Graph Theory

Homework assignment #3

Due date: Sunday, December 24, 2017

Problem 1. Let G be a bipartite graph with bipartition $V(G) = A \cup B$. Define

$$\delta(A) = \max_{S \subseteq A} \big(|S| - |N(S)|\big).$$

Prove that the maximum size of a matching in G is $|A| - \delta(A)$.

Problem 2. An $n \times n$ Latin squere (resp. $r \times n$ Latin rectangle) is an $n \times n$ (resp. $r \times n$) matrix with entries in $\{1, \ldots, n\}$ such that no two entries in the same row or column are the same. Prove that if $r < n$, then every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.

Problem 3. Let k and ℓ be integers. Show that any two partitions of $\{1, \ldots, k\ell\}$ into k-element sets admit a common choice of ℓ representatives.

Problem 4. Let G be a connected graph with an even number of edges. Use Tutte's theorem to prove that the set of edges of G can be partitioned into pairwise disjoint paths of length 2.

Problem 5. Let v be a vertex of a connected graph G and, for $r \geqslant 0$, let G_r be the subgraph of G induced by the vertices at distance exactly r from v. Show that

$$\chi(G) \leqslant \max \left\{ \chi(G_r) + \chi(G_{r+1}) \colon r \geqslant 0 \right\}.$$

Problem 6. Suppose that a graph G is a union of k trees. Prove that $\chi(G) \leqslant 2k$.

Problem 7. Suppose that every pair of odd cycles in a graph G has a common vertex. Show that $\chi(G) \leqslant 5$.

Problem 8. Let G be a graph on n vertices. Prove that $\chi(G) \cdot \chi(\bar{G}) \geqslant n$.

Please do NOT submit written solutions to the following exercises:

Exercise 1. Prove that every bipartite graph with maximum degree Δ is a subgraph of some Δ-regular bipartite graph. Use this fact to give another proof of König's theorem.

Exercise 2. A square matrix $A = (a_{ij})$ of nonnegative real numbers is called *doubly stochastic* if the entries of each row and each column sum up to 1, that is, for every i and j,

$$\sum_i a_{ij} = \sum_j a_{ij} = 1.$$

A doubly stochastic matrix with all entries in $\{0, 1\}$ is called a *permutation matrix.* Prove the *Birkhoff–von Neumann theorem*, which states that every doubly stochastic matrix is a convex combination of permutation matrices.