Lemma

Let $\mathcal{C}_{\text{bad}}$ be the collection of all $C \in \mathcal{C}_{\mathcal{F}}$ such that $e_C/v_C > t$. We will show that there exist positive constants ε and L and a family $\mathcal{S}$ of subgraphs of K_n with the following properties:

(a) Every set in $\mathcal{C}_{\text{bad}}$ contains some element of $\mathcal{S}$.
(b) Every $S \in \mathcal{S}$ satisfies $e_S \geqslant t \cdot v_S + \varepsilon$, or both $v_S \geqslant \log n$ and $e_S \geqslant t \cdot (v_S - 2)$.
(c) For every k, there are at most $(Ln)^k$ graphs $C \in \mathcal{S}$ with $v_C = k$.

We first show that the existence of such a collection $\mathcal{S}$ implies the assertion of the lemma. Indeed, it follows from (a) and the union bound that

$$\mathbb{P}(C \subseteq G_{n,p} \text{ for some } C \in \mathcal{C}_{\text{bad}}) \leqslant \mathbb{P}(S \subseteq G_{n,p} \text{ for some } S \in \mathcal{S}) \leqslant \sum_{S \in \mathcal{S}} p^{e_S}.$$

Moreover, by (b) and (c),

$$\begin{aligned}\sum_{S \in \mathcal{S}} p^{e_S} &\leqslant \sum_k (Ln)^k \cdot \left(p^{tk+\varepsilon} + \mathbb{1}_{k \geqslant \log n} \cdot p^{t(k-2)}\right) \\ &\leqslant \sum_k (c^t L)^k \cdot p^{\varepsilon} + \sum_{k \geqslant \log n} (c^t L)^k \cdot p^{-2t}.\end{aligned}$$

We now choose c sufficiently small so that $c^t L \leqslant \min\{e^{-4t}, 1/2\}$. This way, the first sum above is at most p^{ε} and the second sum is at most $2 \cdot (n^2 p)^{-2t}$, so both tend to zero.

To find the family $\mathcal{S}$, we will first define an exploration process of the clusters. We first fix a labeling of the vertices of K_n, which induces an ordering of all subgraphs according to the lexicographical order. Now, given $C \in \mathcal{C}_{\mathcal{F}}$, we start with $C_0 := \{e_0\}$, where e_0 is the smallest edge of C. As long as $C_i \neq C$, do the following: Since $C \neq C_i$ and C is an $\mathcal{F}$-cluster, there must be a copy of some $F \in \mathcal{F}$ in C that intersects C_i but is not fully contained in C_i. Call such a copy *regular* if it has 2-density exactly t and intersects C_i in exactly one edge e, called its *root*; otherwise, call the copy *degenerate*. We form C_{i+1} from C_i by adding to it one such overlapping copy of F as follows: If there is a regular copy of some $F \in \mathcal{F}$, add the smallest one among the copies rooted at the edge that arrived to C the earliest among all edges of C_i; otherwise, add the smallest degenerate copy of F. Finally, given an integer Γ let

$$\tau = \tau(C) := \min\{i : i \text{ is the } \Gamma\text{-th degenerate step or } v_{C_i} \geqslant \log n \text{ or } C_i = C\}$$

and let

$$\mathcal{S} := \{C_{\tau(C)} : C \in \mathcal{C}_{\text{bad}}\}.$$

This definition guarantees that $\mathcal{S}$ satisfies (a) above; we will show that, for a sufficiently large Γ, it also satisfies (b) and (c).

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[3]This also allows us to finish the proof of the 0-statement in the Rödl–Ruciński theorem earlier.

Claim 3.1. *There exists* $\Gamma = \Gamma(\mathcal{F}) > 0$ *such that item (b) holds.*

Proof. We first argue that (b) holds in the case where $S \in \mathcal{C}_{\mathrm{bad}}$. To this end, note that t is a rational number and therefore there is an integer b that depends only on $\mathcal{F}$ such that bt is an integer. Since both be_S and btv_S are integers and $e_S > tv_S$, as $S \in \mathcal{C}_{\mathrm{bad}}$, we must have $be_S \geqslant btv_S + 1$, which means that (b) holds with $\varepsilon = 1/b$.

Consider an arbitrary $S \in \mathcal{S} \setminus \mathcal{C}_{\mathrm{bad}}$ and let $C \in \mathcal{C}_{\mathrm{bad}}$ be a cluster such that $S = C_{\tau(C)} \neq C$. Let d_i denote the number of degenerate steps taken when constructing C_i. It suffices to show that there exists a positive $\eta = \eta(\mathcal{F})$ such that

$$(e_{C_i} - 1) - t \cdot (v_{C_i} - 2) \geqslant \eta \cdot d_i. \tag{1}$$

Indeed, set $\Gamma := \lceil 2t/\eta \rceil$. Inequality (1) with $i = \tau(C)$ may be rewritten as $e_S - t \cdot v_S \geqslant 1 - 2t + \eta \cdot d_{\tau(C)}$. Consequently, if $d_{\tau(C)} = \Gamma$, then $e_S \geqslant t \cdot v_S + 1 - 2t + \eta \cdot \Gamma \geqslant t \cdot v_S + 1$. Otherwise, if $d_{\tau(C)} < \Gamma$, we only know that $e_S \geqslant t \cdot v_S - 2t + 1$, but it must be that $v_S \geqslant \log n$, since we assumed that $S \neq C$.

We now turn to proving (1); we do so by induction on i. Note that C_0 is an edge and therefore

$$(e_{C_0} - 1) - t \cdot (v_{C_0} - 2) = 0.$$

From there on, we wish to show that the left-hand side of (1) does not change with any regular step, while it grows by at least η with any degenerate step.

Indeed, with every regular step we add a copy of $F \in \mathcal{F}$ without one edge, meaning that $e_{C_i} = e_{C_{i-1}} + e_F - 1$ and $v_{C_i} = v_{C_{i-1}} + v_F - 2$, where, by the regularity condition, $(e_F - 1) - t \cdot (v_F - 2) = 0$. Therefore, a regular step will not change the left-hand side. On the other hand, with any degenerate step we have two options. In our extending step, we used a graph $F \in \mathcal{F}$ which either intersected C_{i-1} in one edge, but had 2-density strictly larger than t, or it intersected C_{i-1} in at least two edges. In the first case, we know that $(e_F - 1) - t \cdot (v_F - 2) > 0$. Since our family $\mathcal{F}$ is finite, we may define η_1 to be the minimum of $(e_F - 1) - t \cdot (v_F - 2)$ over all $F \in \mathcal{F}$ with $m_2(F) > t$. For the second case, if the intersection is in a strict subgraph $F' \subsetneq F$ with $e_{F'} \geqslant 2$, then the left-hand side grows by $(e_F - e_{F'}) - t \cdot (v_F - v_{F'})$. Using the fact that F is strictly 2-balanced and $2 \leqslant e_{F'} < e_F$, we have $(e_{F'} - 1)/(v_{F'} - 2) < (e_F - 1)/(v_F - 2)$ and, consequently,

$$t \leqslant \frac{e_F - 1}{v_F - 2} = \frac{(e_F - e_{F'}) + (e_{F'} - 1)}{(v_F - v_{F'}) + (v_{F'} - 2)} < \frac{e_F - e_{F'}}{v_F - v_{F'}}.$$

So $(e_F - e_{F'}) - t \cdot (v_F - v_{F'}) > 0$, and again we can take η_2 to be the minimum of this quantity over all possible F and F'. Therefore, (1) holds with $\eta = \min\{\eta_1, \eta_2\}$. □

Claim 3.2. *There exists a constant* $L = L(\mathcal{F})$ *such that item (c) holds.*

Proof. Suppose that $S \in \mathcal{S}$ has k vertices. Let $C \in \mathcal{C}_{\mathrm{bad}}$ be such that $S = C_{\tau(C)}$ and consider the exploration process on C. If the i-th step is degenerate, it can be uniquely described by specifying the graph $F \in \mathcal{F}$, the intersection F' of C_{i-1} with F, and the sequence of $v_F - v_{F'}$ vertices of K_n that complete it to this copy of F. If the i-th step is regular, it can be uniquely described by its root in C_{i-1}, the graph $F \in \mathcal{F}$, the edge of F that corresponds to the root, and the sequence of $v_F - 2$ vertices of K_n that complete it to a copy of F.

