Graph Theory

Homework assignment #1

Due date: Sunday, November 20, 2016

Problem 1. Prove that for each $n \geqslant 1$, the number of graphs with vertex set $\{1, \ldots, n\}$ and all degrees even is $2^{\binom{n-1}{2}}$.

Problem 2. Suppose that $n \geqslant 9$. Prove that every n-vertex graph with at least $7n - 27$ edges contains a subgraph with minimum degree at least 8.

Problem 3. Prove that every n-vertex 3-regular graph G admits a bipartition $V(G) = V_1 \cup V_2$ such that the number of edges of G with one endpoint in each V_1 and V_2 is at least n.

Problem 4. Prove that either a graph or its complement is connected.

Problem 5. Let G be a graph with $\delta(G) \geqslant 2$. Show that there is a *connected* graph with the same vertex set and the same degree sequence. More precisely, show that there is a connected graph H with $V(H) = V(G)$ such that $\deg_G(v) = \deg_H(v)$ for all $v \in V(G)$.

Problem 6. Let $d_1, \ldots, d_n$ be positive integers. Prove that there exists a tree with degrees $d_1, \ldots, d_n$ if and only if

$$d_1 + \ldots + d_n = 2n - 2.$$

Problem 7. Prove that every graph G contains each tree with $\delta(G)$ edges as a subgraph.

Problem 8. Compute the number of spanning trees of the complete bipartite graph $K_{m,n}$.

Please do NOT submit written solutions to the following exercises:

Exercise 1. Show that every graph with at least two vertices has two vertices of equal degree.

Exercise 2. Show that every tree with maximum degree $\Delta \geqslant 1$ has at least Δ leaves.

Exercise 3. Show that a graph is bipartite if and only if it contains no odd cycles. In particular, all trees are bipartite.

Exercise 4. Prove the Cauchy–Binet formula:

$$\det AB = \sum_{J \in \binom{[n]}{m}} \det A_J \cdot \det B_J,$$

where A_J is the $m \times m$ submatrix of A consisting of the columns indexed by J and B_J is the $m \times m$ submatrix of B consisting of the rows indexed by J.