Additive combinatorics

Homework assignment #3

Problem 1. Let A, B, and C be finite nonempty sets of positive real numbers. Show that $|A \cdot B + C| \geqslant c\sqrt{|A| \cdot |B| \cdot |C|}$ for some positive constant c.

Problem 2. Let A be a finite set of integers. Without invoking Freiman's $3k-4$ theorem, show that $|A+A| \leqslant 2|A|-1$ if and only if A is an arithmetic progression.

Problem 3. Let A and B be nonempty finite set of elements of an Abelian group.

(a) Show that $|A+B| = |A|$ if and only if $|A-B| = |A|$.

(b) Show that $|A+B| = |A| \cdot |B|$ if and only if $|A-B| = |A| \cdot |B|$.

Problem 4. Let A be a nonempty finite set of elements of an Abelian group.

(a) Show that $|A-A| \leqslant |A+A|^{3/2}$.

(b) Show that $|A+A| \leqslant |A-A|^{3/2}$.

Problem 5. Let G be an Abelian group whose each element has order at most r. Let A be a finite set of elements of G and let H be the subgroup of G generated by A. Suppose that there exists a set B of elements of G with $|B| = |A|$ such that

$$|A+B| \leqslant c|A|$$

for some $c \geqslant 1$. Show that $|H| \leqslant K|A|$ for some constant K that depends only on c and r.

Problem 6. Suppose that $m \geqslant n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{Z}^n$ are integer vectors that span $\mathbb{R}^n$ as a linear space over $\mathbb{R}$. Show that the subgroup of $\mathbb{Z}^n$ generated by $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a lattice in $\mathbb{R}^n$.