Concentration inequalities

Homework assignment #3

Due date: Thursday, January 21, 2015

Problem 1. Suppose that $f\colon \{-1,1\}^n \to \mathbb{R}$ and X is a uniform random element of $\{-1,1\}^n$. Define

$$\mathcal{E}(f) = \frac{1}{2}\sum_{i=1}^{n} \mathbb{E}\left[\left(f(X) - f(\tilde{X}^{(i)})^2\right)\right],$$

where $\tilde{X}^{(i)}$ is obtained from X by replacing the ith coordinate with its independent copy. Recall that the Efron–Stein inequality states that $\mathrm{Var}(f(X)) \leqslant \mathcal{E}(f)$ and that the logarithmic Sobolev inequality implies that $\mathrm{Ent}(f(X)^2) \leqslant 2\mathcal{E}(f)$. Show that the latter implies the former. To simplify the computation, you may assume that $\mathbb{E}[f] = 0$.
Hint: Show that $\mathrm{Ent}\left((1+\varepsilon f)^2\right) = 2\varepsilon^2 \mathrm{Var}(f) + O(\varepsilon^3)$.

Problem 2. Assume that $f\colon \mathbb{R}^n \to \mathbb{R}$ is nondecreasing (in each variable) and let $X_1, \ldots, X_n$ be independent real-valued random variables. Let $Z = f(X) = f(X_1, \ldots, X_n)$ and for each $i \in [n]$, let

$$Z_i = \sup_{x_i'} f(X_1, \ldots, X_{i-1}, x_i', X_{i+1}, \ldots, X_n).$$

Suppose that there exists another nondecreasing function $g\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\sum_{i=1}^{n} (Z - Z_i)^2 \leqslant g(X).$$

Show that for all $t > 0$,

$$\Pr\left(Z < \mathbb{E}[Z] - t\right) \leqslant \exp\left(-\frac{t^2}{2\mathbb{E}[g(X)]}\right).$$

Hint: Use Harris' inequality, which states that if $\alpha, \beta\colon \mathbb{R}^n \to \mathbb{R}$ are nondecreasing functions and $X_1, \ldots, X_n$ are independent real-valued random variables, then $\mathbb{E}[\alpha(X)\beta(X)] \geqslant \mathbb{E}[\alpha(X)]\mathbb{E}[\beta(X)]$.

Problem 3. Let $(\mathcal{X}, d)$ be a metric space, let X be an $\mathcal{X}$-valued random variable distributed according to a probability measure Pr, and let $\alpha\colon [0,\infty) \to [0,1]$ be the corresponding concentration function, that is,

$$\alpha(t) = \sup\Big\{ \Pr\big(d(X,A) \geqslant t\big) \colon\ \Pr(X \in A) \geqslant 1/2 \Big\}.$$

(i) Suppose that $\beta\colon [0,\infty) \to [0,1]$ is such that for every 1-Lipschitz function $f\colon \mathcal{X} \to \mathbb{R}$,

$$\Pr(f(X) \geqslant \mathbb{E}[f(X)] + t) \leqslant \beta(t).$$

Show that $\alpha(t) \leqslant \beta(t/2)$.

(ii) Define the *Laplace functional* by

$$L(\lambda) = \sup_f \mathbb{E}[e^{\lambda f(X)}],$$

where the supremum is taken overy all 1-Lipschitz functions $f\colon \mathcal{X} \to \mathbb{R}$ with $\mathbb{E}[f(X)] = 0$.

(a) Show that for all t,

$$\alpha(t) \leqslant \inf_{\lambda>0} e^{-\lambda t/2} L(\lambda).$$

(b) Denote the diameter of $\mathcal{X}$ by $D = \sup_{x,y\in\mathcal{X}} d(x,y)$ and assume that $D < \infty$. Show that

$$L(\lambda) \leqslant e^{D^2\lambda^2/2}.$$

Hint: If $\mathbb{E}[f(X)] = 0$ and Y is an independent copy of X, then $\mathbb{E}[e^{\lambda f(X)}] \leqslant \mathbb{E}[e^{\lambda(f(X)-f(Y))}]$.

(iii) Suppose that $(\mathcal{X}_1, d_1), \ldots, (\mathcal{X}_n, d_n)$ are metric spaces equipped with probability measures $\Pr_1, \ldots, \Pr_n$, respectively. Let $\Pr = \Pr_1 \times \ldots \times \Pr_n$ be the product measure on the space $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$, a metric space with metric $d(x,y) = \sum_{i=1}^n d_i(x_i, y_i)$. Show that

$$L(\lambda) \leqslant \prod_{i=1}^n L_i(\lambda),$$

where L is the Laplace functional of $\Pr$ and L_i is the Laplace functional of $\Pr_i$.

(iv) Use (ii)b and (iii) to give another proof of the bounded differences inequality (with worse constants).

Problem 4. Prove Harper's vertex isoperimetric theorem in the hypercube in the following three steps:

(i) Let $A \subseteq \{0,1\}^n$. For each $i \in [n]$, define two i-sections of A by

$$A_\varepsilon^{(i)} = \{x_\varepsilon^{(i)} := (x_1, \ldots, x_{i-1}, \varepsilon, x_{i+1}, \ldots, x_n) \colon x_\varepsilon^{(i)} \in A\}, \quad \varepsilon \in \{0,1\}.$$

Let $S_\varepsilon^{(i)}(N)$ denote the set of the first N elements in the simplicial ordering of all vectors x with $x_i = \varepsilon$. Let $C_i(A)$ denote the set whose i-sections are $S_\varepsilon^{(i)}(|A_\varepsilon^{(i)}|)$. (Here, C stands for *compression*.) Prove that $\partial_V(C_i(A)) \leqslant \partial_V(A)$, where $\partial_V(B)$ is the set of vertices in $\{0,1\}^n$ whose (Hamming) distance from B is exactly one.

(ii) Show that each set $A \subseteq \{0,1\}^n$ can be compressed only finitely many times. That is, for every sequence $i_1, i_2, \ldots \in [n]$, if we define $A_0 = A$ and $A_k = C_{i_k}(A_{k-1})$ for every k, then the seqence (A_k) eventually stabilizes.

(iii) Let $A \subseteq \{0,1\}^n$ be a fully compressed set, i.e., $C_i(A) = A$ for each $i \in [n]$. For $x \in \{0,1\}$, define

$$\|x\| = \|x\|_{\ell^1} = |\{i \colon x_i = 1\}|.$$

Prove that either A is the set of the first $|A|$ elements in the simplicial ordering of $\{0,1\}^n$ or:

(a) n is even and

$$A = \{x \colon \|x\| < n/2\} \cup \{x \colon \|x\| = n/2, x_1 = 1\} \cup \{y\} \setminus \{z\},$$

where $y_i = 1$ if and only if $i \in \{2, \ldots, n/2+1\}$ and $z = (1, \ldots, 1) - y$.

(b) n is odd and

$$A = \{x \colon \|x\| < n/2\} \cup \{y\} \setminus \{z\},$$

where $y_i = 1$ if and only if $i \in \{1, 2, \ldots, (n+1)/2\}$ and $z = (1, \ldots, 1) - y$.

Show that both these exceptional sets A have a larger vertex boundary than the set of the first $|A|$ elements in the simplicial ordering.