There are at most n^k ways to choose the ordered sequence of vertices that were added by the exploration process. Since each regular step adds at least one new vertex to the cluster, there are at most k regular steps. Further, since the number of degenerate steps is at most Γ, we have $\tau(C) \leqslant k + \Gamma$. In particular, there are at most $(k + \Gamma) \cdot 2^{k+\Gamma}$

ways to choose $\tau(C)$ and designate which of the $\tau(C)$ steps were regular and which were degenerate. For every degenerate step, there are at most

$$\sum_{F\in\mathcal{F}}\sum_{\ell=2}^{v_F}\binom{v_F}{\ell}\cdot k^\ell \leqslant |\mathcal{F}|\cdot(k+1)^{M_v}$$

ways to choose $F \in \mathcal{F}$ and its intersection with S, where $M_v \coloneqq \max\{v_F : F \in \mathcal{F}\}$. As for the regular steps, since: the root at the i-th step is the edge that was added to C_{i-1} the earliest among all those that are a part of a regular copy; and a copy that is regular at some step j is regular at every earlier step i as long as its root belongs to C_{i-1}, the sequence of 'birth times' of all roots in the exploration process is non-decreasing. In particular, the number of ways to choose the sequence of roots for all the regular steps is at most the number of non-decreasing $[\![e_S]\!]$-valued sequences of length at most k, which is at most $\binom{e_S+k}{k}$. Finally, since every step increases the number of edges of C_i by at most $M_e \coloneqq \max\{e_F : F \in \mathcal{F}\}$, we have

$$e_S \leqslant 1 + \tau(C)\cdot M_e \leqslant 1 + (k+\Gamma)\cdot M_e.$$

To summarise,

$$|\mathcal{S}| \leqslant n^k\cdot(k+\Gamma)\cdot 2^{k+\Gamma}\cdot\left(|\mathcal{F}|\cdot(k+1)^{M_v}\right)^\Gamma\cdot\binom{(k+\Gamma)\cdot M_e+k+1}{k}\cdot\left(|\mathcal{F}|\cdot M_e\right)^k,$$

which is at most $(Ln)^k$, provided that L is sufficiently large. □

Since we have already shown how the existence of a collection $\mathcal{S}$ satisfying items (a)–(c) implies the assertion of the lemma, the proof is now complete.

4. Proof of the Deterministic Lemma

Suppose that a graph G is minimally list-Ramsey with respect to $\mathcal{F}$. It is not hard to see that G must be an $\mathcal{F}$-cluster. We further claim that G must have a certain 'rigidity' property that is slightly stronger than the property that every edge of G is the sole intersection of some two copies of graphs from $\mathcal{F}$.

Definition 4.1. A connected hypergraph H is a *core* if, for every edge $e \in H$ and vertex $v \in e$, there is an edge $e' \in H$ such that $e \cap e' = \{v\}$.

Fact 4.2. *Every minimally non-2-list-colourable hypergraph is a core.*

Proof. Let H be a non-core hypergraph whose every proper subhypergraph is 2-list-colourable. We will show that H is also 2-list-colourable. Indeed, suppose we assign a list of two colours to every vertex of H. Since it is not a core, there is an edge $e \in H$ and a vertex $v \in e$ such that no edge $e' \in H$ intersects e solely in v. Since the hypergraph $H \setminus e$ is 2-list-colourable, we may properly colour its vertices from their lists. If e is not monochromatic under this choice of colours, then we have coloured H. Otherwise, all vertices of e were assigned the same colour. In this case, switch the colour of v to the other option in its list. We claim that this gives a proper colouring of H. Indeed, e is no longer monochromatic; moreover, if this change of colour affected some edge e', then it must contain v. However, if this is the case, the by our assumption e' must intersect e in at least one more vertex v'. But since the colour of v was changed just so it is different than the colour of all other vertices in e, then now v and v' are coloured distinctly, and therefore e' cannot be monochromatic. □

We are now ready to define our rigidity property. We will say that a graph G is an *$\mathcal{F}$-core* if the $\mathcal{F}$-hypergraph of G contains a spanning, connected subhypergraph that is a core[4]. If $\mathcal{F}$ does not contain any graphs with 2-density at most two, the assertion of the Deterministic Lemma will hold under the weaker assumption that G is a connected $\mathcal{F}$-core[5], with the single exceptional case $G = K_6$, which is treated in Claim 4.14 below.

Proposition 4.3. *Suppose that $\mathcal{F}$ is a family of strictly 2-balanced graphs with $m_2(\mathcal{F}) > 2$. If a connected $\mathcal{F}$-core G satisfies $e_G/v_G \leqslant m_2(\mathcal{F})$, then $G = K_6$.*

As there do exist $\mathcal{F}$-cores G with $e_G/v_G \leqslant m_2(\mathcal{F})$ for many families $\mathcal{F}$ with $m_2(\mathcal{F}) \leqslant 2$, this case will require a different argument that uses the full power of the assumption that G is minimally list-Ramsey.

Proposition 4.4. *If G is minimally list-Ramsey for some family $\mathcal{F}$ of strictly 2-balanced graphs, each containing a cycle, then $e_G/v_G > \min\{m_2(\mathcal{F}), 2\}$.*

Finally, for the sake of providing a short, self-contained proof of the 0-statement in the Rödl–Ruciński theorem (Theorem 1.1), we will separately treat the case where $\mathcal{F}$ comprises only one graph. For brevity, we write F-core in place of $\{F\}$-core. The following single-graph analogue of Proposition 4.3 characterises all F-cores G with $e_G/v_G \leqslant m_2(F)$ for all strictly 2-balanced graphs F that contain a cycle, except the triangle. Clearly, K_6 is not Ramsey for K_4 and it is not difficult to prove that every 3-regular graph is a union of two forests (Claim 4.11 below is a generalisation of this statement); showing that no graph G with $e_G/v_G \leqslant m_2(K_3)$ is minimally Ramsey for K_3 is only a little more difficult, see Claim 4.10 below.

Proposition 4.5. *Suppose that F is a strictly 2-balanced graph with $m_2(F) > 1$. If a connected F-core G satisfies $e_G/v_G \leqslant m_2(F)$, then either:*

(1) $F = K_3$,
(2) $F = K_4$ and $G = K_6$, or
(3) $F = C_4$ and G is 3-regular.

The remainder of this section is organised as follows. In Section 4.1, we establish two elementary graph-theoretic lemmas that we use repeatedly while proving Propositions 4.3–4.5. These lemmas appeared in [3, Section 7], but we include the proofs for the sake of being self-contained. In Section 4.2, we treat the single-graph case of the Deterministic Lemma, proving Proposition 4.5, showing that K_6 is not list-Ramsey for K_4, that no 3-regular graph is list-Ramsey for C_4, and that every graph G that is minimally list-Ramsey for K_3 satisfies $e_G/v_G > m_2(K_3)$. Some of the arguments for the single-graph case already appear in [3]. In Section 4.3, we treat the general case of arbitrary (finite) families $\mathcal{F}$. We first prove the easier Proposition 4.3 and argue that K_6 is not list-Ramsey for any family $\mathcal{F}$ with $m_2(\mathcal{F}) \geqslant e_{K_6}/v_{K_6}$. Second, we prove the more intricate Proposition 4.4. We feel that it is worth mentioning that Section 4.3 reuses several claims established in the earlier subsection and thus our proof of the single-graph case of the Deterministic Lemma may be viewed as a warm-up for the general case.

[4]This does not mean that the $\mathcal{F}$-hypergraph of G itself is a core. For example, K_6 is an $\{K_4, K_5\}$-core even though the $\{K_4, K_5\}$-hypergraph of K_6 is not (but the K_4-hypergraph of K_6 is a core)

[5]Since every strictly 2-balanced graph is connected, all minimally list-Ramsey graphs for a family of strictly 2-balanced graphs are connected as well.

4.1. **Some tools.** Given a graph F and a set $W \subseteq V(F)$, it will be convenient to denote by $\bar{e}_F(W)$ the number of edges incident with a vertex of W, i.e., $\bar{e}_F(W) \coloneqq e_F - e_{F-W}$, where $F - W$ denotes the subgraph of F induced by the set $V(F) \setminus W$.

Lemma 4.6 (Helpful Lemma, [3]). *Suppose that F is strictly 2-balanced and that $W \subseteq V(F)$ satisfies $1 \leqslant |W| \leqslant v_F - 3$. Then*

$$\bar{e}_F(W) > m_2(F) \cdot |W|.$$

Proof. Since $F - W$ is a proper subgraph of F with at least three vertices and F is strictly 2-balanced,

$$\frac{e_F - 1 - \bar{e}_F(W)}{v_F - 2 - |W|} = \frac{e_{F-W} - 1}{v_{F-W} - 2} < m_2(F) = \frac{e_F - 1}{v_F - 2},$$

which means that $(e_F - 1) \cdot |W| < (v_F - 2) \cdot \bar{e}_F(W)$ and the result follows. □

Lemma 4.7 (Discharging, [3]). *Suppose that $e_G/v_G \leqslant k + \varepsilon$, where $k \geqslant 1$ is an integer and $\varepsilon \in [0, 1)$. One of the following holds:*

- *(Da) G has a vertex of degree at most $2k$,*
- *(Db) G has a vertex of degree $2k + 1$ with a neighbour of degree at most $2k + 2$ and $\varepsilon \geqslant 1/2$, or*
- *(Dc) G has a vertex of degree $2k+3$ with two neighbours of degree $2k+1$ and $\varepsilon \geqslant 7/8$.*

Proof. Since $\delta(G) \leqslant 2e_G/v_G \leqslant 2k + 2\varepsilon < 2k + 2$, we have $\delta(G) \leqslant 2k + 1$ and, if $\varepsilon < 1/2$, then $\delta(G) \leqslant 2k$. We may thus assume that $\delta(G) = 2k + 1$ and $\varepsilon \geqslant 1/2$, since otherwise (Da) holds. We may further assume that all neighbours of every vertex of degree $2k + 1$ have degrees at least $2k + 3$, since otherwise (Db) holds.

Assign to each $v \in V(G)$ a charge of $\deg(v) - 2(k + \varepsilon)$ and note that the average charge is non-positive. We define the following discharging rule: every vertex of degree $2k + 1$ takes a charge of $\frac{2\varepsilon-1}{2k+1}$ from each of its neighbours. By our assumption, no vertex of degree $2k + 1$ or $2k + 2$ sends charge to any of its neighbours. In particular, the final charge of a vertex of degree $2k + 1$ is

$$2k + 1 - 2(k + \varepsilon) + (2k + 1) \cdot \frac{2\varepsilon - 1}{2k + 1} = 0$$

and the final charge of a vertex of degree $2k + 2$ is $2k + 2 - 2(k + \varepsilon) > 0$.

Since the total charge remains unchanged, the final charge of some vertex of degree at least $2k+3$ must be non-positive. Let v be one such vertex. Suppose that $\deg(v) = 2k+t$, where $t \geqslant 3$, and that v has x neighbours with degree $2k + 1$. Since the final charge of v is

$$2k + t - 2(k + \varepsilon) - x \cdot \frac{2\varepsilon - 1}{2k + 1} \leqslant 0,$$

we have

$$(t - 2)(2k + 1) < \frac{t - 2\varepsilon}{2\varepsilon - 1} \cdot (2k + 1) \leqslant x \leqslant 2k + t,$$

which implies that $t < 3 + 1/k \leqslant 4$. Therefore, $t = 3$ and $x > 2k + 1 \geqslant 3$, which means that some vertex of degree $2k + 3$ has more than three neighbours of degree $2k + 1$. Moreover, we also have $\frac{3-2\varepsilon}{2\varepsilon-1} \cdot (2k + 1) \leqslant 2k + 3$, which implies that $\varepsilon \geqslant \frac{4k+3}{4k+4} \geqslant \frac{7}{8}$. □

4.2. **The single-graph case.** We begin by proving Proposition 4.5 in the case where F is not one of the three exceptional graphs. Note that these are the only strictly 2-balanced graphs with at most four vertices, so disregarding them allows us to apply the Helpful Lemma to F with W of size one or two.

Proof of Proposition 4.5 in the case $v_F \geqslant 5$. Our goal is to prove that no graph G with $e_G/v_G \leqslant m_2(F)$ is an F-core. Suppose by contradiction that G is such a graph and write $m_2(F) = k + \varepsilon$, where $k \geqslant 1$ is an integer and $\varepsilon \in [0, 1)$. We split the argument into three cases, depending on which item of Lemma 4.7 holds.

Case 1. Suppose that G has a vertex v of degree at most $2k$ and let e be an edge incident to v. By assumption, there are two copies of F in G whose sole intersection is the edge e and thus $\deg(v) \geqslant 2\delta(F) - 1$. However, since $v_F > 3$, we may apply the Helpful Lemma to learn that $\delta(F) > m_2(F) \geqslant k$, which implies that $\deg(v) \geqslant 2k + 1$, a contradiction.

Case 2. Suppose now that G has an edge uv with $\deg(u) = 2k+1$ and $\deg(v) \leqslant 2k+2$, and that $\varepsilon \geqslant 1/2$. By our assumption, there are two copies of F in G that intersect only in uv. Since $v_F > 4$, we may use the Helpful Lemma to learn that the endpoints of every edge of F touch strictly more than $2(k + \varepsilon) \geqslant 2k + 1$ edges. Because the copies of F intersect only in uv, this means that $\{u, v\}$ must touch at least $2(2k+2)-1 = 4k+3$ edges in G. However, our assumption implies that $\{u, v\}$ touches at most $\deg(u)+\deg(v)-1 \leqslant 4k + 2$ edges, a contradiction.

Case 3. Finally, suppose that G has a vertex v of degree $2k+3$ connected to two vertices u_1, u_2 of degrees $2k + 1$, and that $\varepsilon \geqslant \frac{7}{8}$. We may assume that $u_1u_2 \notin G$, as otherwise we are in Case 2. By our assumption, we find two copies of F that intersect only in u_1v. If none of them use u_2v, then we may argue exactly as in Case 2, as the edge u_1v is the sole intersection of two copies of F in the graph $G' := G \setminus u_2v$ and $\deg_{G'}(u_1) = 2k + 1$ and $\deg_{G'}(v) = 2k + 2$.

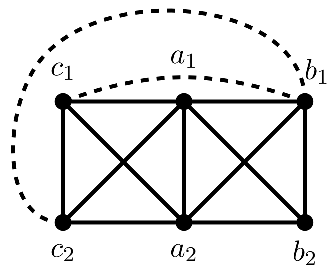
We may thus assume that one of the two copies, say F_1, contains u_2v while the other, F_2, does not. Our assumption that G is a core supplies another copy of F, say F_3, that intersects F_1 precisely in the edge u_2v. Similarly as before, we will show that there are not enough edges in G that touch $\{v, u_1, u_2\}$ to support these three copies of F. Since ε, which is the fractional value of $m_2(F)$, is at least 7/8, the graph F must have at least ten vertices, and we are justified in applying the Helpful Lemma to sets W of up to three vertices. In particular, we may conclude that F_1 contains at least $\lceil 3(k + \varepsilon)\rceil = 3k + 3$ edges touching $\{v, u_1, u_2\}$, that F_2 contains at least $\lceil 2(k + \varepsilon)\rceil = 2k + 2$ edges touching $\{v, u_1\}$, and that F_3 contains at least $\lceil k + \varepsilon\rceil = k + 1$ edges touching u_2. Since, among the edges mentioned, only u_1v and u_2v were counted twice, the set $\{v, u_1, u_2\}$ must touch at least $6k + 4$ edges of G. However, the bounds on the degrees of v, u_1, and u_2 imply that they touch only $\deg(u_1) + \deg(u_2) + \deg(v) - 2 = 6k + 3$ edges. □

In order to complete the proof of Proposition 4.5, we are now going to identify the F-cores in the case where $F \in \{K_4, C_4\}$. Further, to conclude the proof of the Deterministic Lemma in the case $\mathcal{F} = \{F\}$, we will have to show how to colour each K_4-core and C_4-core and prove that every graph G that is minimally list-Ramsey with respect to K_3 satisfies $e_G/v_G > 2 = m_2(K_3)$.

Claim 4.8. *The only connected K_4-core G with $e_G/v_G \leqslant m_2(K_4)$ is $G = K_6$.*

Proof. Suppose that G is a connected K_4-core with $e_G/v_G \leqslant m_2(K_4) = 5/2$. Since G must satisfy $\delta(G) \geqslant 2\delta(K_4)-1 = 5 \geqslant 2e_G/v_G$, see Case 1 in the proof of Proposition 4.5,

G must be 5-regular. Now, let $a_1a_2 \in G$ be an edge, and let $b_1b_2a_1a_2$ and $c_1c_2a_1a_2$ be two copies of K_4 whose edge sets intersect solely in a_1a_2; in particular, the vertices b_1, b_2, c_1, c_2 are all distinct.



Now, the edge a_1b_1 should also participate in two copies of K_4 where it is the sole intersection, a_1b_1vv' and a_1b_1uu'. However the vertex a_1 is of degree 5 so $\{v, v', u, u'\} = \{a_2, b_2, c_1, c_2\}$ meaning that b_1 is adjacent to c_1 and c_2. Repeating the same argument for a_1b_2, we get that b_2 is adjacent to c_1 and c_2 meaning that $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ induces a K_6. Since G is connected and 5-regular, we have $G = K_6$. $\square$

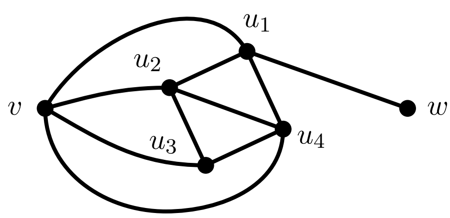
Claim 4.9. *If G is a connected C_4-core with $e_G/v_G \leqslant m_2(C_4)$, then G is 3-regular.*

Proof. Suppose that G is a connected C_4-core with $e_G/v_G \leqslant m_2(C_4) = 3/2$. Since $\delta(G) \geqslant 2\delta(C_4) - 1 = 3 \geqslant 2e_G/v_G$, the graph G must be 3-regular. $\square$

The proof of Proposition 4.5 is now complete. Since it is easily checked that K_6 is not Ramsey for K_4 and K_5 is not Ramsey for K_3, the following two claims complete the proof of the 0-statement in Theorem 1.1.

Claim 4.10. *If $G \neq K_5$ is minimally (list-)Ramsey graph for K_3, then $e_G/v_G > m_2(K_3)$.*

Proof. Suppose that a graph G is minimally (list-)Ramsey for K_3 and satisfies $e_G/v_G \leqslant m_2(K_3) = 2$. If, for some $v \in V(G)$, there was an orientation of the edges of $G[N(v)]$ with maximum out-degree at most one, we could then extend every K_3-free colouring of $G - v$ to G as follows: For every $u \in N(v)$, the edge uv gets a colour that is different from the colour of the out-edge from u. (Since each triangle involving v contains an edge of $G[N(v)]$, this colouring is K_3-free.) As every graph with at most four edges has such an orientation, we may assume that $e(N(v)) \geqslant 5$ for every $v \in V(G)$; in particular, $\delta(G) \geqslant 4 \geqslant 2e_G/v_G$, so G must be 4-regular.



If $e(N(v)) > 5$ for some $v \in V(G)$, then $G = K_5$, since G is 4-regular and connected. We may thus further assume that $e(N(v)) = 5$ for every v. Pick some v and denote $N(v) = \{u_1, u_2, u_3, u_4\}$ so that $u_1u_3 \notin G[N(v)]$. Since G is 4-regular, there must be a $w \in V(G)\backslash(\{v\} \cup N(v))$ such that $N(u_1) = \{v, w, u_2, u_4\}$. Moreover, w is not adjacent to either of v, u_2, and u_4, as they all have 4 neighbours in $\{v\} \cup N(v)$, and thus $e(N(u_1)) \leqslant 3$, a contradiction. $\square$

Claim 4.11. *Every graph with maximum degree three can be 2-list-coloured in such a way that every colour class is a forest.*

Proof. Suppose that this were not true and let G be a smallest graph with maximum degree three that does not admit such a cycle-free colouring for some choice of lists of size two assigned to its edges. If some vertex v had degree at most two or its three incident edges not all have the same lists, then we could extend any colouring of $G - v$ by assigning different colours to the edges incident with v; this was we would not create any monochromatic cycles. Therefore, G must be 3-regular and, as G is clearly connected, all lists must be identical, say $\{\text{red}, \text{blue}\}$. Let $v_1 v_2 \dots v_\ell v_1$ be an arbitrary cycle in G and denote by $e_1, \dots, e_\ell$ the edges incident with $v_1, \dots, v_\ell$, respectively, that are not on this cycle. We can extend any cycle-free colouring of $G - \{v_1, \dots, v_\ell\}$ to a cycle-free colouring of G by colouring all the edges of the path $v_1 \dots v_\ell$ and the edge e_1 red and the edges $v_1 v_\ell$ and $e_2, \dots, e_\ell$ blue. □

We have now completed the proof of the 0-statement in Theorem 1.1. The following two simple claims are what remains to be proved to obtain the complete statement of the Deterministic Lemma and, as a result, the proof of the 0-statement in the list-colouring version of Theorem 1.1.

Claim 4.12. *K_6 is not list-Ramsey with respect to K_4.*

Proof. Given an assignment of lists of size two, pick a random colouring uniformly from the lists. There are $\binom{6}{4} = 15$ copies of K_4 in K_6, so the expected number of monochromatic K_4s is at most $15 \cdot 2^{1-6} < 1$, meaning that there is a proper list-colouring. □

Claim 4.13. *K_5 is not list-Ramsey with respect to K_3.*

Proof. If some colour, say red, contains a 5-cycle, then we may colour this 5-cycle red and the complementary 5-cycle not red. If some colour class, say red, contains an edge, say e, not in a triangle, then we may colour $K_5 \setminus e$ without monochromatic triangles (this is possible as no proper subgraph K_5 is list-Ramsey for K_3, by Claim 4.10) and colour e red. If none of the above is true, then each colour induces one of the following graphs: K_3, K_4, K_4^-, $K_5 \setminus K_3$, or two triangles sharing a vertex. If some colour, say red, induces $K_5 \setminus K_3$, then we colour $K_{2,3}$ with red, the remaining edge of $K_5 \setminus K_3$ with not red and the edges of the K_3 in the complement with two different colours other than red. If one of the colours, say red, induces K_4 or K_4^-, then colour a C_4 with red and its diagonal(s) with a colour other than red. Each of the remaining, uncoloured four edges can close at most one monochromatic triangle, as red is not available anywhere outside of the K_4 we have already coloured; thus we may colour them one-by-one. This leaves the case where every colour class is either K_3 or two triangles sharing a vertex. But this is impossible, since 3 does not divide $2e(G) = 20$. □

4.3. **General families.** We first prove the Deterministic Lemma for families $\mathcal{F}$ with $m_2(\mathcal{F}) > 2$. To this end, we first prove Proposition 4.3 and then show that K_6 is not list-Ramsey with respect to any family $\mathcal{F}$ with $m_2(\mathcal{F}) \geqslant 5/2 = e_{K_6}/v_{K_6}$.

Proof of Proposition 4.3. Write $m_2(\mathcal{F}) = k + \varepsilon$, where $k \geqslant 2$ is an integer and $\varepsilon \in [0, 1)$, and suppose that G is a connected $\mathcal{F}$-core with $e_G/v_G \leqslant k + \varepsilon$. It suffices to show that $K_4 \in \mathcal{F}$ and G is a K_4-core, since then Claim 4.8 will allow us to conclude that $G = K_6$. We split the argument into three cases, depending on which item of Lemma 4.7 holds. Since K_3, C_4, and K_4 are the only strictly 2-balanced graphs with at most four vertices, the assumption that $m_2(\mathcal{F}) > 2$ implies that the only graph with fewer than five vertices that can belong to $\mathcal{F}$ is K_4.

Case 1. Suppose that G has a vertex v of degree at most $2k$ and let e be an edge incident to v. Since G is an $\mathcal{F}$-core, there are two copies of some $F, F' \in \mathcal{F}$ in G that intersect solely in e, and thus $\deg(v) \geqslant \delta(F) + \delta(F') - 1$. Since $v_F, v_{F'} > 3$, we can apply the Helpful Lemma to learn that $\delta(F), \delta(F') \geqslant k + 1$, which gives $\deg(v) \geqslant 2k + 1$, a contradiction.

Case 2. Suppose now that G has an edge uv with $\deg(u) = 2k + 1$ and $\deg(v) \leqslant 2k + 2$ and that $\varepsilon \geqslant 1/2$. By assumption, there are two copies of some $F, F' \in \mathcal{F}$ that intersect only in uv. If $v_F, v_{F'} > 4$, we could use the Helpful Lemma to learn that the endpoints of every edge of F and F' touch strictly more than $2m_2(\mathcal{F}) \geqslant 2(k + \varepsilon) \geqslant 2k + 1$ edges. This would mean that $\{u, v\}$ touch at least $2(2k+2) - 1 = 4k + 3 > \deg(u) + \deg(v) - 1$ edges in G, a contradiction. Thus, one of F, F' is a K_4. Consequently, we have

$$e_G/v_G \leqslant m_2(\mathcal{F}) \leqslant m_2(K_4) = 5/2$$

As Case 1 rules out $\delta(G) \leqslant 4$, the graph G must be 5-regular and $m_2(\mathcal{F}) = 5/2$.

Finally, we show that G is a K_4-core. If this were not true, then some edge uv of G would be the sole intersection of two copies of some graphs from $\mathcal{F}$, not both of them K_4. The Helpful Lemma implies that, for every $F \in \mathcal{F}$ with $v_F > 4$, the endpoints of every edge of F touch strictly more than 5 edges. On the other hand, the endpoints of every edge of K_4 touch precisely 5 edges. This means that at least 10 edges of G would have to touch $\{u, v\}$, a contradiction.

Case 3. Finally, suppose that $\varepsilon \geqslant 7/8$ and that there are vertices u, v_1, v_2 with $\deg(u) = 2k + 3$, $\deg(v_1) = \deg(v_2) = 2k + 1$, and $uv_1, uv_2 \in G$. We may assume that $v_1v_2 \notin G$, since otherwise we would be in Case 2. By assumption, there are two copies of some $F_1, F_2 \in \mathcal{F}$ that intersect only in uv_1. Applying the Helpful Lemma, we learn that $\delta(F_1) > m_2(F_1)$ and $\delta(F_2) > m_2(F_2)$. If $m_2(F_i) \geqslant k + 1$ for some $i \in \{1, 2\}$, then we would have $\deg(v_1) \geqslant \delta(F_1) + \delta(F_2) - 1 \geqslant 2k + 2$, a contradiction. Therefore,

$$k + 7/8 \leqslant k + \varepsilon = m_2(\mathcal{F}) \leqslant m_2(F_i) < k + 1$$

for each $i \in \{1, 2\}$, and thus the definition of $m_2(\cdot)$ implies that $v_{F_1}, v_{F_2} \geqslant 10$.

We can therefore apply the Helpful Lemma and learn that the endpoints of every edge of F_1 and F_2 touch strictly more than $2m_2(\mathcal{F}) \geqslant 2(k + \varepsilon) \geqslant 2k + 1$ edges. Because our copies of F_1 and F_2 intersect only in uv and $\{u, v\}$ touches exactly $4k + 3$ edges of G, one of the copies, say of F_2, contains uv_2. By the Helpful Lemma, this copy of F_2 must therefore use strictly more than $3m_2(F_2) \geqslant 3(k + \varepsilon) \geqslant 3k + 2$ edges touching $\{u, v_1, v_2\}$.

Finally, since G is an $\mathcal{F}$-core, there is another copy of some graph $F_3 \in \mathcal{F}$ that intersects our copy of F_2 solely in uv_2. The Helpful Lemma gives $\delta(F_3) \geqslant \lceil m_2(F_3) \rceil \geqslant k + 1$. The union of our copies of F_1, F_2, and F_3 must contain at least $3k + 3 + 2k + 2 + k + 1 - 2 = 6k + 4$ edges touching $\{u, v_1, v_2\}$. However, the bounds on the degrees of u, v_1, and v_2 in G imply that they touch only $6k + 3$ edges, a contradiction. □

Claim 4.14. *K_6 is not list-Ramsey with respect to any family $\mathcal{F}$ with $m_2(\mathcal{F}) \geqslant 5/2$.*

Proof. If all lists assigned to the edges of K_6 are identical, then we can colour so that one colour class is $K_{3,3}$ and the other $2 \cdot K_3$; both these graphs have 2-density two. We may thus assume that the lists incident to some vertex v are not all identical. Any $\mathcal{F}$-free colouring of $K_6 - v$ (at least one such colouring exists by Proposition 4.3) can now be extended to K_6 by colouring the edges incident with v so that no colour is repeated more than twice. Since every graph F with $m_2(F) \geqslant 5/2$ has minimum degree at least three, this colouring is also $\mathcal{F}$-free. □

Finally, we prove Proposition 4.4, thus establishing the Deterministic Lemma for families $\mathcal{F}$ with $m_2(\mathcal{F}) \leqslant 2$.

Proof of Proposition 4.4. Suppose by contradiction that a graph G is a minimally list-Ramsey for some family $\mathcal{F}$ of graphs, each containing a cycle, and satisfies $e_G/v_G \leqslant \min\{m_2(\mathcal{F}), 2\}$. Observe first that $\delta(G) \geqslant 3$ and that, for every vertex v of degree three, the lists assigned to the edges incident with v are identical. Indeed, if this were not true, we could extend any $\mathcal{F}$-free colouring of $G - v$ to G by assigning different colours to all edges incident with v; such a colouring would also be $\mathcal{F}$-free as every graph in $\mathcal{F}$ has minimum degree at least two. We may also assume that $m_2(\mathcal{F}) \geqslant e_G/v_G > 3/2$, as otherwise G is 3-regular and Claim 4.11 implies that it is not list-Ramsey for $\mathcal{F}$. In particular, since $m_2(C_4) = 3/2$, the only graphs with fewer than five vertices that can belong to $\mathcal{F}$ are K_3 and K_4.

Case 1. $e_G/v_G \in (3/2, 2)$. We apply Lemma 4.7 to G with $k + \varepsilon = e_G/v_G$. Since we have already ruled out the possibility that $\delta(G) \leqslant 2$, there are only two subcases.

Case 1a. Suppose first that G has an edge uv with $\deg(u) = 3$ and $\deg(v) \leqslant 4$. Recall that the three lists assigned to the edges incident with u are the same, say $\{\text{red}, \text{blue}\}$. Consider an arbitrary $\mathcal{F}$-free colouring of $G - u$. Since v has at most three neighbours in $G - u$, it is incident with at most one blue edge or at most one red edge; without loss of generality, this rare colour is blue. Let w be the other endpoint of the blue edge incident with v. Colour two edges incident with u blue and one edge red so that uw, if it is an edge of G at all, is not blue.

If this colouring was Ramsey for $\mathcal{F}$, we would see a monochromatic copy F of a graph from $\mathcal{F}$ that contains u. As F has minimum degree two, it cannot be red and, if it is blue, it must also contain both v and w and the other blue neighbour of u (which is distinct from w). However, since $m_2(F) > 3/2$, it cannot contain a path of length three whose two centre vertices have degree two; indeed $\delta(K_4) > 2$ and the Helpful Lemma implies that, for every $F \in \mathcal{F}$ with $v_F > 4$ and every pair $\{a, b\}$ of vertices of F, we have $\bar{e}_F(\{a, b\}) > 3$.

Case 1b. Suppose now that $\varepsilon \geqslant 7/8$ and that G has two edges vu_1 and vu_2, where v is of degree 5 and u_1, u_2 are non-adjacent vertices of degree 3. Recall that, for both $i \in \{1, 2\}$, the three lists assigned to the edges incident with u_i are identical. Consider an arbitrary $\mathcal{F}$-free colouring of $G - \{u_1, u_2\}$. Since v has degree three in this graph, for each $i \in \{1, 2\}$, there is a colour c_i belonging to the lists seen at u_i that appears on at most one edge incident with v (it may happen that $c_1 = c_2$). Now, for each $i \in \{1, 2\}$, colour two edges incident to u_i with c_i (and the third edge in the other available colour) so that there is no triangle whose all edges are coloured c_i; this is possible as v had at most one incident edge coloured c_i and u_1 and u_2 are not adjacent.

If this colouring was Ramsey for $\mathcal{F}$, we would see a monochromatic copy F of a graph from $\mathcal{F}$ that contains u_i for some $i \in \{1, 2\}$. As F has minimum degree two, it must be coloured c_i and it must contain both v and at least one more neighbour of v connected by an edge coloured c_i. If there is only one such neighbour, then F would contain a path of length three whose two centre vertices have degree two, which is impossible (see Case 1a). Otherwise, it must be that $c_1 = c_2$, there are exactly two such neighbours, and

$\bar{e}_F(\{v, u_1, u_2\}) = 5$. However, since $m_2(F) \geqslant m_2(\mathcal{F}) \geqslant 1+\varepsilon \geqslant 2-1/8$ but $F \neq K_3$, then either $m_2(F) \geqslant 2$, in which case $\delta(F) \geqslant 3$, or $15/8 \leqslant m_2(F) < 2$, in which case F has at least ten vertices. In both cases, the Helpful Lemma implies that $\bar{e}_F(\{v, u_1, u_2\}) \geqslant 6$, a contradiction.

Case 2. $e_G/v_G = 2$. Since we have already ruled out the possibility that $\delta(G) \leqslant 2$, either $\delta(G) = 3$ or G is 4-regular. Let u be a vertex of smallest degree in G that minimises $e(N(u))$ and consider an arbitrary $\mathcal{F}$-free colouring of $G-u$. We will show that it can be extended to an $\mathcal{F}$-free colouring of G, unless $G = K_5$. As every $F \in \mathcal{F}$, other than the triangle, has minimum degree at least three, it is enough to show there is a colouring of the edges incident with u that uses each colour at most twice and has no monochromatic triangles containing u. It is straightforward to check that such a colouring exists when $\deg(u) = 3$, so we will assume that G is 4-regular.

Case 2a. $e(N(u)) \leqslant 4$. Denote the four neighbours of u by $v_1, \ldots, v_4$. If the four lists assigned to $uv_1, \ldots, uv_4$ are identical, say $\{\text{red}, \text{blue}\}$, then we can colour as follows: Since $N(u)$ is not a clique, we may assume that $v_1v_2 \notin G$. If $v_3v_4 \in G$ and it is coloured red, colour uv_3, uv_4 blue and uv_1, uv_2 red; otherwise, colour uv_3, uv_4 red and uv_1, uv_2 blue. It is straightforward to check that no monochromatic triangle was created.

If the four lists assigned to $uv_1, \ldots, uv_4$ are not identical, then some colour, say red, appears in the lists of at most two edges incident with u. If red appears only once, say in the list of uv_1, then we colour uv_1 red and the remaining three edges as in the case where $\deg(u) = 3$. Suppose now that red appears in two lists, say of uv_1 and uv_2. Since $N(u)$ induces at most four edges, either $v_3v_4 \notin G$ or one of v_1, v_2 is not adjacent to one of v_3, v_4. If $v_3v_4 \notin G$, then we colour uv_1 red, uv_2 with a colour c different than red and both v_3 and v_4 with a colour different than c; it is easy to check that no colour is used more than twice and that no triangle is monochromatic. Otherwise, we may assume without loss of generality that $v_2v_4 \notin G$. Now, colour uv_1 red, uv_2 with a colour c different than red, uv_3 with a colour $c' \neq c$, and uv_4 with a colour different than c'. Again, it is easy to check that no colour was used more than twice and that no triangle is monochromatic.

Case 2b. $e(N(u)) \geqslant 5$. Since G is connected, 4-regular, and $e(N(v)) \geqslant 5$ for all $v \in V(G)$, it must be K_5, see Claim 4.10. Thus, it is enough to show that K_5 is not list-Ramsey for any family $\mathcal{F}$ with $m_2(\mathcal{F}) \geqslant 2$. To this end, we first claim that every 2-balanced graph with 2-density at least 2 that is contained in K_5 contains a triangle. Indeed, if F is strictly 2-balanced and $m_2(F) \geqslant 2$, then $v_F \geqslant 3$ and $e_F \geqslant 2(v_F-2)+1 = 2v_F - 3$, but every triangle-free graph F satisfies $e_F \leqslant v_F^2/4$; it is easy to check that $2v_F - 3 > v_F^2/4$ when $v_F \in \{3, 4, 5\}$. Consequently, any triangle-free colouring of K_5 will automatically be $\mathcal{F}$-free for every family $\mathcal{F}$ with $m_2(\mathcal{F}) \geqslant 2$. Finally, the fact that K_5 is 2-list-colourable with respect to K_3 was proved in Claim 4.13. □

5. Allowing forests

When we allow our family to contain graphs F with $m_2(F) = 1$, i.e., forests, the assertion of Theorem 1.6 no longer holds unconditionally. We see the most drastic change in behaviour when our family contains a star forest; in this case, it is not hard to see that the threshold is at $n^{-(1+1/s)}$, with $s = (r-1)(\Delta-1)+1$ and where Δ is the smallest maximum degree of a star forest in the family. Otherwise, the threshold is located at n^{-1}, but in some cases the 0-statement will not hold if we only assume that $p \leqslant cn^{-1}$ for any constant $c > 0$.

In the case when $m_2(\mathcal{F}) = 1$, we can no longer make the reduction to strictly 2-balanced graphs, as this would leave us with K_2, which is unavoidable (unless the graph is empty). Instead, we will replace the Probabilistic Lemma with the following well-known fact about sparse random graphs (see [6, Section 5]), which can be readily proved using an argument similar to the one we used to prove the Probabilistic Lemma.

Lemma 5.1. *There exists a constant $c > 0$ such that a.a.s. whenever $p \leqslant cn^{-1}$ every connected component of $G_{n,p}$ has at most one cycle.*

In view of the lemma, in order to prove the bulk of Theorem 1.8, it suffices to devise a colouring scheme for graphs whose all connected components are either trees or unicyclic graphs, i.e., graphs that consist of exactly one cycle, with trees rooted at some of its vertices.

Proposition 5.2. *Let $\mathcal{F}$ be a family of graphs and let $r \geqslant 2$ be an integer. Then every graph G with $m(G) \leqslant 1$ is not r-list-Ramsey with respect to $\mathcal{F}$ unless $r = 2$ and $\mathcal{F}$ contains both a $\mathcal{B}$-graph and a $\mathcal{C}^*$-graph.*

Further, since every graph G with $m(G) = 1$ appears in $G_{n,p}$ with probability $\Omega(1)$ whenever $p = \Omega(n^{-1})$, the following proposition completes the proof of Theorem 1.8.

Proposition 5.3. *For every family $\mathcal{F}$ of graphs that contains both a $\mathcal{B}$-graph and a $\mathcal{C}^*$-graph, there exists a graph G with $m(G) \leqslant 1$ that is list-Ramsey with respect to $\mathcal{F}$.*

Finally, we turn to the usual (non-list) Ramsey threshold for families of graphs that contain both a $\mathcal{B}$-graph and a $\mathcal{C}^*$-graph. The classification there is slightly more complex, and is described using an auxiliary hypergraph.

Definition 5.4. Given a family of graphs $\mathcal{F}$ that contains both a $\mathcal{B}$-graph and a $\mathcal{C}^*$-graph, we define an auxiliary hypergraph $\mathcal{A} = \mathrm{Aux}(\mathcal{F})$ in the following way. The vertices of $\mathcal{A}$ are all odd cycles $C_{2\ell+1}$, where $\ell \geqslant 1$ is an integer. Further, for every $F \in \mathcal{F}$ that is a $\mathcal{C}^*$-graph whose non-star components are $F^1, \ldots, F^m$, we add to $\mathcal{A}$ an edge for every choice of odd cycles $C^1, \ldots, C^m$ such that $F^i \subseteq C^i$ for all i.

For example, let C^1 and C^2 be two odd cycles of different lengths and let B be a broom with at least two hairs. Then, taking a family $\mathcal{F}_1$ which consists of the broom B and the disjoint union $C^1 \cup C^2$ will give us $\mathrm{Aux}(\mathcal{F}_1)$ which is isomorphic to the graph K_2. If we take another odd cycle C^3 such that the lengths are still distinct, then $\mathcal{F}_2 = \{B, C^1 \cup C^2, C^1 \cup C^3, C^2 \cup C^3\}$ will have $\mathrm{Aux}(\mathcal{F}_2) \cong K_3$.

The following proposition completes the proof of Theorem 1.9.

Proposition 5.5. *Let $\mathcal{F}$ be a family of graphs that contains both a $\mathcal{B}$-graph and a $\mathcal{C}^*$-graph. There exists a graph G with $m(G) \leqslant 1$ that is 2-Ramsey for $\mathcal{F}$ if and only if $\mathcal{F}$ contains a star forest or $\mathrm{Aux}(\mathcal{F})$ is not 2-colourable.*

Carrying on with our example from before, $\mathrm{Aux}(\mathcal{F}_1)$ is 2-colourable, while $\mathrm{Aux}(\mathcal{F}_2)$ is not, meaning that the Deterministic Lemma would hold for the former and fail for latter.

Remark. Note that the combination of these propositions implies a discrepancy in the behaviour of the thresholds for list-Ramsey and usual Ramsey. Indeed, for $\mathcal{F}_1$ both thresholds appear at n^{-1}, but while the usual Ramsey threshold is semi-sharp, for every $c > 0$ there is a positive probability that $G_{n,c/n}$ is still 2-list-Ramsey for $\mathcal{F}_1$.

We prove Propositions 5.2–5.5 in the three subsections that follow.

5.1. **Proof of Proposition 5.2.** A connected graph G with $m(G) \leqslant 1$ whose every edge is assigned a list of $r \geqslant 2$ colours is called *nontrivial* if it contains an odd cycle, $r = 2$, and all the lists are identical; otherwise, we call G *trivial.*

Claim 5.6. *Suppose that every edge of a connected graph G with $m(G) \leqslant 1$ is assigned a list of $r \geqslant 2$ colours. There is a colouring from these lists such that every monochromatic subgraph is a star forest unless G is nontrivial.*

Proof. Fix an arbitrary orientation of G with out-degree at most one. Our aim is to choose colours for all edges of G in such a way that the colour of an edge uv, oriented from u to v, is different from the colour of the out-edge from v; call such a colouring *nonrepetitive.* Since each connected graph that is not a star contains either a path of length three or a triangle and since each orientation of the path of length three and of K_3 with out-degree at most one contains a directed path of length two, the colour classes of a nonrepetitive colouring are guaranteed to be star forests.

It is not hard to see that G admits a nonrepetitive colouring if and only if it contains a cycle and the cycle admits a nonrepetitve colouring. Indeed, if a graph H admits a nonrepetitve colouring, then the graph obtained from H by attaching to it a new leaf v whose incident edge is directed from v to H also admits a nonrepetitive colouring; moreover, our G can be built from a (directed) cycle or a single vertex using a sequence of such operations.

Suppose now that G is a (directed) cycle. If either $r \geqslant 3$ or not all colour lists are identical, then we can construct a nonrepetitve colouring as follows. Denote the vertices of G by $v_1, \dots, v_\ell$ so that its directed edges are (v_i, v_{i+1}), for $i \in \{1, \dots, \ell\}$, where the addition is modulo ℓ. Our assumption implies that there is an i and a colour c that belongs to the list of $v_i v_{i+1}$ such that the list of $v_{i+1} v_{i+2}$ contains at least two colours different than c; without loss of generality, we may assume that $i = 1$. We now colour $v_1 v_2$ with c, then the edges $v_\ell v_1, v_{\ell-1} v_\ell, \dots, v_3 v_4$, and finally the edge $v_2 v_3$ with a colour different than c and the colour chosen for $v_3 v_4$. Finally, if $r = 2$ and all the lists are identical, then a nonrepetitve colouring exists as long as G is an even cycle. □

Proof of Proposition 5.2. Suppose first that $\mathcal{F}$ does not contain a $\mathcal{C}^*$-graph. We claim that one can colour G so that every monochromatic component is a star or an odd cycle and thus every monochromatic subgraph is a $\mathcal{C}^*$-graph. Indeed, Claim 5.6 allows us to colour every trivial component of G. For each nontrivial component, we colour its cycle red and each edge not on the cycle according to the parity of its distance from the cycle so that the edges with one endpoint on the cycle are coloured blue, see Figure 1. This way, the monochromatic graphs are disjoint unions of odd cycles and stars.

Suppose now that $\mathcal{F}$ does not contain a $\mathcal{B}$-graph. We claim that one can colour G so that every monochromatic component is a star or a broom and thus every monochromatic subgraph is a $\mathcal{B}$-graph. Again, Claim 5.6 allows us to colour every trivial component of G. For each nontrivial component, which contains an odd cycle $v_1 \dots v_{2\ell+1}$, we colour as follows (see Figure 1). The edges $v_1 v_2, v_3 v_4, \dots, v_{2\ell-1} v_{2\ell}$ are red and the remaining edges of the cycle are blue (so that the only monochromatic path of length two is the blue path $v_{2\ell} v_{2\ell+1} v_1$); each edge not on the cycle, which lies on a tree rooted at some v_i, is coloured according to the parity of its distance from v_i so that the edges with one endpoint on the cycle are coloured differently than the edge $v_i v_{i+1}$. This way, all but one monochromatic components are stars, with the only exception being the (blue) component of $v_{2\ell}v_{2\ell+1}v_1$, which is a broom. □

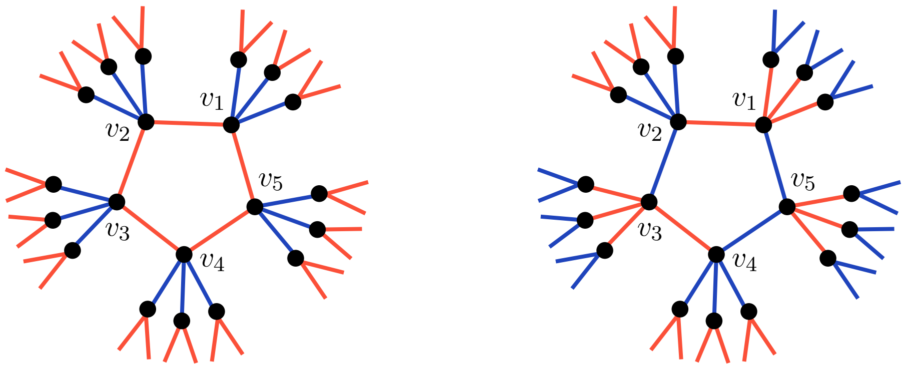


FIGURE 1. Colourings resulting in $\mathcal{C}^*$-graphs and $\mathcal{B}$-graphs.

5.2. **Proof of Proposition 5.3.** The situation gets a little more complex once we allow brooms and $\mathcal{C}^*$-graphs to mingle. As we will see in this subsection, for every family $\mathcal{F}$ of graphs that contains both $\mathcal{B}$-graphs and $\mathcal{C}^*$-graphs, there are graphs G with $m(G) \leqslant 1$ that are list-Ramsey for $\mathcal{F}$.

Definition 5.7. Given a broom B with b hairs, an odd cycle C of length ℓ, and a star S with s rays, we define the following two graphs:

- $F(B, C)$ is obtained from C by attaching a star with $2b - 1$ rays to each vertex;
- $T(B, S)$ is a rooted tree with two layers, where the root has degree $2 \cdot \max\{b + 1, s\} - 1$ and each of its children has s children.

Obviously $m(F(B, C)) = 1$ and $m(T(B, S)) \leqslant 1$. We will show that for any odd cycle and broom, there is a graph of that ilk that is Ramsey for them.

Claim 5.8. *For every broom B and odd cycle C, the graph $F(B, C)$ is 2-Ramsey with respect to $\{B, C\}$.*

Proof. Let b denote that number of hairs in B. Fix some $\{\text{red}, \text{blue}\}$-colouring of $F(B, C)$. We may assume that the cycle in $F(B, C)$ is not monochromatic, since otherwise there is nothing left to prove. Let $P_1, ..., P_{2k}$ be all the maximal monochromatic subpaths of the cycle, labeled in such a way that P_i lies between P_{i-1} and P_{i+1}. By the pigeonhole principle, some b (out of $2b - 1$) rays of the star attached to every endpoint of each of these paths are coloured the same; colour each endpoint with such majority colour. If both endpoints of some P_i have the same colour as P_i, then there is a monochromatic copy of B, so we may assume that this is not the case. This means that the colouring must have the following regular form ('right' can be changed to 'left'): Each P_i and its right endpoint have the same colour, which depends on the parity of i. However, some P_i must have length at least two (since the cycle is odd), but then P_i and the star at its right endpoint contain a monochromatic copy of B. □

Claim 5.9. *For every broom B and star S, any $\{red, blue\}$-colouring of the edges of $T(B, S)$ admits either a monochromatic copy of B or copies of S in both colours.*

Proof. By the pigeonhole principle, in any $\{\text{red}, \text{blue}\}$-colouring of $T(B, S)$, its root v must be connected to $m := \max\{b + 1, s\}$ vertices, say $u_1, \ldots, u_m$, in one of the colours, say red. We thus already have a red star with m rays, and in particular a red copy of S. If the edge connecting u_1 to any of its s children is red, then we have a red copy of the broom with $m-1$ hairs, and in particular a red copy of B. Otherwise, all the edges connecting u_1 to its children are blue, meaning that we have a blue copy of S. □

Proof of Proposition 5.3. Suppose that $\mathcal{F}$ contains a $\mathcal{B}$-graph X and a $\mathcal{C}^*$-graph Y. We will construct a graph G with $m(G) = 1$, with a list of two colours assigned to each edge, such that any colouring of G from these lists yields a monochromatic copy of either X or Y.

Let B be a broom with sufficiently many hairs so that every component of X is a subgraph of B and suppose that the connected components of Y are stars $\{S^1, \dots, S^k\}$ and (subgraphs of) odd cycles $\{C^1, \dots, C^m\}$. Define, for all $j \in [\![k+m]\!]$,

$$A_j := \begin{cases} T(B, S^j), & j \leqslant k, \\ F(B, C^{j-k}), & j > k, \end{cases}$$

and let A be the disjoint union of all A_j for $j \in [\![k+m]\!]$.

We first define a graph G', together with an assignment of colours, in the following way. For every $\{a, b\} \in \binom{[\![k+m+1]\!]}{2}$, add to G' a disjoint copy of A whose every edge receives the same colour list $\{a, b\}$. We claim that every colouring of G' from these lists either has a monochromatic copy of B or a monochromatic copy of Y. Indeed, suppose we colour from the lists without creating a monochromatic copy of B. By Claim 5.9, for every $j \leqslant k$ and each pair $\{a, b\}$, the colouring of the corresponding copy of $A_j = T(B, S^j)$ will contain a monochromatic copy of S^j in either colour a or colour b.[6] This means thatm, for every j, there is a copy of S^j in all but at most one colour from $[\![k+m+1]\!]$. The same argument applies to every C^j. Since there are $m+k+1$ colours and $m+k$ graphs, there must be some colour that contains a copy of all S^j and all C^j. Consequently, every colouring of G' from the lists has a monochromatic copy of either B or Y.

Suppose now that X has t connected components. Define G to be the disjoint union of $(t-1)(k+m+1)+1$ copies of G' together with their list assignments. Any colouring of G from these lists, either has a monochromatic copy of Y in one of the copies of G' or a monochromatic copy of B, in some colour in $[\![k+m+1]\!]$, in each copy of G'. In the latter case, by the pigeonhole principle, there is a colour containing t disjoint copies of B, thus introducing a monochromatic copy of X. □

5.3. **Proof of Proposition 5.5.** Suppose first that $\mathcal{A} := \mathrm{Aux}(\mathcal{F})$ is 2-colourable and that $\mathcal{F}$ does not contain a star forest. We wish to show that any G with $m(G) \leqslant 1$ admits a $\{\mathrm{red}, \mathrm{blue}\}$-colouring of its edges without a monochromatic copy of any member of $\mathcal{F}$. Fix a proper 2-colouring $\varphi \colon V(\mathcal{A}) \to \{\mathrm{red}, \mathrm{blue}\}$ of the auxiliary hypergraph $\mathcal{A}$ and denote the connected components of G by $G_1, \dots, G_m$. For every $i \in [\![m]\!]$, we colour G_i as follows:

- If G_i contains an odd cycle C, colour the edges of the cycle with $\varphi(C)$ and the remaining edges according to the parity of their distance from the cycle, so that the monochromatic components are C and stars.
- Otherwise, if G_i contains no odd cycle, colour it as in Claim 5.6, so that both monochromatic graphs are star forests.

[6]Actually, Claim 5.9 guarantees that the colouring of A_j contains monochromatic copies of S^j in both colours, but we do not need this stronger statement here.

We claim that this colouring if $\mathcal{F}$-free. Indeed, every monochromatic subgraph must be a $\mathcal{C}^*$-graph. Furthermore, if $F \in \mathcal{F}$ became monochromatic and its non-star components are $F^1, \dots, F^m$, there were odd cycles $C^1, \dots, C^m$, with $F^i \subseteq C^i$ for each i, that all received the same colour. However, this would mean that the edge $\{C^1, \dots, C^m\} \in \mathcal{A}$ was monochromatic, a contradiction.

Suppose now that either $\mathcal{F}$ contains a star forest or $\mathcal{A}$ is not 2-colourable. If the former holds, and $\mathcal{F}$ contains a union of m stars, each having at most s rays, then the disjoint union of $2m-1$ copies of a star with $2s-1$ rays, which has density less than one, is clearly Ramsey for $\mathcal{F}$. We may therefore suppose that the latter holds, i.e., that $\mathcal{A}$ is not 2-colourable. Using compactness, we can find a finite subhypergraph $\mathcal{A}' \subseteq \mathcal{A}$ that is already not 2-colourable.

Let X be an arbitrary $\mathcal{B}$-graph from $\mathcal{F}$ and let B be a broom with sufficiently many hairs so that each component of X is a subgraph of B. Define G' to be a graph whose connected components are $\{F(B,C) : C \in V(\mathcal{A}')\}$. We claim that every $\{\text{red}, \text{blue}\}$-colouring of the edges of G' admits a monochromatic copy of either B or the disjoint union $C^1 \cup \dots \cup C^k$ for some $\{C^1, \dots, C^k\} \in \mathcal{A}'$. Indeed, since each component of G' has a monochromatic copy of B or the corresponding cycle, by Claim 5.8, we either have a monochromatic copy of B or a monochromatic copy of every $C \in V(\mathcal{A}')$. However, since the hypergraph $\mathcal{A}'$ is not 2-colourable, there must be some collection $C^1, \dots, C^k$ of cycles that form an edge $\mathcal{A}'$ and were all coloured the same.

In order to define G, consider the subfamily $\mathcal{F}' \subseteq \mathcal{F}$ that consists of all $\mathcal{C}^*$-graphs that contribute an edge to $\mathcal{A}'$. Since $\mathcal{F}'$ is finite, we may give the next definitions:

- Let S be a star with sufficiently many rays so that S contains every star in each $F \in \mathcal{F}'$.
- For every graph A, let $r(A)$ denote the maximum, over all $F \in \mathcal{F}'$, number of components F^0 of F such that $F^0 \subseteq A$.

Now, let $r := \max\{r(C) \colon C \in V(\mathcal{A}')\}$ and let x denote the number of components of X. Finally, define G to be the disjoint union of $2(r-1) \cdot e(\mathcal{A}') + 2x - 2$ copies of G' and $r(S) + 2x - 2$ copies of $T(B,S)$. Since every component of G has at most one cycle, we clearly have $m(G) \leqslant 1$. To finish the proof, it suffices to show that G is Ramsey for $\mathcal{F}$.

Consider any $\{\text{red}, \text{blue}\}$-colouring of G. We have already shown that every G'-component contains a monochromatic copy of either B or the disjoint union $C^1 \cup \dots \cup C^k$ for some $\{C^1, \dots, C^k\} \in \mathcal{A}'$. Moroever, by Claim 5.9, each $T(B,S)$-component contains either a monochromatic copy of B or monochromatic copies of S in both colours. If there are $2x-1$ monochromatic copies of B, some x amongst them have the same colour and therefore we get a monochromatic copy of X. Otherwise, there are at least $2(r-1) \cdot e(\mathcal{A}')$ copies of G' and $r(S)$ copies of $T(B,S)$ that do not contain a monochromatic copy of B. This means that we have $r(S)$ copies of S in both colours and $2(r-1) \cdot e(\mathcal{A}')$ monochromatic edges of $\mathcal{A}'$. By the pigeonhole principle, some edge $\{C^1, \dots, C^k\} \in \mathcal{A}$ is repeated r times in the same colour, say red. Let $F \in \mathcal{F}'$ be a graph that generated this edge. We claim that G has a monochromatic copy of F. Indeed, every component of F is either a star, and thus contained in S, or contained in some C^i for $i \in [\![k]\!]$. Moreover, F has at most $r(S)$ star components and at most r components that are contained in every C^i. We conclude that there is a red copy of F.

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